



This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

- + *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.
- + *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.
- + *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.
- + *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at <http://books.google.com/>



3 3433 06907213 4

1

UNIVERSITY OF CALIFORNIA PUBLICATIONS

PUBLICATIONS

OF THE

112 ✓
LICK OBSERVATORY

PRINTED BY AUTHORITY OF THE REGENTS OF THE UNIVERSITY



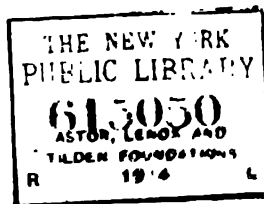
VOLUME VII

SACRAMENTO

FRIEND WM. RICHARDSON

SUPERINTENDENT OF STATE PRINTING

1913



ORGANIZATION OF THE BERKELEY ASTRONOMICAL DEPARTMENT.

(STUDENTS' OBSERVATORY.)

BENJAMIN IDE WHEELER	- - - - -	<i>President of the University</i>
A. O. LEUSCHNER	- -	<i>Professor of Astronomy and Director of the Students' Observatory, Dean of the Graduate School</i>
R. T. CRAWFORD	- -	<i>Associate Professor of Practical Astronomy</i>
S. EINARSSON	- - -	<i>Instructor in Practical Astronomy</i>
SETH B. NICHOLSON	- - - - -	<i>Instructor in Astronomy</i>
SOPHIA H. LEVY	- - - - -	<i>University Fellow in Astronomy</i>
ANNA R. KIDDER	- - - - -	<i>University Fellow in Astronomy</i>
MRS. S. B. NICHOLSON	- - - - -	<i>Assistant in Astronomy</i>

CONTRIBUTIONS
OF THE
BERKELEY ASTRONOMICAL DEPARTMENT,
(STUDENTS' OBSERVATORY.)
UNIVERSITY OF CALIFORNIA,

I.

A. O. LEUSCHNER,
DIRECTOR, STUDENTS' OBSERVATORY.

SAVED BY THE BELL

PREFACE.

Parts 1, 2 and 3 of this volume were completed and printed in the fall of 1902. A number of separates were distributed by the Students' Observatory at that time. Parts 4, 5 and 6 were completed in May of 1905, spring of 1906, and March of 1907, respectively, but on account of pressure of work in the State Printing Office at Sacramento the last of these parts did not come off the press until the spring of 1909. Unfortunately, since the material for the completion of the volume was then ready and the prospects for having the remainder printed without delay seemed bright, no separates of Parts 4 to 6 were distributed at the proper time. The manuscript of Part 7 and practically also of Part 9, with which as Part 8 it was originally intended to conclude the volume was ready to be printed in April, 1909, but owing to the lack of University funds at the State Printing Office the printing of these papers was unavoidably delayed. It, therefore, seemed expedient to extend the scope of the volume so as to include the material which in the mean time had become available, as Parts 8 and 10, and to add the tables at the end of Part 7.

It may be of interest to point out that the opportunities for research are greatly limited in this department by our extensive duties of instruction, but this disadvantage is offset to some extent by the voluntary assistance given by graduate students in the numerical work incident to the investigations contained in these pages, particularly in the construction of tables and determination of orbits. The orbits included in Parts 8 and 10 as typical examples of the Short Methods form only a fraction of those derived with the assistance of students, mostly in connection with the regular courses of instruction.

The *Short Method*, originally proposed in Part 1, appears in a revised and greatly extended form in Parts 7 and 9.

I desire to take this opportunity to express to Director W. W. CAMPBELL of the Lick Observatory my appreciation of his coöperation in providing for the publication of papers by the staff of the Berkeley Astronomical Department by giving them space in the established publications of the Lick Observatory. By this arrangement it has become unnecessary to maintain a separate series of astronomical publications for the Berkeley Astronomical Department.

JUNE, 1911.

A. O. LEUSCHNER.

TABLE OF CONTENTS.

	PAGE.
PART 1. A SHORT METHOD OF DETERMINING ORBITS FROM THREE OBSERVATIONS. A FIRST PAPER, by A. O. LEUSCHNER, -	I
PART 2. ELEMENTS OF ASTEROID 1900 GA, by A. O. LEUSCHNER AND ADELAIDE M. HOBE, - - - - -	21
PART 3. PRELIMINARY ELEMENTS OF COMET 1900 III, by R. H. CUR- TISS AND C. G. DALL, - - - - -	39
PART 4. TABLES FOR THE REDUCTION OF PHOTOGRAPHIC MEASURES, by BURT L. NEWKIRK, - - - - -	47
PART 5. INVESTIGATION OF THE REPSOLD MEASURING APPARATUS, by BURT L. NEWKIRK, - - - - -	135
PART 6. ON ASTRONOMICAL REFRACTION, by R. T. CRAWFORD, - - -	159
PART 7. SHORT METHODS OF DETERMINING ORBITS. SECOND PAPER, by A. O. LEUSCHNER, - - - - -	217
PART 8. APPLICATIONS OF LEUSCHNER'S SHORT METHODS OF DETER- MINING ORBITS, by R. T. CRAWFORD, - - - - -	377
PART 9. SHORT METHODS OF DETERMINING ORBITS. THIRD PAPER: THE DIRECT SOLUTION OF ORBITS OF DISTURBED BODIES, by A. O. LEUSCHNER, - - - - -	455
PART 10. DIRECT SOLUTION OF PRELIMINARY DISTURBED ORBITS OF THE SEVENTH AND EIGHTH SATELLITES OF JUPITER, BY LEUSCHNER'S METHOD, by R. T. CRAWFORD, - - - - -	485

A SHORT METHOD
OF
DETERMINING ORBITS FROM THREE OBSERVATIONS.

BY A. O. LEUSCHNER,
DIRECTOR OF THE STUDENTS' OBSERVATORY,
BERKELEY ASTRONOMICAL DEPARTMENT.

~~Not Indexed~~

A SHORT METHOD OF DETERMINING ORBITS FROM THREE OBSERVATIONS.

By A. O. LEUSCHNER.

A few years ago Professor HARZER developed a very elegant method of determining orbits, without previous hypothesis as to the eccentricity, on the basis of a limited number of observations; five, in general, representing the greatest advantages (A. N. 3371). At least two essential points are gained by HARZER over LAPLACE'S method contained in *Méc. cél.*, T. I., première partie, livre II, Chap. IV: the first is a practical gain, and involves an adjustment of the effects on the residuals, due to the coefficients of the higher powers, when the geocentric coördinates are developed as a function of the time by TAYLOR'S theorem; the second is a theoretical advantage, and confines the effects of perturbations to the accelerations and higher derivatives by the use of heliocentric coördinates, by means of which the higher derivatives may be expressed in terms of the coördinates, velocities, and accelerations at the normal date more readily than by means of geocentric coördinates.

From a theoretical point of view the method gives perfect satisfaction in the case to which it has been applied by HARZER. There are, however, practical difficulties involved which make it doubtful whether astronomers will adopt this method in preference to the short methods of determining circular elements for planets, to OLBERS'S and v. OPPOLZER'S methods of deriving parabolic orbits for comets, or even to the more general methods of GAUSS and OPPOLZER for the determination of orbits without previous hypothesis as to the eccentricity. Some of the difficulties, all of them of the utmost importance to computers, are as follows:

(A.) It is necessary to solve a system of five simultaneous equations for each of the two observed coördinates, before and after corrections for parallax and

*It has been thought best by all concerned that the results of astronomical researches made under the auspices of the University of California should be published in as few mediums as possible. I shall therefore be glad to give appropriate places in the established Lick Observatory publications to standard papers written by members of the Berkeley Astronomical Department.—W. W. CAMPBELL, Director.

aberration have been applied. In all, therefore, at least four such systems will have to be solved, and more if the first attempt does not give the geocentric distances with sufficient accuracy for the final determination of the parallax and aberration. This is all the more serious, as the greatest accuracy must be observed in determining the unknowns. In HARZER'S example the first attempt at the geocentric distances gave the final corrections for parallax and aberration. Further, not four, but only two systems of simultaneous equations needed to be solved, as it was possible to avoid the solution of the third and fourth systems by a somewhat less laborious process. This process is applicable whenever the parallax corrections in α and $\tan \delta$ are expressible in the linear forms

$$\delta\alpha = \eta_0 + \theta_a \eta_1; \quad \delta \tan \delta = \zeta_0 + \theta_a \zeta_1$$

where θ_a is the interval between an observed date t_a and the normal date. If the observations are made at observatories widely separated in latitude, or if the hour-angles differ greatly, the linear relations will not be sufficiently accurate, particularly not when, as in HARZER'S example, the calculation is conducted with seven-place logarithms. In such cases—which are as likely to occur as not—recourse must be had to the solution of a third and fourth system of five simultaneous equations.

(B.) No hints are given for the solution of the equation of the seventh degree which gives the geocentric distance at the normal date. This equation will have to be solved at least twice, before and after parallax and aberration are applied, and the necessary approximations will, in general, involve a large amount of numerical work.

(C.) After the residuals of the five observations have been determined on the basis of the last approximation for the geocentric distances, it becomes necessary to determine the coefficients of the six differential corrections to the rectangular heliocentric coördinates and their velocities at the normal date for ten observation equations. The total number of these coefficients is, therefore, sixty. If it is intended that the results shall satisfy only three of the available observations, thirty-six coefficients are still necessary. Although the solution for the six unknown quantities from the observation equations has been rendered quite simple, the amount of labor involved in determining the coefficients themselves does not seem warranted when the results are not based on normal places.

Freed from these disadvantages the methods of LAPLACE and HARZER will permit of the determination of an orbit, without previous hypothesis as to the eccentricity, more readily than OLBERS'S method for the determination of parabolic elements in the case of comets.

The essential improvements to be sought for are:

I. The restriction of the number of observations to three, the minimum number necessary for the solution of the problem. The slight gain in accuracy

resulting from the introduction of two additional observations is more than offset by the shortening of the numerical work.

II. A reduction of the number of fundamental data to be approximated. These data consist in (*a*) the two observed geocentric coördinates, their velocities and accelerations; (*b*) the rectangular heliocentric coördinates and their velocities—all for the normal date. This reduction involves no loss in accuracy.

III. A short method of obtaining preliminary values of the quantities to be approximated, which will do away with the solution of simultaneous equations.

IV. A short method of solving the equation of the seventh degree.

V. A short method of determining the final corrections to the data from which the elements are computed.

HARZER has based the determination of the fundamental data (*a*) on five observations to secure a greater accuracy in them by taking into account the third and fourth derivatives of the right ascension and declination with regard to the time. This procedure assumes that the errors of observation are considerably less appreciable than the effect of these higher derivatives. For, in cases where these effects are comparable with the errors of observation, less accurate values of the required fundamental data (*b*) will result from the accurate representation of all five observations than are attained by neglecting these derivatives and by deriving the most probable values of the required normal coördinates, velocities, and accelerations from the five observation equations available for each coördinate.

It is of course impossible to decide beforehand when the effect of the higher derivatives and the errors of observation are comparable. But it seems that for short intervals three positions can be made to give approximations for the fundamental data (*a*) sufficiently accurate to permit of a rough approximation of the fundamental data (*b*), so that it is advisable to restrict the number of observations to the minimum number necessary for a solution of the problem. This restriction (I) may be made to include a reduction in the number of fundamental data to be approximated (II) and a short method of obtaining first approximations for the same (III). The first three improvements are thus secured at one and the same time. For, let us assume for a moment that we have at our disposal an ephemeris giving α and δ referred to the mean equinox at the beginning of the year for every w mean solar days. Then the velocity α' in α at an instant midway between two ephemeris epochs is given in terms of successive differences by the following formula of numerical differentiation, the unit of the time being chosen as $1/k$ mean solar days:

$$\alpha' = \frac{d\alpha}{d\theta} = \frac{1}{wk} \left\{ f'(a + [i + \frac{1}{2}]w) - \frac{1}{24} f'''(a + [i + \frac{1}{2}]w) + \frac{3}{640} f^{(5)}(a + [i + \frac{1}{2}]w) - \dots \right\} \quad (1)$$

and δ' is given by a similar formula. Let us further assume that the successive differences diminish numerically, which is equivalent to the assumption that the

curves representing the right ascension and declination as a function of the time are free from maxima, minima, points of inflection, etc., within the range of the ephemeris.

As a first approximation to (1) we may then write:

$$\alpha' = \frac{1}{wk} f' (a + [i + \frac{1}{2}] w).$$

We now proceed to apply this expression to the observations. Let $\theta_{...}$ and $\theta_.$ denote the time intervals between the first and second, and second and third observations, respectively, expressed in $1/k$ mean solar days, so that

$$\theta_{...} = k(t_{..} - t_.); \quad \theta_ = k(t_{...} - t_{..}),$$

where $t_.$, $t_{..}$, $t_{...}$ are the dates of the three given observations. Assume $w = t_{..} - t_.$, $t_{...} - t_{..}$ successively, and let $\alpha_.$, $\alpha_{..}$, $\alpha_{...}$, $\delta_.$, $\delta_{..}$, $\delta_{...}$ be the observed right ascensions and declinations referred to the mean equinox at the beginning of the year. Then we shall have

$$f'_{...} (a + [i + \frac{1}{2}] w) = \alpha_{..} - \alpha_ = (\Delta\alpha)_{...}; \quad f'_ (a + [i + \frac{1}{2}] w) = \alpha_{...} - \alpha_{..} = (\Delta\alpha)_.,$$

and we may write as a first approximation

$$\alpha_{...}' = \frac{(\Delta\alpha)_{...}}{\theta_{...}}; \quad \alpha_.' = \frac{(\Delta\alpha)_.}{\theta_}. \quad (2a)$$

Similarly

$$\delta_{...}' = \frac{\delta_{..} - \delta_}{\theta_{...}} = \frac{(\Delta\delta)_{...}}{\theta_{...}}; \quad \delta_.' = \frac{\delta_{...} - \delta_{..}}{\theta_} = \frac{(\Delta\delta)_.}{\theta_}. \quad (2b)$$

$\alpha_{...}'$, $\delta_{...}'$; $\alpha_.'$, $\delta_.'$ are approximations to the velocities in α and δ at the instants

$$T_{...} = \frac{t_ + t_{..}}{2}; \quad T_ = \frac{t_{..} + t_{...}}{2}.$$

In general, these velocities differ from their true values, first, on account of the errors of observation; secondly, because they are not corrected for parallax and aberration; thirdly, because the third and higher differences have been neglected. No allowance can be made for the errors of observation, the problem before us being the exact representation of three observations without previous hypothesis regarding the eccentricity. Parallax, aberration, third and higher differences, if appreciable, may be allowed for subsequently.

Letting, further,

$$T = \frac{1}{2}(T_{...} + T_) = \frac{1}{2}(t_ + 2t_{..} + t_{...}),$$

we have for this instant from (2a) and (2b)

$$\alpha' = \frac{1}{2}(\alpha_{...}' + \alpha_'); \quad \delta' = \frac{1}{2}(\delta_{...}' + \delta_'), \quad (3)$$

with the same degree of accuracy as (2a) and (2b).

Let us next assume that we have at our disposal a table giving α' and δ' for every w mean solar days. The accelerations α'' , δ'' at the instant T are then obtained from (2a) and (2b) by replacing the coördinates by the velocities. Accordingly we write

$$wk = \theta = k(T_i - T_{iii}) = k\left(\frac{t_{ii} - t_i}{2} + \frac{t_{iii} - t_{ii}}{2}\right) = \frac{1}{2}(\theta_{iii} + \theta_i),$$

$$\Delta\alpha' = \alpha'_i - \alpha'_{iii}; \quad \Delta\delta' = \delta'_i - \delta'_{iii}$$

and obtain

$$\alpha'' = \frac{\Delta\alpha'}{\theta}; \quad \delta'' = \frac{\Delta\delta'}{\theta}. \quad (4)$$

α'' , δ'' differ from their true values for the same three reasons that were given in the case of α' and δ' . It should be observed that whenever the effect of the third and higher differences is inappreciable within the range of the observations, the values of α' , δ' , α'' , δ'' computed by (3) and (4) require correction only on account of parallax and aberration.

By adopting the second observation as the normal place and correcting α' and δ' for the acceleration during the interval from T to $t_0 = t_{ii}$ by the formulæ

$$\alpha_0' = \alpha' + k(t_{ii} - T)\alpha''; \quad \delta_0' = \delta' + k(t_{ii} - T)\delta'' \quad (5)$$

we obtain the following system of fundamental data (a), with which we may proceed to solve the equation of the seventh degree for ρ or z :

$$\alpha_0 = \alpha_{ii}, \delta_0 = \delta_{ii}; \quad \alpha_0' = \alpha_0', \delta_0' = \delta_0'; \quad \alpha_0'' = \alpha'', \delta_0'' = \delta'' \quad (6)$$

The choice of the second observation for the normal, or better zero position, has the effect of reducing the number of quantities to be approximated from six to four, for both groups (a) and (b), and the method here given for obtaining preliminary values of the velocities and accelerations does away with the necessity of solving any system of simultaneous equations.

The nearer the actual velocities and accelerations determined by (4) and (5) are to their true values, the smaller will be the corrections to the fundamental data (b) which will have to be derived later. By plotting the observations we may determine readily whether the assumptions on which formulæ (4) and (5) are based hold approximately in a given case. If they do not, the velocities and accelerations may be estimated from the plot with sufficient accuracy to furnish the basis for the determination of the final values. Formulæ (4) and (5) can not, therefore, be used mechanically.

The first three of the improvements for which we were seeking having been determined, it now becomes necessary to facilitate the solution of the equation of the seventh degree (see A. N. 3371, pages 181-182).

Mr. ROGER SPRAGUE has already shown in A. N. 3669, at my suggestion, that a first approximation to the roots of the equation of the seventh degree may

be obtained from OPPOLZER'S table XIIIa, *Bahnbestimmung*, Vol. I. In its present form the table gives $z = \frac{r}{R}$ at most to three decimal places. I hope, however, soon to be able to extend the table so that z may be interpolated more accurately. When this shall have been accomplished, the geocentric distance at the normal date may be taken directly from the table on the basis of two arguments depending on the fundamental data (α). The so-called trials or successive approximations to the geocentric or heliocentric distances will then, in general, no longer be necessary.

In the meantime differential relations may be used for the correction of the approximate value of z taken from table XIIIa.

The equation of the seventh degree is given in the following form in A. N. 3669, page 387, equation (12):

$$-\frac{m'}{R^4} - z = \frac{-\frac{m'}{R^4}}{(1 - 2z \cos \psi + z^2)^{\frac{1}{2}}}$$

in which

$$m' = \frac{\kappa}{\cos \delta}; \quad \kappa = -R \cos D \frac{(\tan \delta \cos (A - \alpha) - \tan D) \alpha' + \sin (A - \alpha) (\tan \delta)'}{\alpha'^2 \tan \delta - \alpha'' (\tan \delta)' + \alpha' (\tan \delta)''}$$

R , A , and D being the sun's geocentric polar coördinates at the normal date, referred to the equator. It is more convenient, however, to use the form

$$f(z) - m^2 = (z^2 - 2z \cos \psi + 1)^{\frac{1}{2}} (z - m)^2 - m^2 = 0 \quad (7)$$

or

$$f(z) - m^2 = \mu^2 \nu^2 - m^2 = 0 \quad (8)$$

where

$$m = -\frac{m'}{R^4} = -\frac{\kappa}{\cos \delta R^4} \quad (9)$$

The arguments with which z is taken from table XIIIa are $\frac{1}{m}$ and ψ , the geocentric angle between the planet or comet and the sun. Let the approximate value of z taken from the table be denoted by z_1 and let

$$(z_1^2 - 2z_1 \cos \psi + 1)^{\frac{1}{2}} (z_1 - m)^2 - m^2 = \Delta f(z_1)$$

or

$$\mu_1^2 \nu_1^2 - m^2 = M_1 \quad (10)$$

It will then be necessary to determine a correction Δz_1 to z_1 in such a manner that $z_2 = z_1 + \Delta z_1$ will satisfy (7). By differentiating $f(z)$ with respect to z and by identifying $df(z)$ with M_1 , we obtain for the required correction

$$\Delta z_1 = \frac{-M_1}{2\mu_1^2 \nu_1 [\mu_1 + 3\nu_1 (z_1 - \cos \psi)]} \quad (11)$$

In many cases $z_2 = z_1 + \Delta z_1$ will be found to satisfy equation (7), so that $M_2 = 0$. If M_2 is not equal to zero, then the process must be repeated. It is easily seen that the numerical operations involved in solving for the geocentric distance according

to the foregoing method are almost insignificant in comparison with the corresponding operations even in OLBERS'S method.

From the final value of $z = z'' = z_0$ we find:

$$\rho_0 = \rho'' = R_s z_0 \quad (12)$$

The next step will consist in the determination of the geocentric distances ρ , and ρ''' or $\sigma = \rho \cos \delta$, and $\sigma''' = \rho''' \cos \delta'''$. We shall need the velocity σ'_0 and the acceleration σ''_0 . These may be computed by HARZER'S formulæ. After the approximate values of the three geocentric distances have been computed, the observations are to be corrected for parallax and aberration.

As it is intended to make use of the heliocentric rectangular coördinates and their velocities at the normal date (fundamental data (*b*)) for the purpose of deriving the elements and an ephemeris, we stop to consider whether we shall at once proceed to the determination of these quantities, or whether the whole process of determining ρ_0 , ρ'_0 or σ_0 , σ'_0 shall be repeated on the basis of the corrected observations. The latter course has been adopted by HARZER, and also in the two examples which follow. Some labor, however, could have been saved by omitting the second approximation, particularly in HARZER'S example and in the comet orbit, part 3, as in both these cases the first approximation of the geocentric distances furnished sufficiently accurate corrections for parallax and aberration. The small changes in the fundamental data (*b*) resulting from these corrections could have been determined just as well from the observation equations as part of the final corrections of the heliocentric coördinates and their velocities.

The determination of these final corrections can rarely be avoided. Even in HARZER'S example, in which the fourth derivatives of the right ascension and declination with respect to the time were taken into account, it was possible to improve the heliocentric coördinates and their velocities. There is every reason to believe that this was due to the small errors of observation rather than to the neglect of the fifth and higher derivatives. In the process outlined in these pages even the third derivatives are neglected, and, although this method is intended only for short intervals, a second approximation of the geocentric distances would rarely lead to heliocentric coördinates and velocities of such accuracy that the first and third places leave no residuals. This need not trouble us so long as we are able to set up simple formulæ for the determination of such corrections to the fundamental data (*b*) as will reduce the residuals to zero. These formulæ constitute the last of the improvements for which we are seeking.

I have so far assumed that it is intended to satisfy the given observations within the limits of accuracy of a six- or seven-place computation, but this may not always be desirable. In cases of newly-discovered comets, for example, the computer generally will aim at results only sufficiently accurate to furnish a fairly close finding ephemeris. In such cases a direct approximation of the geocentric

distances may frequently take the place of the determination of corrections to the fundamental data (δ) by the differential formulæ derived below. We find this to be true in the case of our second example, in which, with two-day intervals, the residuals of the first and third places, resulting from the second approximation for the geocentric distances, are only

$$(O - C) \begin{cases} \cos \delta, \Delta \alpha, = -0''.30, & \Delta \delta, = +0''.1 \\ \cos \delta_{\text{III}}, \Delta \alpha_{\text{III}} = -0''.28, & \Delta \delta_{\text{III}} = +0''.8 \end{cases}$$

When two approximations have been made it is sometimes possible to improve the corrections for parallax and aberration. This should not be neglected, if it is intended to accurately represent the observations.

From the value of $\rho_{\text{II}} = \rho_0$ resulting from the first or second approximation of the geocentric distances, as the case may be, and from the corrected values of α_0, δ_0 , the rectangular geocentric coördinates ξ_0, η_0, ζ_0 are now to be derived. To compute the velocities $\xi'_0, \eta'_0, \zeta'_0$ we avail ourselves, in addition, of the last values of ρ'_0 or $\delta'_0, \alpha'_0, \delta'_0$ or $(\tan \delta)_0'$.

ξ_0, η_0, ζ_0 give the heliocentric rectangular coördinates x_0, y_0, z_0 with the aid of the sun's rectangular coördinates X_0, Y_0, Z_0 which have already been used in determining R_0, A_0, D_0 . To obtain the velocities x'_0, y'_0, z'_0 we determine X'_0, Y'_0, Z'_0 for the zero (second) date, uncorrected for aberration, by numerical differentiation on the basis of the X, Y, Z , tabulated in the *American Ephemeris and Nautical Almanac*, or the *Berlin Jahrbuch*.

$X, Y, Z, X_{\text{III}}, Y_{\text{III}}, Z_{\text{III}}$ may be interpolated at the same time, care being taken to use the times t , and t_{III} , uncorrected for aberration, as it is supposed that in reducing the three given right ascensions and declinations to the beginning of the year the aberration terms have been taken into account. From the heliocentric coördinates and velocities we easily derive r_0 and r'_0 .

We now have all the necessary data to compute $\alpha, \delta, \alpha_{\text{III}}, \delta_{\text{III}}$ by HARZER'S formulæ (A. N. 3371, pages 183-184) and to compare them with the observed values. Let

$$\left. \begin{aligned} f_t &= 1 - \theta_{\text{III}}^2 \frac{1}{2r_0^3} - \theta_{\text{III}}^2 \frac{r'_0}{2r_0^4} + \dots; & g_t &= -\theta_{\text{III}} + \theta_{\text{III}}^3 \frac{1}{6r_0^3} - \dots \\ f_{\text{III}} &= 1 - \theta_t^2 \frac{1}{2r_0^3} + \theta_t^2 \frac{r'_0}{2r_0^4} + \dots; & g_{\text{III}} &= \theta_t - \theta_t^3 \frac{1}{6r_0^3} - \dots \end{aligned} \right\} \quad (13)$$

then

$$x_t = f_t x_0 + g_t x'_0; \quad x_{\text{III}} = f_{\text{III}} x_0 + g_{\text{III}} x'_0$$

and

$$\left. \begin{aligned} \rho_t \cos \delta_t \cos \alpha_t &= \xi_t = X_t + x_t = X_t + f_t x_0 + g_t x'_0 \\ \rho_t \cos \delta_t \sin \alpha_t &= \eta_t = Y_t + y_t = Y_t + f_t y_0 + g_t y'_0 \\ \rho_t \sin \delta_t &= \zeta_t = Z_t + z_t = Z_t + f_t z_0 + g_t z'_0 \\ \rho_{\text{III}} \cos \delta_{\text{III}} \cos \alpha_{\text{III}} &= \xi_{\text{III}} = X_{\text{III}} + x_{\text{III}} = X_{\text{III}} + f_{\text{III}} x_0 + g_{\text{III}} x'_0 \\ \rho_{\text{III}} \cos \delta_{\text{III}} \sin \alpha_{\text{III}} &= \eta_{\text{III}} = Y_{\text{III}} + y_{\text{III}} = Y_{\text{III}} + f_{\text{III}} y_0 + g_{\text{III}} y'_0 \\ \rho_{\text{III}} \sin \delta_{\text{III}} &= \zeta_{\text{III}} = Z_{\text{III}} + z_{\text{III}} = Z_{\text{III}} + f_{\text{III}} z_0 + g_{\text{III}} z'_0 \end{aligned} \right\} \quad (14)$$

Let the residuals in α and δ , ($O - C$), be denoted by $\partial\alpha, \partial\delta, \partial\alpha_{\text{III}}, \partial\delta_{\text{III}}$. We proceed to derive linear differential relations which shall enable us to determine such corrections to our fundamental data (b) as will reduce these residuals to zero. Through the adoption of the second as the zero observation the number of these fundamental data has been reduced to four. The quantities requiring correction in HARZER'S method are $x_0, y_0, z_0, x'_0, y'_0, z'_0$, while here they are ρ_0, x'_0, y'_0, z'_0 .

For the sake of convenience in deriving our formulæ we shall omit for the present the subscripts referring to the first and third places. We commence with the well-known relations:

$$\begin{aligned} \rho \partial \alpha &= -\sin \alpha \partial x + \cos \alpha \partial y \\ \rho \partial \delta &= -\sin \delta (\cos \alpha \partial x + \sin \alpha \partial y) + \cos \delta \partial z \end{aligned} \quad \left. \vphantom{\begin{aligned} \rho \partial \alpha &= -\sin \alpha \partial x + \cos \alpha \partial y \\ \rho \partial \delta &= -\sin \delta (\cos \alpha \partial x + \sin \alpha \partial y) + \cos \delta \partial z \end{aligned}} \right\} (15)$$

By differentiating (13), we have ($x, y, z = \omega$),

$$\begin{aligned} \partial \omega &= f \partial \omega_0 + \omega_0 \partial f + g \partial \omega'_0 + \omega'_0 \partial g \\ \partial f &= \frac{3}{2} \frac{\partial r_0}{r_0^4} + \frac{\partial r'_0}{2r_0^4} - \frac{2f r'_0}{r_0^5} \partial r_0 - \dots \\ \partial g &= \frac{1}{2} \frac{\partial r_0}{r_0^4} + \dots \end{aligned} \quad \left. \vphantom{\begin{aligned} \partial \omega &= f \partial \omega_0 + \omega_0 \partial f + g \partial \omega'_0 + \omega'_0 \partial g \\ \partial f &= \frac{3}{2} \frac{\partial r_0}{r_0^4} + \frac{\partial r'_0}{2r_0^4} - \frac{2f r'_0}{r_0^5} \partial r_0 - \dots \\ \partial g &= \frac{1}{2} \frac{\partial r_0}{r_0^4} + \dots \end{aligned}} \right\} (16)$$

In general, it will be sufficient to neglect all but the first term of ∂f and ∂g . With ten-day intervals and a correction $\partial r'_0 = .01$, which is larger than we must expect, the greatest of the neglected terms will average less than .00005, a quantity which ordinarily may be neglected in determining differential corrections. Should our starting values of ρ_0, x'_0, y'_0, z'_0 be so far from the truth as to make appreciable higher terms than those considered, it will be more convenient to make a second approximation to the differential corrections. Since α_0, δ_0 are constants, we also have

$$\begin{aligned} \partial x_0 &= \cos \alpha_0 \cos \delta_0 \partial \rho_0 = \frac{\xi_0}{\rho_0} \partial \rho_0, & \partial y_0 &= \sin \alpha_0 \cos \delta_0 \partial \rho_0 = \frac{\eta_0}{\rho_0} \partial \rho_0 \\ \partial z_0 &= \sin \delta_0 \partial \rho_0 = \frac{z_0}{\rho_0} \partial \rho_0, \end{aligned} \quad (17)$$

and

$$\partial r_0 = \frac{x_0 \partial x_0 + y_0 \partial y_0 + z_0 \partial z_0}{r_0} = \left\{ \frac{x_0 \xi_0}{r_0 \rho_0} + \frac{y_0 \eta_0}{r_0 \rho_0} + \frac{z_0 z_0}{r_0 \rho_0} \right\} \partial \rho_0 \quad (18)$$

It is not advisable to introduce $\frac{\partial \log \rho_0}{\text{Mod.}}$ in place of $\frac{\partial \rho_0}{\rho_0}$ into these formulæ, as the neglected terms of the development of $\partial \log \rho_0$ in terms of $\partial \rho_0$ are apt to be appreciable. The coefficient of $\partial \rho_0$ is the cosine of the angle at the comet or planet included between ρ_0 and r_0 . If we denote this angle by β , we have

$$\cos \beta = \frac{\rho_0}{r_0} \cos \psi; \quad \partial r_0 = \cos \beta \partial \rho_0. \quad (19)$$

Introducing (17) and (19) into (16), we obtain ($\xi_0, \eta_0, z_0 = \omega_0$)

$$\partial\omega = f_\omega \partial\rho_0 + g \partial\omega_0' \quad (20)$$

in which

$$f_\omega = f \frac{\omega_0}{\rho_0} + \frac{\cos \beta}{2r_0^4} (3\theta^2 \omega_0 + \theta^4 \omega_0'), \quad (21)$$

or separately for the three coördinates

$$\left. \begin{aligned} f_x &= \frac{f}{\rho_0} \xi_0 + \frac{\cos \beta}{2r_0^4} (3\theta^2 x_0 + \theta^4 x_0'), & \partial x &= f_x \partial\rho_0 + g \partial x_0', \\ f_y &= \frac{f}{\rho_0} \eta_0 + \frac{\cos \beta}{2r_0^4} (3\theta^2 y_0 + \theta^4 y_0'), & \partial y &= f_y \partial\rho_0 + g \partial y_0', \\ f_z &= \frac{f}{\rho_0} z_0 + \frac{\cos \beta}{2r_0^4} (3\theta^2 z_0 + \theta^4 z_0'), & \partial z &= f_z \partial\rho_0 + g \partial z_0'. \end{aligned} \right\} \quad (22)$$

Generally, the last of the three terms in the f expressions may be neglected.

From (15) and (22) we readily deduce for the first and third places:

$$\left. \begin{aligned} A_i &= \frac{1}{\rho_i} [\cos \alpha_i f_{y_i} - \sin \alpha_i f_{x_i}], \\ B_i &= -\frac{1}{\rho_i} [\sin \delta_i (\sin \alpha_i f_{y_i} + \cos \alpha_i f_{x_i}) - \cos \delta_i f_{z_i}], & C_i &= \frac{g_i}{\rho_i}, \end{aligned} \right\} \quad (23)$$

$$\left. \begin{aligned} A_{iii} &= \frac{1}{\rho_{iii}} [\cos \alpha_{iii} f_{y_{iii}} - \sin \alpha_{iii} f_{x_{iii}}], \\ B_{iii} &= -\frac{1}{\rho_{iii}} [\sin \delta_{iii} (\sin \alpha_{iii} f_{y_{iii}} + \cos \alpha_{iii} f_{x_{iii}}) - \cos \delta_{iii} f_{z_{iii}}], & C_{iii} &= \frac{g_{iii}}{\rho_{iii}}; \end{aligned} \right\} \quad (24)$$

$$\left. \begin{aligned} \partial_i \alpha_i &= A_i \partial\rho_0 - C_i \sin \alpha_i \partial x_0' + C_i \cos \alpha_i \partial y_0', \\ \partial_i \alpha_{iii} &= A_{iii} \partial\rho_0 - C_{iii} \sin \alpha_{iii} \partial x_0' + C_{iii} \cos \alpha_{iii} \partial y_0'; \end{aligned} \right\} \quad (25)$$

$$\left. \begin{aligned} \partial \delta_i &= B_i \partial\rho_0 - C_i \sin \delta_i \cos \alpha_i \partial x_0' - C_i \sin \delta_i \sin \alpha_i \partial y_0' + C_i \cos \delta_i \partial z_0', \\ \partial \delta_{iii} &= B_{iii} \partial\rho_0 - C_{iii} \sin \delta_{iii} \cos \alpha_{iii} \partial x_0' - C_{iii} \sin \delta_{iii} \sin \alpha_{iii} \partial y_0' + C_{iii} \cos \delta_{iii} \partial z_0'. \end{aligned} \right\} \quad (25)$$

Eliminating $\partial z_0'$ from equations (25), $\partial x_0'$, $\partial y_0'$ from equations (24) we have:

$$\begin{aligned} C_{iii} \cos \delta_{iii} \partial \delta_i - C_i \cos \delta_i \partial \delta_{iii} &= (B_i C_{iii} \cos \delta_{iii} - B_{iii} C_i \cos \delta_i) \partial\rho_0 \\ &\quad - C_i C_{iii} (\sin \delta_i \cos \delta_{iii} \cos \alpha_i - \sin \delta_{iii} \cos \delta_i \cos \alpha_{iii}) \partial x_0' \\ &\quad - C_i C_{iii} (\sin \delta_i \cos \delta_{iii} \sin \alpha_i - \sin \delta_{iii} \cos \delta_i \sin \alpha_{iii}) \partial y_0'. \end{aligned} \quad (26)$$

$$\left. \begin{aligned} C_{iii} \sin \alpha_{iii} \partial_i \alpha_i - C_i \sin \alpha_i \partial_i \alpha_{iii} &= (A_i C_{iii} \sin \alpha_{iii} - A_{iii} C_i \sin \alpha_i) \partial\rho_0 + C_i C_{iii} \sin (\alpha_{iii} - \alpha_i) \partial y_0', \\ C_{iii} \cos \alpha_{iii} \partial_i \alpha_i - C_i \cos \alpha_i \partial_i \alpha_{iii} &= (A_i C_{iii} \cos \alpha_{iii} - A_{iii} C_i \cos \alpha_i) \partial\rho_0 + C_i C_{iii} \sin (\alpha_{iii} - \alpha_i) \partial x_0'. \end{aligned} \right\} \quad (27)$$

From (26) $C_i C_{iii} \partial x_0'$, and $C_i C_{iii} \partial y_0'$ may be eliminated by means of (27). In the eliminant the coefficients of $-\frac{C_{iii} \partial_i \alpha_i}{\sin (\alpha_{iii} - \alpha_i)}$ and $\frac{A_i C_{iii}}{\sin (\alpha_{iii} - \alpha_i)}$ on the one, and of $\frac{C_i \partial_i \alpha_{iii}}{\sin (\alpha_{iii} - \alpha_i)}$ and $-\frac{A_{iii} C_i}{\sin (\alpha_{iii} - \alpha_i)}$ on the other hand, are

$$\sin \delta_i \cos \delta_{iii} \cos (\alpha_{iii} - \alpha_i) - \sin \delta_{iii} \cos \delta_i$$

and

$$\sin \delta_i \cos \delta_{iii} - \sin \delta_{iii} \cos \delta_i \cos (\alpha_{iii} - \alpha_i),$$

respectively. As $(\alpha_{\text{III}} - \alpha_0)$ is known, we can, of course, decide in a given case whether or not the accuracy which is being observed in determining the differential corrections warrants the assumption $\cos(\alpha_{\text{III}} - \alpha_0) = 1$. For the present we shall introduce in all four coefficients

$$\cos(\alpha_{\text{III}} - \alpha_0) = 1 - 2 \sin^2 \frac{1}{2}(\alpha_{\text{III}} - \alpha_0).$$

Solving the eliminant for $\partial \rho_0$ we finally obtain

$$\partial \rho_0 = \frac{\sin(\alpha_{\text{III}} - \alpha_0) [C_{\text{III}} \cos \delta_{\text{III}} \partial \delta_0 - C_0 \cos \delta_0 \partial \delta_{\text{III}}] - \sin(\delta_{\text{III}} - \delta_0) [C_{\text{III}} \partial \alpha_0 - C_0 \partial \alpha_{\text{III}}] - \Delta N}{\sin(\alpha_{\text{III}} - \alpha_0) [C_{\text{III}} \cos \delta_{\text{III}} B_0 - C_0 \cos \delta_0 B_{\text{III}}] - \sin(\delta_{\text{III}} - \delta_0) [C_{\text{III}} A_0 - C_0 A_{\text{III}}] - \Delta D} \quad (28)$$

in which

$$\Delta N = 2 \sin^2 \frac{1}{2}(\alpha_{\text{III}} - \alpha_0) [C_{\text{III}} \cos \delta_{\text{III}} \sin \delta_0 \partial \alpha_0 + C_0 \cos \delta_0 \sin \delta_{\text{III}} \partial \alpha_{\text{III}}]$$

and

$$\Delta D = 2 \sin^2 \frac{1}{2}(\alpha_{\text{III}} - \alpha_0) [C_{\text{III}} \cos \delta_{\text{III}} \sin \delta_0 A_0 + C_0 \cos \delta_0 \sin \delta_{\text{III}} A_{\text{III}}]$$

in which the last term in the numerator and in the denominator may be omitted whenever $2 \sin^2 \frac{1}{2}(\alpha_{\text{III}} - \alpha_0)$ is less than one unit of the last place carried in computing $\partial \rho_0$ from this formula.

With the value of $\partial \rho_0$ resulting from (28), $\partial x_0'$ and $\partial y_0'$ may be computed by means of (27). As a check on the computation of the differential corrections, $\partial z_0'$ should be obtained from each of the formulæ (25).

From $\partial \rho_0$, $\partial x_0'$, $\partial y_0'$, $\partial z_0'$, we next compute

$$\partial \xi_0 = \partial x_0, \quad \partial \eta_0 = \partial y_0, \quad \partial \zeta_0 = \partial z_0, \quad \partial \xi_{\text{III}} = \partial x_{\text{III}}, \quad \partial \eta_{\text{III}} = \partial y_{\text{III}}, \quad \partial \zeta_{\text{III}} = \partial z_{\text{III}},$$

in which the corrections to the heliocentric coördinates are given by (22). The corrected geocentric coördinates give the values of ρ_0 , α_0 , δ_0 , ρ_{III} , α_{III} , δ_{III} , which correspond to the corrected values of ρ_0 , x_0' , y_0' , z_0' .

If the computed right ascensions and declinations agree with their observed values, we may proceed to the computation of the heliocentric coördinates at the zero date. From the corrected heliocentric coördinates and velocities we then derive the elements by ENCKE'S formulæ in the form in which OPPOLZER gives them, *Bahnbestimmung*, Vol. II, pp. 93 and 99.

If the agreement of the observed and computed right ascensions and declinations is not satisfactory, the corrections to the fundamental data (b) should be redetermined by substituting the new residuals in place of the old ones in (28), (27), (25); but, as a rule, it will not be necessary to recompute any of the auxiliary quantities (23).

It will rarely happen that the value of $\partial \rho_0$, determined from (28), is so large as to make necessary a recomputation of the parallax and aberration. Should such a case arise, however, it will usually be sufficient to apply the new parallax to α_0 and δ_0 and to recompute x_0 , y_0 , z_0 . The epoch $t_0 = t_0$ should be rechecked for aberration.

In the majority of cases the first approximation of the geocentric distances

will be a little nearer to the truth, if, at the beginning of the calculation, the second observation be corrected for the parallax at distance unity.

As the resulting elements are referred to the equator, they must be transformed to the ecliptic, in order that they may be compared readily with those of previously known planets or comets.

The whole computation has been referred to the equator so as to avoid transformations. In some rare cases, however, it may become advisable to refer the observations to some arbitrary fundamental plane.

An ephemeris may be computed in the usual way or by (14), which formulæ sometimes will be more convenient.

Chief among the advantages of the method here outlined are the ease with which such corrections to ρ_0, x'_0, y'_0, z'_0 , may be determined as will cause the residuals due to the original values of these quantities to disappear, and the possibility of determining these corrections directly from the residuals. On that account, it is of no great consequence if the originally adopted velocities and accelerations in α and δ are only approximate. By means of the differential formulæ here introduced, it is generally possible to determine the final values of the fundamental data with greater numerical accuracy than would be possible by applying the original integral formulæ.

For the sake of convenience in following the foregoing directions for the computation of an orbit from three observations, $t, \alpha, \delta; t_u, \alpha_u, \delta_u; t_m, \alpha_m, \delta_m$, the necessary formulæ are collected below. It is assumed that the observations are referred to the beginning of the year.

I.

From one of the Astronomical Ephemerides interpolate the solar coördinates for the dates $t, t_u = t_0, t_m$.

$$X_t, Y_t, Z_t; \quad X_u, Y_u, Z_u; \quad X_m, Y_m, Z_m.$$

At the same time obtain $X'_0 = X'_u, Y'_0 = Y'_u, Z'_0 = Z'_u$, by either of the following formulæ of numerical differentiation:

$$kw \frac{df(l)}{dl} = f'(a + iw) + n f''(a + iw) + N_1^3(n) f'''(a + iw) + \dots;$$

$$l = a + [i + n] w = t_u.$$

$$kw \frac{df(l)}{dl} = f'(a + [i + \frac{1}{2}] w) + m f''(a + [i + \frac{1}{2}] w) + M_1^3(m) f'''(a + [i + \frac{1}{2}] w) + \dots;$$

$$l = a + [i + \frac{1}{2} + m] w = t_m.$$

according as to whether t_0 lies nearer to a tabulated argument or to the mean of two such arguments. n and m are $< \pm 0.25$. $N_1^3(n)$ or $M_1^3(m)$ may be taken from table I or II, OPPOLZER, *Bahnbestimmung*, etc., Vol. II, page 515 or 523, but,

in general, the terms in which they occur may be neglected. When the solar coördinates are tabulated for every 12 hours, $w = 1/2$; $\log 1/kw = 2.0654486$.

$$R \cos D \cos A = X_{\alpha}; \quad R \cos D \sin A = Y_{\alpha}; \quad R \sin D = Z_{\alpha}.$$

It will do no harm to correct the second observation for the parallax corresponding to distance unity.

II.*

$$\alpha_{\alpha\alpha}' = \frac{15 \sin 1''}{k} \frac{\alpha_{\alpha\alpha} - \alpha_{\alpha}}{t_{\alpha\alpha} - t_{\alpha}}; \quad \alpha_{\alpha}' = \frac{15 \sin 1''}{k} \frac{\alpha_{\alpha\alpha} - \alpha_{\alpha}}{t_{\alpha\alpha} - t_{\alpha}}; \quad \delta_{\alpha\alpha}' = \frac{\sin 1''}{k} \frac{\delta_{\alpha\alpha} - \delta_{\alpha}}{t_{\alpha\alpha} - t_{\alpha}}; \quad \delta_{\alpha}' = \frac{\sin 1''}{k} \frac{\delta_{\alpha\alpha} - \delta_{\alpha}}{t_{\alpha\alpha} - t_{\alpha}}$$

$$\alpha_0'' = \frac{30 \sin 1''}{k^2} \frac{\alpha_{\alpha\alpha} - \alpha_{\alpha} - \alpha_{\alpha} - \alpha_{\alpha}}{t_{\alpha\alpha} - t_{\alpha} - t_{\alpha} - t_{\alpha}}; \quad \delta_0'' = \frac{2 \sin 1''}{k^2} \frac{\delta_{\alpha\alpha} - \delta_{\alpha} - \delta_{\alpha} - \delta_{\alpha}}{t_{\alpha\alpha} - t_{\alpha} - t_{\alpha} - t_{\alpha}}$$

$$\alpha_0' = \frac{1}{2} (\alpha_{\alpha\alpha}' + \alpha_{\alpha}') + \frac{k}{4} [(t_{\alpha\alpha} - t_{\alpha}) - (t_{\alpha\alpha} - t_{\alpha})] \alpha_0''; \quad \delta_0' = \frac{1}{2} (\delta_{\alpha\alpha}' + \delta_{\alpha}') + \frac{k}{4} [(t_{\alpha\alpha} - t_{\alpha}) - (t_{\alpha\alpha} - t_{\alpha})] \delta_0''$$

$$(\tan \delta)_0' = \sec^2 \delta_{\alpha\alpha} \delta_0'; \quad (\tan \delta)_0'' = \sec^2 \delta_{\alpha\alpha} [2 \tan \delta_{\alpha\alpha} (\delta_0')^2 + \delta_0'']$$

$$\log \frac{15 \sin 1''}{k} = 7.626 \, 084 \, 7 - 10; \quad \log \frac{\sin 1''}{k} = 6.449 \, 993 \, 4 - 10; \quad \log \frac{k}{4} = 7.633 \, 521 \, 4 - 10;$$

$$\log \frac{30 \sin 1''}{k^2} = 9.691 \, 533 \, 3 - 10; \quad \log \frac{2 \sin 1''}{k^2} = 8.515 \, 442 \, 0 - 10.$$

III.

$$N = (\alpha_0')^2 \tan \delta_{\alpha\alpha} - \alpha_0'' (\tan \delta)_0' + \alpha_0' (\tan \delta)_0'';$$

$$\kappa = - \frac{R \cos D}{N} \left\{ [\tan \delta_{\alpha\alpha} \cos (A - \alpha_{\alpha}) - \tan D] \alpha_0' + \sin (A - \alpha_{\alpha}) (\tan \delta)_0' \right\}$$

$$\cos \psi = \sin \delta_{\alpha\alpha} \sin D + \cos \delta_{\alpha\alpha} \cos D \cos (A - \alpha_{\alpha}); \quad \frac{1}{m} = - \frac{R^2 \cos \delta}{\kappa}.$$

IV.

With $\frac{1}{m}$ and ψ take $z = z_1$ from table XIIIa, OPPOLZER, *Bahnbestimmung*, Vol. I.

$$\mu_1 = z_1^2 - 2z_1 \cos \psi + 1; \quad \nu_1 = z_1 - m; \quad \mu_1^2 \nu_1^2 - m^2 = M_1;$$

$$z_2 = z_1 + \frac{-M_1}{2 \mu_1^2 \nu_1 [\mu_1 + 3 \nu_1 (z_1 - \cos \psi)]}$$

Continue these approximations until $M = 0$; z , without subscript, being the final value.

* When the geocentric motion is such that the higher differences in α and δ are very appreciable, these formulæ must receive proper modification, depending upon the special conditions prevailing in individual cases. See page 7.

V.

$$\rho_0 = Rz ; \quad \sigma_0 = \rho_0 \cos \delta_{\alpha} ;$$

$$\lambda = \frac{R \cos D}{2N} \left\{ [\tan \delta_{\alpha} \cos (A - \alpha_{\alpha}) - \tan D] \alpha_0'' + \sin (A - \alpha_{\alpha}) [(\alpha_0')^2 \tan \delta_{\alpha} + (\tan \delta)_{\alpha}'] \right\}$$

$$\sigma_0' = \frac{\lambda}{\kappa} \sigma_0 ; \quad \sigma_0'' = \frac{\sigma_0}{\kappa} \left[R \cos D \cos (A - \alpha_{\alpha}) - \sigma_0 \right] - \sigma_0 \left[\frac{1}{R^2} - (\alpha_0')^2 \right] ;$$

$$\theta_i = k(t_{\alpha\alpha} - t_{\alpha}) ; \quad \theta_{\alpha\alpha} = k(t_{\alpha\alpha} - t_i) ;$$

$$\rho_i = \frac{1}{\cos \delta_i} \left[\sigma_0 - \theta_{\alpha\alpha} \sigma_0' + \theta_{\alpha\alpha}^2 \frac{\sigma_0''}{2} \right] ; \quad \rho_{\alpha\alpha} = \frac{1}{\cos \delta_{\alpha\alpha}} \left[\sigma_0 + \theta_i \sigma_0' + \theta_i^2 \frac{\sigma_0''}{2} \right] .$$

With ρ_0 correct the second observation for parallax and with $\rho_i, \rho_0, \rho_{\alpha\alpha}$ correct the three dates of observation for aberration, also θ_i and $\theta_{\alpha\alpha}$.

VI.

$$\begin{aligned} x_0 &= \sigma_0 \cos \alpha_{\alpha} - X_{\alpha} ; & y_0 &= \sigma_0 \sin \alpha_{\alpha} - Y_{\alpha} ; & z_0 &= \sigma_0 \tan \delta_{\alpha} - Z_{\alpha} ; & r_0^2 &= x_0^2 + y_0^2 + z_0^2 \\ &= \xi_0 - X_{\alpha} ; & &= \eta_0 - Y_{\alpha} ; & &= \zeta_0 - Z_{\alpha} ; & & \\ x_0' &= \cos \alpha_{\alpha} \sigma_0' - \sin \alpha_{\alpha} \sigma_0 \alpha_0' - X_0' ; & & & y_0' &= \sin \alpha_{\alpha} \sigma_0' + \cos \alpha_{\alpha} \sigma_0 \alpha_0' - Y_0' ; \\ z_0' &= \tan \delta_{\alpha} \sigma_0' + \sigma_0 (\tan \delta)_{\alpha}' ; & & & r_0 \alpha_0' &= x_0 x_0' + y_0 y_0' + z_0 z_0' ; \end{aligned}$$

$$\begin{aligned} f_i &= 1 - \theta_{\alpha\alpha}^2 \frac{1}{2r_0^3} - \theta_{\alpha\alpha}^3 \frac{r_0'}{2r_0^4} + \dots ; & g_i &= -\theta_{\alpha\alpha} + \theta_{\alpha\alpha}^3 \frac{1}{6r_0^3} - \dots \\ f_{\alpha\alpha} &= 1 - \theta_i^2 \frac{1}{2r_0^3} + \theta_i^3 \frac{r_0'}{2r_0^4} + \dots ; & g_{\alpha\alpha} &= \theta_i - \theta_i^3 \frac{1}{6r_0^3} + \dots \end{aligned}$$

$$\begin{aligned} \rho_i \cos \delta_i \cos \alpha_i &= X_i + f_i x_0 + g_i x_0' = \xi_i ; & \rho_{\alpha\alpha} \cos \delta_{\alpha\alpha} \cos \alpha_{\alpha\alpha} &= X_{\alpha\alpha} + f_{\alpha\alpha} x_0 + g_{\alpha\alpha} x_0' = \xi_{\alpha\alpha} \\ \rho_i \cos \delta_i \sin \alpha_i &= Y_i + f_i y_0 + g_i y_0' = \eta_i ; & \rho_{\alpha\alpha} \cos \delta_{\alpha\alpha} \sin \alpha_{\alpha\alpha} &= Y_{\alpha\alpha} + f_{\alpha\alpha} y_0 + g_{\alpha\alpha} y_0' = \eta_{\alpha\alpha} \\ \rho_i \sin \delta_i &= Z_i + f_i z_0 + g_i z_0' = \zeta_i ; & \rho_{\alpha\alpha} \sin \delta_{\alpha\alpha} &= Z_{\alpha\alpha} + f_{\alpha\alpha} z_0 + g_{\alpha\alpha} z_0' = \zeta_{\alpha\alpha} \end{aligned}$$

With these last values of $\rho_i, \rho_{\alpha\alpha}$ correct the second and third observations for parallax. Form the residuals, $(O - C)$, $\partial \alpha_i = \partial \alpha_i \cos \delta_i, \partial \delta_i, \partial \alpha_{\alpha\alpha} = \partial \alpha_{\alpha\alpha} \cos \delta_{\alpha\alpha}, \partial \delta_{\alpha\alpha}$. If these residuals are sufficiently small for the purpose in hand, the elements and an ephemeris may be computed at once by VIII. If it is intended, however, to represent the observations more accurately within the limits of the tables, the necessary corrections to ρ_0, x_0', y_0', z_0' may be determined by means of [VII].

[VII.]

$$\cos \beta = \frac{\rho_0 - R_0 \cos \psi}{r_0};$$

$$f_{x_i} = f_{\rho_0} \frac{\xi_0}{\rho_0} + \frac{\cos \beta}{2r_0^4} (3\theta_{iii}^2 x_0 - \theta_{iii}^2 x_0'); \quad f_{x_{iii}} = f_{\rho_0} \frac{\xi_0}{\rho_0} + \frac{\cos \beta}{2r_0^4} (3\theta_i^2 x_0 + \theta_i^2 x_0')$$

$$f_{y_i} = f_{\rho_0} \frac{\eta_0}{\rho_0} + \frac{\cos \beta}{2r_0^4} (3\theta_{iii}^2 y_0 - \theta_{iii}^2 y_0'); \quad f_{y_{iii}} = f_{\rho_0} \frac{\eta_0}{\rho_0} + \frac{\cos \beta}{2r_0^4} (3\theta_i^2 y_0 + \theta_i^2 y_0')$$

$$f_{z_i} = f_{\rho_0} \frac{\zeta_0}{\rho_0} + \frac{\cos \beta}{2r_0^4} (3\theta_{iii}^2 z_0 - \theta_{iii}^2 z_0'); \quad f_{z_{iii}} = f_{\rho_0} \frac{\zeta_0}{\rho_0} + \frac{\cos \beta}{2r_0^4} (3\theta_i^2 z_0 + \theta_i^2 z_0').$$

$$A_i = \frac{1}{\rho_i} [\cos \alpha_i f_{y_i} - \sin \alpha_i f_{x_i}],$$

$$B_i = -\frac{1}{\rho_i} [\sin \delta_i (\sin \alpha_i f_{y_i} + \cos \alpha_i f_{x_i}) - \cos \delta_i f_{z_i}], \quad C_i = \frac{g_i}{\rho_i},$$

$$A_{iii} = \frac{1}{\rho_{iii}} [\cos \alpha_{iii} f_{y_{iii}} - \sin \alpha_{iii} f_{x_{iii}}],$$

$$B_{iii} = -\frac{1}{\rho_{iii}} [\sin \delta_{iii} (\sin \alpha_{iii} f_{y_{iii}} + \cos \alpha_{iii} f_{x_{iii}}) - \cos \delta_{iii} f_{z_{iii}}], \quad C_{iii} = \frac{g_{iii}}{\rho_{iii}}.$$

$$\partial \rho_0 = \frac{N - \Delta N}{D - \Delta D},$$

in which

$$N = \sin (\alpha_{iii} - \alpha_i) [C_{iii} \cos \delta_{iii} \partial \delta_i - C_i \cos \delta_i \partial \delta_{iii}] - \sin (\delta_{iii} - \delta_i) [C_{iii} \partial \alpha_i - C_i \partial \alpha_{iii}]$$

$$D = \sin (\alpha_{iii} - \alpha_i) [C_{iii} \cos \delta_{iii} B_i - C_i \cos \delta_i B_{iii}] - \sin (\delta_{iii} - \delta_i) [C_{iii} A_i - C_i A_{iii}]$$

$$\Delta N = 2 \sin^2 \frac{1}{2} (\alpha_{iii} - \alpha_i) [C_{iii} \cos \delta_{iii} \sin \delta_i \partial \alpha_i + C_i \cos \delta_i \sin \delta_{iii} \partial \alpha_{iii}]$$

$$\Delta D = 2 \sin^2 \frac{1}{2} (\alpha_{iii} - \alpha_i) [C_{iii} \cos \delta_{iii} \sin \delta_i A_i + C_i \cos \delta_i \sin \delta_{iii} A_{iii}]$$

or also

$$\partial \rho_0 = \frac{\sin (\alpha_{iii} - \alpha_i) [N_i^1 - N_{iii}^1] - \sin (\delta_{iii} - \delta_i) [N_i^2 - N_{iii}^2] - 2 \sin^2 \frac{1}{2} (\alpha_{iii} - \alpha_i) [N_i^3 + N_{iii}^3]}{\sin (\alpha_{iii} - \alpha_i) [D_i^1 - D_{iii}^1] - \sin (\delta_{iii} - \delta_i) [D_i^2 - D_{iii}^2] - 2 \sin^2 \frac{1}{2} (\alpha_{iii} - \alpha_i) [D_i^3 + D_{iii}^3]}.$$

$$C_i C_{iii} \sin (\alpha_{iii} - \alpha_i) \partial x_0' = C_{iii} \cos \alpha_{iii} \partial \alpha_i - C_i \cos \alpha_i \partial \alpha_{iii} - (C_{iii} \cos \alpha_{iii} A_i - C_i \cos \alpha_i A_{iii}) \partial \rho_0 = (1) - (2) - [(3) - (4)] \partial \rho_0$$

$$C_i C_{iii} \sin (\alpha_{iii} - \alpha_i) \partial y_0' = C_{iii} \sin \alpha_{iii} \partial \alpha_i - C_i \sin \alpha_i \partial \alpha_{iii} - (C_{iii} \sin \alpha_{iii} A_i - C_i \sin \alpha_i A_{iii}) \partial \rho_0 = (1) - (2) - [(3) - (4)] \partial \rho_0$$

$$\left. \begin{aligned} C_i \cos \delta_i \partial z_0' &= \partial \delta_i - B_i \partial \rho_0 + C_i \sin \delta_i \cos \alpha_i \partial x_0' - C_i \sin \delta_i \sin \alpha_i \partial y_0' \\ C_{iii} \cos \delta_{iii} \partial z_0' &= \partial \delta_{iii} - B_{iii} \partial \rho_0 + C_{iii} \sin \delta_{iii} \cos \alpha_{iii} \partial x_0' + C_{iii} \sin \delta_{iii} \sin \alpha_{iii} \partial y_0' \end{aligned} \right\} \text{check}$$

Multiply $\partial \rho_0, \partial x_0', \partial y_0', \partial z_0'$ by $\sin 1''$ to reduce them to circular measure. As a test on the accuracy of the corrections derived from these linear differential relations, compute

$$\begin{aligned} \rho_i \cos \delta_i \cos \alpha_i &= \xi_i + f_{x_i} \partial \rho_0 + g_i \partial x_0'; & \rho_{iii} \cos \delta_{iii} \cos \alpha_{iii} &= \xi_{iii} + f_{x_{iii}} \partial \rho_0 + g_{iii} \partial x_0' \\ \rho_i \cos \delta_i \sin \alpha_i &= \eta_i + f_{y_i} \partial \rho_0 + g_i \partial y_0'; & \rho_{iii} \cos \delta_{iii} \sin \alpha_{iii} &= \eta_{iii} + f_{y_{iii}} \partial \rho_0 + g_{iii} \partial y_0' \\ \rho_i \sin \delta_i &= \zeta_i + f_{z_i} \partial \rho_0 + g_{iii} \partial z_0'; & \rho_{iii} \sin \delta_{iii} &= \zeta_{iii} + f_{z_{iii}} \partial \rho_0 + g_{iii} \partial z_0' \end{aligned}$$

If the computed values of α_* , δ_* , α_{**} , δ_{**} represent the observed values within the desired limits of accuracy, the elements and an ephemeris may be computed by VIII. Otherwise, the differential corrections must be recomputed by [VII]. It will not, however, be necessary to change any of the auxiliary quantities.

Whenever new values of ρ_* , $\rho_{**} = \rho_0$, ρ_{***} have been derived, the corrections for parallax and aberration should be recomputed, if necessary. If the observations as previously corrected are properly represented, and if the final values of ρ_* , ρ_0 , ρ_{***} indicate the possibility of further improving the corrections for parallax and aberration, the second observation alone needs to be thus improved. In this case compute the final values:

$$x_0 = \rho_0 \cos \delta_{**} \cos \alpha_{**} - X_{**}; \quad y_0 = \rho_0 \cos \delta_{**} \sin \alpha_{**} - Y_{**}; \quad z_0 = \rho_0 \sin \delta_{**} - Z_{**},$$

using the final values for ρ_0 , α_{**} , δ_{**} .

Otherwise correct the original values of x_0 , y_0 , z_0 by

$$\partial x_0 = \xi_0 \frac{\partial \rho_0}{\rho_0}; \quad \partial y_0 = \eta_0 \frac{\partial \rho_0}{\rho_0}; \quad \partial z_0 = \zeta_0 \frac{\partial \rho_0}{\rho_0},$$

respectively, and in either case correct the original values of x'_0 , y'_0 , z'_0 by $\partial x'_0$, $\partial y'_0$, $\partial z'_0$, respectively.

VIII.

In the following formulæ all coördinates refer to the zero (second) date. The subscripts are omitted throughout:

$$\begin{aligned} \sqrt{p} \cos i &= xy' - yx'; & r \sin u &= \frac{z}{\sin i}; \\ \sqrt{p} \sin i \sin \Omega &= yz' - zy'; & r \cos u &= x \cos \Omega + y \sin \Omega \\ \sqrt{p} \sin i \cos \Omega &= xz' - zx'; & r^2 &= x^2 + y^2 + z^2, \text{ check} \\ r' &= \frac{1}{r} (xx' + yy' + zz') & \omega &= u - v \\ e \sin v &= r' \sqrt{p} & \pi &= \omega + \Omega \\ e \cos v &= \frac{p}{r} - 1 & a &= \frac{p}{1 - e^2} \end{aligned}$$

<p style="text-align: center;"><i>Ellipse.</i></p> $\tan \frac{1}{2} E = \sqrt{\frac{1-e}{1+e}} \tan \frac{1}{2} v$ $M = E - \frac{e}{\sin 1''} \sin E$ $\mu = k'' a^{-\frac{1}{2}}$ <p>$\log k'' = 3.550 \ 006 \ 6$</p> <p>Epoch = t_{**}</p>	<p><i>Parabola.</i></p> $q = \frac{p}{2}$ <p>with v as argument, take M_{**} from OPPOLZER, vol. I, table IV.</p> $T = t_{**} - M_{**} q^{\frac{1}{2}}$	<p style="text-align: center;"><i>Hyperbola.</i></p> $\tan \frac{1}{2} F = \sqrt{\frac{e-1}{e+1}} \tan \frac{1}{2} v$ $T = t_{**} + \frac{a^{\frac{1}{2}}}{k} \left[e \tan F - \frac{\log \tan (45^\circ + \frac{1}{2} F)}{\text{Mod.}} \right]$
--	---	---

An ephemeris may be computed in the usual way, or, when the intervals are not too long, by means of the proper formulæ of VI. The number of terms to be taken into account in f and g depends, of course, on the intervals between ephemeris date and epoch. Additional terms may be written out easily on the basis of HARZER'S formulæ, A. N. 3371, pages 183-184.

When the orbit computed by the foregoing formulæ does not represent later observations with sufficient accuracy, the elements may be corrected easily so as to satisfy a single additional observation (or normal place) falling within the limits within which an ephemeris may be computed from formulæ VI. Since $\theta = k(t_\alpha - t_0)$ and $k = 0.017 \dots$, θ will be very nearly equal to unity for an interval of a little over 50 mean solar days. In general, for this interval, the f and g formulæ will no longer be applicable for determining an ephemeris. Just what the limiting value is in a given case depends also on r_0 and r'_0 , but as these quantities are already known, the applicability of the f and g expressions, and the number of terms to be considered in them, is easily decided in each individual case.

To correct the elements so as to represent a later observation, we compute for this observation the residuals due to the final elements, as in VI. If, then, these residuals be introduced in [VII] in place of $\partial\alpha_{\dots}$ and $\partial\delta_{\dots}$ we shall obtain such corrections $\partial\rho_0, \partial x'_0, \partial y'_0, \partial z'_0$ to the final values of ρ_0, x'_0, y'_0, z'_0 , of the original calculation, as will cause the residuals of the new observation to vanish. The auxiliary quantities contained in VI and [VII] should be computed with those values of $x_0, y_0, z_0, r_0, x'_0, y'_0, z'_0, r'_0$, etc., which correspond to the elements to which corrections are sought. If these elements exactly satisfied the original observations, then we put $\partial\alpha = \partial\delta = 0$ in the formulæ which give the differential corrections $\partial\rho_0, \partial x'_0, \partial y'_0, \partial z'_0$. If the first observation was not accurately represented by the finally adopted orbit, the proper residuals $\partial'\alpha'$ and $\partial'\delta'$ must be introduced in [VII]. With the newly corrected heliocentric coördinates and velocities the elements are computed as in the ordinary case.

The method here outlined for computing an orbit is principally intended to aid in the rapid determination of the best orbit that can be passed through three observations made at short intervals. Such observations are now almost always available within a few days after the discovery of a new planet or comet. In the case of an object, however, remaining above the horizon the greater part of the

night, it is quite practicable to apply our formulæ to observations made in a single night, provided that the object be not too difficult to measure and that the geocentric motion be sufficiently rapid to enable the observer to determine observationally the velocity and acceleration in α and δ by making settings three times during the night as many hours apart as may be possible. In cases where the necessary observations can not be obtained in a single night, two nights may be sufficient, etc. The point images of asteroids secured by photography seem to be particularly promising in this direction.

ELEMENTS OF ASTEROID 1900 GA.

BY A. O. LEUSCHNER AND ADELAIDE M. HOBE,
BERKELEY ASTRONOMICAL DEPARTMENT.

Doc. Ind. 1001

ELEMENTS OF ASTEROID 1900 GA.

BY A. O. LEUSCHNER AND ADELAIDE M. HOBE.

Asteroid 1900 GA was discovered June 28th, 1900, by the late Director JAMES E. KEELER, of the Lick Observatory, while photographing the region of the sky near Saturn with the CROSSLEY Reflector. The asteroid was again photographed on June 29th, June 30th, and July 2d. On all four dates trails were secured, the exposure time being two hours on June 28th, and one hour on each of the other dates. In addition, point images of the asteroid were secured on June 30th and July 2d. The plates were measured by Mr. H. K. PALMER, and he considers the object the most difficult ever measured by him. The observations were published by Director W. W. CAMPBELL in *Astronomische Nachrichten*, No. 3708.

About a month after discovery, Director KEELER sent us the positions of the middle of the trail of June 28th, and of the point images, and expressed the opinion that an indeterminate case seemed to be involved in the solution for the orbit from these observations. Unfortunately, no more observations could then be obtained, as the asteroid had moved out of the range of the reflector, and as it was too faint for visual observations, besides being in the thick of the Milky Way.

It should be stated here that, although the asteroid was photographed on four nights, the available data for a determination of the orbit consist only of three positions, which lie very nearly in the same great circle. Very little, if any, weight can be attached to the position derived from the trail of June 29th, as this trail fell near the edge of the plate and was exceedingly difficult to measure. On June 28th, the time which corresponds to the middle of the trail is in doubt by over a minute. Owing to the slow motion of the planet, this uncertainty of the time is of no great consequence.



Doc 1000

ELEMENTS OF ASTEROID 1900 GA.

BY A. O. LEUSCHNER AND ADELAIDE M. HOBE.

Asteroid 1900 GA was discovered June 28th, 1900, by the late Director JAMES E. KEELER, of the Lick Observatory, while photographing the region of the sky near Saturn with the CROSSLEY Reflector. The asteroid was again photographed on June 29th, June 30th, and July 2d. On all four dates trails were secured, the exposure time being two hours on June 28th, and one hour on each of the other dates. In addition, point images of the asteroid were secured on June 30th and July 2d. The plates were measured by Mr. H. K. PALMER, and he considers the object the most difficult ever measured by him. The observations were published by Director W. W. CAMPBELL in *Astronomische Nachrichten*, No. 3708.

About a month after discovery, Director KEELER sent us the positions of the middle of the trail of June 28th, and of the point images, and expressed the opinion that an indeterminate case seemed to be involved in the solution for the orbit from these observations. Unfortunately, no more observations could then be obtained, as the asteroid had moved out of the range of the reflector, and as it was too faint for visual observations, besides being in the thick of the Milky Way.

It should be stated here that, although the asteroid was photographed on four nights, the available data for a determination of the orbit consist only of three positions, which lie very nearly in the same great circle. Very little weight can be attached to the position derived from the trail of June 29th, as the trail fell near the edge of the plate and was exceedingly difficult to measure. On June 28th, the time which corresponds to the middle of the trail is off by over a minute. Owing to the slow motion of the plate, the time is of no great consequence.

The available data are as follows:

Mt. Hamilton M. T.	Exposure Time.	Asteroid's App. α .	Asteroid's App. δ .	No. of Comp. Stars.
1900, June 28 ^d { 9 ^h 54 ^m 25 ^s { 11 54 25	2 hours	18 ^h 3 ^m 19 ^s .80 18 3 15.60	22 [°] 28' 59".4 — 22 29 50 .4	2
June 29 { 10 13 { 11 13	1 hour	18 2 36.66 18 2 35.43	— 22 40 49 .5 — 22 41 11 .3	2
June 30 { 10 30.0 { 11 30.0	1 hour	18 1 54.48 18 1 53.12	— 22 52 42 .6 — 22 53 7 .6	5
June 30 11 52 4	5 minutes	18 1 52.16	— 22 53 22 .1	5
July 2 { 10 15 30 { 11 15 30	1 hour	18 0 34.46 18 0 33.31	— 23 15 58 .9 — 23 16 32 .8	7
July 2 11 22 50	10 minutes	18 0 32.28	— 23 16 37 .1	7

The problem of determining the orbit of the asteroid presents many points of interest, and has led to the derivation of "A Short Method of Determining Orbits from Three Observations," published in Part I of this volume.

A few preliminary tests, in which the calculations were made by Miss A. M. HOBE and Mr. S. C. PHIPPS, made it evident that the existing methods for determining orbits from three observations, without hypothesis as to the eccentricity, could not be used to advantage in this case.

In GAUSS'S and OPPOLZER'S methods, for instance, we find $\log K$ (the logarithm of the coefficient of the geocentric distances) = 4.62_m, so that with six-place logarithms this coefficient is comparable with the uncertainty of the calculation. The geocentric distance is, therefore, practically indeterminate by these methods. In a seven-place calculation it is nearly so, and the results of the first hypothesis do not furnish data sufficiently accurate for a second hypothesis, etc.

In problems of this sort the greatest disadvantage arises from the fact that the numerical operations of the successive hypotheses are so complicated that it is difficult to form an estimate of the degree of their convergency. This is true also of GIBBS'S Vector Method. In none of the existing methods are the quantities upon which a new hypothesis is to be based given directly as functions of the residuals of the preceding hypothesis.

In the present problem we encounter the additional difficulties that the observations extend over only four days, that the total motion in right ascension and declination is only 2^m 45^s and 47', respectively, and that owing to the faintness of the object the available positions are less reliable than is ordinarily the case.

The uncertainty of the orbit, due to the insufficiency of the available material, can not be overcome by any method of solution, no matter on what theoretical considerations it may be based. The best that can be done is to accurately represent the given material. The theoretical difficulties involved in this task can be overcome by a proper choice of method. The consideration that an indeterminate function may be evaluated by the use of derivatives, suggests the introduction of the geocentric velocities and accelerations. The methods of LAPLACE and HARZER furnish convenient expressions which contain the geocentric distance, its velocity and acceleration as a function of the corresponding right ascension and declination, their velocities and accelerations; but for reasons which are apparent from the preceding article, these methods are otherwise not suited for the solution of the present problem. The method outlined in the preceding pages has proved very efficient in this case, as well as in the cases of comet 1900 III and comet 1901 I, to which it has been applied.

It was at first hoped that it might be possible to derive the necessary velocities and accelerations in the two observed geocentric coördinates directly from the trails, and for this purpose all the necessary data were kindly furnished by Director CAMPBELL, but it was found to be impracticable.

The numerical operations involved in determining the orbit are given here in full, as they are intended to serve as an illustration of the method derived in Part I. The arguments are in strict accordance with the notation adopted in the formulæ collected under the headings I-VIII.

Ordinarily the calculation will be much briefer than the present case would seem to indicate. Thus, systems II-V have been computed three times in order to demonstrate that nothing is gained by making three direct approximations for the geocentric distance. One approximation would have been entirely sufficient. It should be noted, also, that almost the entire calculation [VII] is based on logarithms already known. The only new logarithms which need to be taken from the tables are the addition and subtraction logarithms involved.

Of the three places on which the calculation is based, the first represents the position of the middle of the trail of June 28th. It was derived by taking the mean of the coördinates of the termini and of the corresponding times. The places for June 30th and July 2d were derived by combining the coördinates of the middle of the trail with the coördinates of the point-image observed on the same date, the latter coördinates being given twice the weight of the former. Whether this manner of combining the observations possesses the greatest advantages can only be decided by future observations, if the asteroid should be observed again. The adopted positions are:

Date, Gr. M.T.	α app.	δ app.
1900, June 28.792359	18 ^h 3 ^m 17 ^s .700	— 22° 29' 24".90
June 30.820341	1 52.707	53 13 .10
July 2.803450	0 32.815	— 23 16 30 .02

For the position of the middle of the trail of June 29th we have

$$\text{June } 29.784431 \quad 18^{\text{h}} 2^{\text{m}} 36^{\text{s}}.045 \quad - 22^{\circ} 41' 0''.40$$

This observation was given zero weight. The other three places were reduced to the beginning of the year by the ordinary formulæ on the basis of the data given in the *American Ephemeris*, f' being neglected. The resulting mean places, which are not corrected for the aberration term, depending on the product of the aberration into the eccentricity of the earth's orbit, are given in II.

I.

From the *American Ephemeris and Nautical Almanac* for 1900 we obtain by interpolation the solar coördinates referred to the beginning of the year, as follows:

Date, Gr. M. T., 1900, June	28.792359 _s	30.820341 _s	32.803450 _s
X'	- 0.1246766	- 0.1586609 _s	- 0.1917110
Y'	+ 0.9257300	+ 0.9213667 _s	+ 0.9160654 _s
Z'	+ 0.4015994	+ 0.3997058 _s	+ 0.3974057

For the determination of the velocities X'_0 , Y'_0 , Z'_0 we have

$$m = 0.1406830 \quad \log m = 9.148242$$

	X'	Y'	Z'
$\log f''(a + [i + \frac{1}{2}]w)$	5.0540	5.813414 _n	5.450634 _n
$\log m f''(a + [i + \frac{1}{2}]w)$	4.2022	4.961656 _n	4.598876 _n
$\log f'(a + [i + \frac{1}{2}]w)$	7.9221465 _n	7.0787286 _n	6.7160869 _n
B	3.7199	A 2.11707	2.11721
A	0.0000828	B 0.0033041 _s	0.0033031
$\log (X'_0, Y'_0, Z'_0)$	9.9875127 _n	9.1474817 _n	8.7848390 _n
$\log \cos A$	9.229692 _n	$\log \cos D$	9.963549
$\log X_0$	9.200470 _n	$\log R \cos D$	9.970778
$\log Y_0$	9.964432	$\log Z_0$	9.601740
$\log \sin A$	9.993654	$\log \sin D$	9.594511
$\log \tan A$	0.763962 _n	$\log \tan D$	9.630962
A	99° 46' 14".21	D	23° 8' 52".59
α	270 27 3.44	$\log R$	0.007229
$A - \alpha$	- 170 40 49.23		

The second observation was not corrected for the parallax corresponding to the unit of distance.

II.

Date, Gr. M. T.		α (1900.0).	δ (1900.0).
June 28.792359s		18 ^h 3 ^m 13 ^s .254	- 22° 29' 28".95s
	2.027982	- 85".024s	- 1428".04s
June 30.820341s	0.044873	1 48.229s	53 17.00
	1.983109	- 79.921s	- 1396.76s
July 2.803450s		0 28.308	- 23 16 33.76s
	4.011091 (sum)		

$\log (\alpha_n - \alpha_i)$	1.929544n	$\delta'_i - \delta_{iii}'$	- 0.0000446
$\log (t_n - t_i)$	0.307064	$\log (\alpha'_i - \alpha_{iii}')$	7.836830
$\log (\delta_n - \delta_i)$	3.154742n	$\log (t_{iii} - t_i)$	0.603262
$\log (\alpha_{iii} - \alpha_n)$	1.902663sn	$\log (\delta'_i - \delta_{iii}')$	5.649335n
$\log (t_{iii} - t_n)$	0.297347	$\log \alpha_0''$	9.299016s
$\log (\delta_{iii} - \delta_n)$	3.145123n	$\log [(t_n - t_i) - (t_{iii} - t_n)]$	8.652
$\log \alpha_{iii}'$	9.248565n	$\log \delta_0''$	7.111521sn
$\log \delta_{iii}'$	9.297671sn	$\log \frac{k}{4} [(t_n - t_i) - (t_{iii} - t_n)] \alpha_0''$	5.585
$\log \alpha_i'$	9.231401sn	$\log \frac{1}{2} (\alpha_{iii}' + \alpha_i')$	9.240067sn
$\log \delta_i'$	9.297769sn	B	3.655
α_{iii}'	- 0.1772412	C	9.999904
α_i'	- 0.1703732	$\log \alpha_0'$	9.239971sn
δ_{iii}'	- 0.1984593	$\log \frac{k}{4} [(t_n - t_i) - (t_{iii} - t_n)] \delta_0''$	3.40n
δ_i'	- 0.1985039	$\log \frac{1}{2} (\delta_{iii}' + \delta_i')$	9.297721 n
$\alpha_{iii}' + \alpha_i'$	- 0.3476144	A	4.10
$\delta_{iii}' + \delta_i'$	- 0.3969632	B	0.000001
$\alpha_i' - \alpha_{iii}'$	+ 0.0068680	$\log \delta_0'$	9.297722 n

δ_0'' , being less than a tenth of a second of arc per day, might have been put equal to zero without diminishing the accuracy of the first approximation of σ_0 , σ_0' , σ_0'' , but it was retained in the calculation in order to render the example more complete. It should be stated, however, that the values of α_0' , α_0'' , and δ_0'' , used below, differ slightly from those derived above. The differences are due to the fact that in the original calculation right ascensions and declinations, differing from those printed above by a few one thousands of a second of arc, were used. These differences serve to show that the accelerations are subject, generally, to more or less uncertainty, depending on their magnitude as compared with the errors of observation. This uncertainty is of no consequence in our method, as later we may readily correct the fundamental data (δ) so as to accurately represent the foregoing observations.

$\log \delta_0'$	9.297722n	$\log 2 \tan \delta_n (\delta_0')^2$	8.521962n
$\log \sec^2 \delta_n$	0.071228	$\log \delta_0''$	7.112040n
$\log (\tan \delta)_0'$	9.368950n	A	8.590078
$\log \tan \delta_n$	9.625488n	B	0.016579
$\log (\delta_0')^2$	8.595444	sum	8.538541n
		$(\tan \delta)_0''$	8.609769n

III.

$\log (\alpha'_0)^3$	7.719907 _n	$\log \sin (A - \alpha'_0)$	9.209361 _n
$\log \tan \delta'_0$	9.625488 _n	$\log \sin (A - \alpha'_0) (\tan \delta'_0)$	8.578311
$\log \alpha''_0$	9.298855	A	8.700145
$\log (\tan \delta'_0)'$	9.368950 _n	B	0.021245
$\log \alpha'_0$	9.239969 _n	$\log R \cos D$	9.970778
$\log (\tan \delta'_0)''$	8.609769 _n	$\text{colog } N$	1.253148
$\log \alpha'_0 (\tan \delta'_0)''$	7.849738	$\log \kappa$	9.823482 _n
$\log [-\alpha''_0 (\tan \delta'_0)']$	8.667805	$\log \cos \delta'_0$	9.964386
A	9.181933	$\log \cos D$	9.963549
B	0.061464	$\log \sin \delta'_0$	9.589874 _n
sum	8.729269	$\log \sin D$	9.594511
$\log (\alpha'_0)^3 \tan \delta'_0$	7.345395	$\log \sin \delta'_0 \sin D$	9.184385 _n
A	8.616126	$\log \cos \delta'_0 \cos \delta \cos (A - \alpha'_0)$	9.922165 _n
B	0.017583	A	9.262220
N	8.746852	B	0.072949
$\log \cos (A - \alpha'_0)$	9.994230 _n	$\log \cos \psi$	9.995114 _n
$\log \tan \delta'_0$	9.625488 _n	ψ	171° 25'.3
$\log \tan \delta'_0 \cos (A - \alpha'_0)$	9.619718	$\log R^4$	0.028916
$\log \tan D$	9.630962	$\text{colog } \kappa$	0.176518 _n
B	0.011244	$\log \frac{1}{m}$	0.169820
A	8.418769	$\frac{1}{m}$	1.4785
$\log [\tan \delta'_0 \cos (A - \alpha'_0) - \tan D]$	8.038487 _n	z_1 (OPPOLZER, table XIIIa)	0.4551
$\log \alpha'_0 [\quad]$	7.278456		

IV.

$\log \cos \psi$	9.995114 _n	$\log \nu_1^2 \mu_1^3$	9.660896	9.660361 ₃
$\log z_1$	9.658107	$\log m^2$	9.660360	9.660360
$\log 2z_1 \cos \psi$	9.954251 _n	B	0.000536	0.000001 ₃
$\log z_1^2$	9.316214	A	7.091667	4.55
A	9.361963	$\log M_1$	6.7520	4.21
B	0.089949	A	9.6630	
$\log (\mu_1 - 1)$	0.044200	B	0.1644	
B	0.279492	$\log (z_1 - \cos \psi)$	0.1595	
$\log \mu_1$	0.323692	$\log 3 \frac{\nu_1}{\mu_1}$	9.4983	
$\log m$	9.830180	$\log 3 \frac{\nu_1}{\mu_1} (z_1 - \cos \psi)$	9.6578 _n	
B	0.172073	$\log [1 + \frac{3\nu_1}{\mu_1} (z_1 - \cos \psi)]$	0.7366	
A	9.686803	$\log 2 \mu_1^3 \nu_1$	0.6170 _n	
$\log \nu_1$	9.344910 _n	$\log \mathcal{A} z_1$	6.3984	3.86
$\log \nu_1^2$	8.689820	$\mathcal{A} z_1$	0.000250	0.0000007
$\log \mu_1^3$	0.971076	z_2	0.455350	0.455351

V.

$\log R$	0.007229	$\log \frac{R \cos D}{2 N}$	0.922896
$\log Z$	9.658346	$\log \lambda$	8.734717
$\log \rho_0$	9.665575	$\text{colog } \kappa$	0.176518 _n
$\log \cos \delta_0$	9.964386	$\log \sigma_0'$	8.541196 _n
$\log \sigma_0$	9.629961	$\text{colog } R^3$	9.9783
$\log (\alpha_0')^2$	8.479938	$\log (\alpha_0')^2$	8.4799
$\log \tan \delta_0$	9.625488 _n	B	1.4984
$\log (\alpha_0')^2 \tan \delta_0$	8.105426 _n	C	9.9860
$(\tan \delta)_0''$	8.609769 _n	$\log \sigma_0 [R^{-3} - (\alpha_0')^2]$	9.5943
A	9.495657	$\log R \cos D \cos (A - \alpha_0)$	9.9650 _n
B	0.118292	A	9.6650
$\log \sin (A - \alpha_0)$	9.209361 _n	B	0.1650
$\log \sin (A - \alpha_0) [(\alpha_0')^2 \tan \delta_0 + (\tan \delta)_0'']$	7.937422	$\log \frac{\sigma_0}{\kappa} [R \cos D \cos (A - \alpha_0) - \sigma_0]$	9.9365
$\log \alpha_0'' [\tan \delta_0 \cos (A - \alpha_0) - \tan D]$	7.337342 _n	B	0.3422
B	0.600080	A	0.0788
C	9.874399	$\log \sigma_0''$	9.6731

	I.	III.		I.	III.
$\log \theta$	8.5426 ₅	8.5329	$\log (\pm \theta \sigma_0' + \theta^2 \frac{\sigma_0''}{2})$	7.1759	6.9600 _n
$\log \theta^2$	7.0853	7.0658	A	7.5459	B 2.6700
$\log \pm \theta \sigma_0'$	7.0838	7.0741 _n	B	0.00152	C 9.99907
$\log \theta^2 \frac{\sigma_0''}{2}$	6.4574	6.4379	$\log \sigma$	9.63148	9.62903
A	9.3736	B 0.6362	$\log \cos \delta$	9.96564	9.96313
B	0.0921	C 9.8859	$\log \rho$	9.66584	9.66590

Owing to the uncertainty of some of the fundamental data (a) in this example, particularly of the accelerations, we are justified in assuming that the first approximation does not result in sufficiently accurate geocentric distances to enable us to determine the final corrections for parallax and aberration. It would, therefore, be best to proceed at once to the computation of the heliocentric rectangular coördinates and their velocities, after applying a preliminary parallax correction to the second observation. Nevertheless, we shall correct all three places for parallax and aberration, and shall redetermine the geocentric velocities and accelerations, and, from these, the values of σ_0 , σ_0' , etc., in order that we may be able to show that little or nothing can be gained by a second approximation of this kind. With the values of ρ , derived above, the original data (coördinates of termini of trails and point-images) were separately corrected for parallax and aberration, and then recombined into three places as before, with the following result:

Date, Gr. M. T.	α (1900.0).	δ (1900.0).
June 28.789686	18 ^h 3 ^m 13 ^s .065 ₈	- 22° 29' 12".87
June 30.817670	1 48 .269 ₅	53 0 .61
July 2.800777	0 28 .277	- 23 16 17 .30

From these we derive

$\log \alpha_0'$	9.239575 _n	$\log (\tan \delta)_n'$	9.368864 _n	$\log \kappa$	9.856427 _n
$\log \alpha_0''$	9.257362	$\log (\tan \delta)_n''$	8.621354 _n	$\log \lambda$	8.790774

and further

$\log \rho_0$	9.723653	$\log \sigma_0'$	8.622400 _n	$\log \rho_1$	9.72396
$\log \sigma_0$	9.688053	$\log \sigma_0''$	9.7057	$\log \rho_{\infty}$	9.72389

These values of ρ differ from the former to such an extent that it is necessary to recompute the corrections for parallax and aberration. It might, therefore, seem that the approximations should be continued until the parallax and aberration cease to vary. But, in the present case, κ and ρ_0 must remain uncertain, even if the observations were fully corrected. This may easily be seen from the computation of the first approximation of κ . It is, then, evident that, after the parallax and aberration cease to vary, everything depends on whether the accelerations are constant or not within the interval between our first and last dates of observation. If they fail to be constant, even by a small amount, the value of ρ_0 , to be computed later from the residuals of the first and third places, will be considerable, and may affect the parallax and aberration. The approximations for the geocentric velocities and accelerations and for the corresponding value of ρ_0 should, therefore, now be discontinued; but in order that the reader may not be led to conclude that the large numerical value of ρ_0 , which will be found later, is due to the fact that the approximations have not been carried sufficiently far, we shall continue the same until the corrections practically cease to vary. Thus, from the last values of ρ_1 , ρ_0 , and ρ_{∞} , printed above, we first obtain the following corrected positions:

Date, Gr. M. T.	α (1900.0).	δ (1900.0).
June 28.789303	18 ^h 3 ^m 13 ^s .089	- 22° 29' 14".89
June 30.817287	1 48.265	53 2.66
July 2.800394 ₃	0 28.280 ₃	- 23 16 19.36

These positions give

$\log \alpha_0'$	9.239625 _n	$\log \kappa$	9.852281 _n	$\log \sigma_0'$	8.612542 _n
$\log \alpha_0''$	9.262634	$\log \lambda$	8.783846	$\log \sigma_0''$	9.7019
$\log (\tan \delta)_n'$	9.368874 _n	$\log \rho_0$	9.716579	$\log \rho_1$	9.71689
$\log (\tan \delta)_n''$	8.619945 _n	$\log \sigma_0$	9.680977	$\log \rho_{\infty}$	9.71683

With the foregoing values of ρ we obtain the following corrected positions:

Date, Gr. M. T.	α (1900.0).	δ (1900.0).
June 28.789352	18 ^h 3 ^m 13 ^s .086 ₃	- 22° 29' 14".65 ₃
June 30.817336	1 48.265 ₃	53 2.42 ₃
July 2.800443	0 28.280	- 23 16 19.12 ₃

Further approximations would not produce any sensible change in the parallax or aberration. We, therefore, proceed to the computation of the heliocentric coördinates and velocities, and of the residuals of the first and third places, etc.

VI.

$\log \cos \alpha_n$	7.8961509	$\log \cos \alpha_n \sigma_n'$	6.5086929 _n	B	1.7957751
$\log \sin \alpha_n$	9.9999865 _n	$\log \eta_0 \alpha_0'$	8.9205890	A	0.0070066
$\log \tan \delta_n$	9.6254026 _s	A	2.4118961	$\log \tan \delta_n \sigma_n'$	8.2379446
$\log \xi_n$	7.5771279	B	0.0016790	$\log \sigma_0 (\tan \delta)_n'$	9.0498515 _n
$\log \eta_0$	9.6809635 _n	$\log \sin \alpha_n \sigma_n'$	8.6125285	B	0.8119068
$\log \zeta_0$	9.3063796 _{sn}	$\log \xi_0 \alpha_0'$	6.8167534 _n	A	0.0727339

ξ_0, η_0, ζ_0	+0.0037768	-0.4796931	-0.2024788
X_n, Y_n, Z_n	-0.1586609 _s	+0.9213667 _s	+0.3997058 _s
x_0, y_0, z_0	+0.1624377 _s	-1.4010598 _s	-0.6021846 _s
$\log (x_0, y_0, z_0)$	9.2106869 _s	0.1464566 _{sn}	9.7797296 _{sn}
$\log (\xi_0', \eta_0', \zeta_0')$	8.9222680 _n	8.6055219	8.9771176 _n
$\log (X_0', Y_0', Z_0')$	9.9875127 _n	9.1474817 _{sn}	8.7848390 _n
B	1.0652447	A 0.5419598 _s	0.1922786
A	0.0390780	B 0.1096138 _s	0.4464514
$\log (x_0', y_0', z_0')$	9.9484347	9.2570956	8.5306662 _n

$\log x_0^2$	8.4213739	$\log x_0 x_0'$	9.1591216 _s	$\log r_0 r_0'$	8.9472556 _{sn}
$\log y_0^2$	0.2929133	$\log y_0 y_0'$	9.4035522 _s	$\log r_0$	0.1857169
A	1.8715394	A	0.2444306	$\log r_0'$	8.7615387 _{sn}
B	0.0057989	B	0.3661270	$\log r_0^3$	0.55715
$\log (x_0^2 + y_0^2)$	0.2987122	$\log (x_0 x_0' + y_0 y_0')$	9.0374252 _n	$\log r_0^4$	0.74
$\log z_0^2$	9.5594593	$\log z_0 z_0'$	8.3103958 _s	$\log \frac{1}{2r_0^3}$	9.14182
A	0.7392529	B	0.7270294	$\log \frac{r_0'}{2r_0^4}$	7.72 _n
B	0.0727215 _s	A	0.0901696	$\log \frac{1}{6r_0^3}$	8.66470
$\log r_0^2$	0.3714337 _s				

	I.	III.		I.	III.
$(t_n - t_i), (t_{nn} - t_n)$	2.0279840	1.9831070	gx_0'	-0.0309781	+0.0302928
$\log \Delta t$	0.3070645	0.2973461	Y'	+0.9257300	+0.9160654 _s
$\log \theta$	8.5426455	8.5329271	fy_0	-1.4008239	-1.4008335 _s
$\log \theta^2$	7.08529	7.06585	gy_0'	-0.0063054 _s	+0.0061659 _s
$\log \theta^2 \frac{1}{2r_0^3}$	6.22711	6.20767	Z	+0.4015994	+0.3974057
$\log \theta^3 \frac{r_0'}{2r_0^4}$	3.35 _n	3.32 _n	fz_0	-0.6020832	-0.6020874
$\theta^2 \frac{1}{2r_0^3}$	+0.0001687	+0.0001613	gz_0'	+0.0011838	-0.0011576
$\theta^3 \frac{r_0'}{2r_0^4}$	2	2	ξ	+0.0067557	+0.0009933
f	0.9998315	0.9998385	η	-0.4813993 _s	-0.4786021 _s
$\log \theta^3 \frac{1}{6r_0^3}$	4.293	4.263	ζ	-0.1993000	-0.2058393
$\theta^3 \frac{1}{6r_0^3}$	0.0000020	0.0000018	$\log \cos \alpha$	8.1471221 _s	7.3171049
θ	0.0348855 _s	0.0341136	$\log \xi$	7.8296704	6.9970804
g	-0.0348835 _s	+0.0341118	$\log \eta$	9.6825055 _n	9.6799746 _{sn}
$\log f$	9.9999268 _n	9.9999298 _s	$\log \sin \alpha$	9.9999572 _{sn}	9.9999991 _n
$\log g$	8.5426206 _{sn}	8.5329047	$\log \cot \alpha$	8.1471649 _n	7.3171057 _{sn}
$\log fx_0$	9.2106138	9.2106168	α	270° 48' 14".420	270° 7' 8".085
$\log fy_0$	0.1463835 _n	0.1463865 _n	$\log \cos \delta$	9.9656541 _s	9.9631460
$\log fz_0$	9.7796565 _n	9.7796595 _n	$\log \rho \cos \delta$	9.6825482 _s	9.6799755
$\log gx_0'$	8.4910553 _n	8.4813394	$\log \zeta$	9.2995073 _n	9.3135283 _n
$\log gy_0'$	7.799716 _n	7.790000	$\log \sin \delta$	9.5826132 _n	9.5966988 _n
$\log gz_0'$	7.073287	7.063571 _n	$\log \tan \delta$	9.6169590 _{sn}	9.6335528 _n
X	-0.1246766	-0.1917110	δ	-22° 29' 15".467	-23° 16' 18".276 _s
fx_0	+0.1624104	+0.1624115	$\log \rho$	9.7168941	9.7168296

	I.	III.		I.	III.
$\log f_x \partial \rho_0$	6.33533 ₅	6.33558 ₅	corrected ζ	-0.2106565 ₅	-0.2175383 ₅
$\log f_y \partial \rho_0$	8.43640 _{5n}	8.43639 _{5n}	$\log \cos \alpha$	8.1473846	7.3133893 ₅
$\log f_z \partial \rho_0$	8.06182 _{5n}	8.06181 _{5n}	$\log \xi$	7.8540052 ₅	7.0173673
$\log g \partial x'_0$	6.23794	6.22822 _{5n}	$\log \eta$	9.7065778 _{5n}	9.7039770 _n
$\log g \partial y'_0$	6.08470 _{5n}	6.07499	$\log \sin \alpha$	9.9999572 _n	9.9999991 _n
$\log g \partial z'_0$	6.23874	6.22902 _{5n}	$\log \cot \alpha$	8.1474274 _n	7.3133903 _n
$f_x \partial \rho_0$	+0.0002164	+0.0002166	α	270° 48' 16".169	270° 7' 4".438 ₅
$g \partial x'_0$	+0.0001729 ₅	-0.0001691	$\log \cos \delta$	9.9656548 ₅	9.9631452
ξ	+0.0067557	+0.0009933	$\log \rho \cos \delta$	9.7066206 ₅	9.7039779
corrected ξ	+0.0071450 ₅	+0.0010408	$\log \zeta$	9.3235750 _n	9.3375358 _n
$f_y \partial \rho_0$	-0.0273152	-0.0273146	$\log \sin \delta$	9.5826092 _n	9.5967031 _n
$g \partial y'_0$	-0.0001215	+0.0001188 ₅	$\log \tan \delta$	9.6169543 _{5n}	9.6335579 _n
η	-0.4813993 ₅	-0.4786021 ₅	δ	-22° 29' 14".673	-23° 16' 19".155
corrected η	-0.5088360 ₅	-0.5057979	$\log \rho$	9.7409658	9.7408327
$f_z \partial \rho_0$	-0.0115298 ₅	-0.0115296	$O - C \begin{cases} \Delta \alpha & + 0".13 \\ \Delta \delta & + 0.02 \end{cases}$		-0".24 +0.03
$g \partial z'_0$	+0.0001733	-0.0001694 ₅			
ζ	-0.1993000	-0.2058393			

In view of the comparative uncertainty of the observations, these residuals are more than satisfactory. It should be noted, however, that the residuals in declination are actually larger than those derived above by about 0".8, on account of change in the parallax, due to the considerable change which has taken place in the geocentric distances. We shall allow for the changes in parallax in computing the final heliocentric rectangular coördinates. With the values of $\log \rho$, and $\log \rho_{\text{true}}$, found above, and with $\log \rho_{\text{true}} = \log (\rho_0 + \partial \rho_0) = 9.74062$, we first obtain the following newly-corrected positions:

Date, Gr. M. T.	α (1900.0).	δ (1900.0).
June 28.789181	18 ^h 3 ^m 13".096	-22° 29' 15".43
June 30.817165	1 48.264	53 3.21
July 2.800272	0 28.281	-23 16 19.91

With the second of these positions and with $\log \rho_{\text{true}} = \log (\rho_0 + \partial \rho_0) = 9.7406228$ the heliocentric rectangular coördinates are now re-computed as in VI; X_{true} , Y_{true} , and Z_{true} remaining the same as in the original computation. The velocities x' , y' , z' are obtained by applying to x'_0 , y'_0 , z'_0 the corrections $\partial x'_0$, $\partial y'_0$, $\partial z'_0$, determined above. We then complete the recomputation of VI, and thus obtain the new residuals. It will be noted that by following the process just outlined, the change in parallax is fully taken into account in the rectangular coördinates and in the residuals. The velocities, however, are not changed, this being considered unnecessary, as all three declinations were changed by the new parallax by about the same amount, and as the change in the right ascensions was very small. The various corrected quantities follow below:

				I.	III.	
$\log x$	9.2112614	$\log x'$	9.9460031	α	270° 48' 16".13	270° 7' 4".40
$\log y$	0.1548393 _n	$\log y'$	9.2653868	δ	- 22 29 15.49	- 23 16 19.98
$\log z$	9.7879651 _n	$\log z'$	8.5899909 _n	$O - C \left\{ \begin{array}{l} \cos \delta \Delta \alpha + 0".28 \\ \Delta \delta + 0.06 \end{array} \right.$		- 0".16
$\log r$	0.1939908	$\log r'$	8.7867019 _n			+ 0.07

A further reduction of the residuals is unwarranted, and would not lead to more accurate elements.

VIII.

A	1.6241942	$\log \sqrt{p}$	0.1486498	φ	16° 22' 55".02
$\log xy'$	8.4766482	$\log p$	0.2972996	$\frac{1}{2} \varphi$	8 11 27 .51
$\log yx'$	0.1008424 _n	B	0.4416705	$\log \cot (45^\circ + \frac{1}{2} \varphi)$	9.8740997
B	0.0101972	$\log x \cos \Omega$	9.1907378	$\log \tan \frac{1}{2} v$	9.1945212 _n
A	0.3085217	$\log y \sin \Omega$	9.6324083 _n	$\log \tan \frac{1}{2} E$	9.0686210 _n
$\log yz'$	8.7448302	A	0.1949644	$\frac{1}{2} E$	6° 40' 47".62 ₃
$\log zy'$	9.0533519 _n	$\log \cos u$	9.2434531 _n	E	- 13 21 35 .25
B	0.1736083	$\log r \cos u$	9.4374439 _n	$\log \cos \varphi$	9.9820010
B	1.9327159	$\log r \sin u$	0.1872236 _n	$\log \sin E$	9.3637345 _n
$\log xz'$	7.8012523 _n	$\log \sin u$	9.9932328 _n	$\log e''$	4.7647346
$\log zx'$	9.7339682 _n	$\log \tan u$	0.7497797	$\log \Delta M$	4.1284691 _n
A	0.0051005	u	259° 54' 41".78	ΔM	- 3° 44' 2".16
$\log \sin \Omega$	9.4775690	$\log r$	0.1939908 check	M	- 9 37 33 .09
$\log \sqrt{p} \sin i \sin \Omega$	9.2269603	$\log \frac{p}{r}$	0.1033088	ω	277 42 4 .19
$\log \sqrt{p} \sin i \cos \Omega$	9.7288677	B	0.6742779	π	295 10 38 .48
$\log \cos \Omega$	9.9794764	$\log \sin v$	9.4850422 _n	$\log \cos^2 \varphi$	9.9640020
$\log \tan \Omega$	9.4980926	$\log \sin \varphi \sin v$	8.9353517 _n	$\log a$	0.3332976
Ω	17° 28' 34".29	$\log \sin \varphi \cos v$	9.4290308	a	2.1542574
$\log \sin i$	9.6007415	$\log \cos v$	9.9787213	$\log a^{\frac{1}{2}}$	0.4999464
$\log \sqrt{p} \sin i$	9.7493913	$\log \tan v$	9.5063208 _n	$\log \mu$	3.0500602
$\log \sqrt{p} \cos i$	0.1110396	v	- 17° 47' 22".41 ₅	$\log P$	3.0625448
$\log \cos i$	9.9623898	$\frac{1}{2} v$	- 8 53 41 .21	μ	1122".174
$\log \tan i$	9.6383516	$\log \sin \varphi$	9.4503095	P	1154 ^d .901
i	23° 30' 8".64	e	0.2820392		

The elements which determine the position of the orbit in space are referred to the equator. After transforming them to the ecliptic in the usual way and determining the reductions to the equinox of 1902.0, we obtain the following collected results, where the reductions of the constants to the equator for 1901.0 are one half of those for 1902.0. The quantities referring to 1900.0 are given more accurately than the others, as it is intended to use them in an accurate redetermination, by means of an ephemeris, of the residuals of the places on which the calculation was based.

Epoch 1900, June 30.817165, Gr. M. T.	1900.0.	Reduction to 1902.0.
M 350° 22' 16".91	Ω 97° 36' 55".57	+ 1' 32".8
φ 16 22 55 .01	i 6 56 23 .06	- 0 .2
a 2.154257	ω 196 8 5 .52	+ 7 .7
μ 1122".174		
P 1154 ^d .901		

	1900.0.	Reduction to 1901.0.	Reduction to 1902.0.
A'	23° 48' 21".04	+ 50".6	+ 1' 41".2
B'	296 38 58 .86	+ 49 .4	+ 1 38 .8
C'	277 42 4 .19	+ 48 .4	+ 1 36 .9
$\log \sin a$	9.9968631	+ 0.000000	+ 0.000000
$\log \sin b$	9.9660620	+ 5	+ 10
$\log \sin c$	9.6007415	- 29	- 58

According to these elements, opposition and perihelion occurred in 1900 on June 23.09 and July 31.70, Gr. M. T., respectively.

As a check on the calculation, the residuals of all three places were computed directly from the constants to the equator by the usual ephemeris method. The residuals of the middle of the trail, photographed on June 29th, are also here given. For the sake of comparison, the final residuals derived by VI are also printed below. To make the latter strictly comparable with the former, the aberration terms depending on the product of the aberration into the eccentricity of the earth's orbit were computed according to OPPOLZER, *Bahnbestimmung*, Vol. I, page 115. In this final comparison of computed and observed apparent geocentric places f' was also taken account of.

	$\cos \delta \Delta \alpha (O - C).$		$\Delta \delta (O - C).$	
	From Ephemeris.	From VI.	From Ephemeris.	From VI.
June 28	+0".42	+0".40	+0".06	+0".06
[June 29	+1.74	—	+2.78	—]
June 30	+0.11	+0.16	-0.02	±0.00
July 2	+0.19	+0.15	+0.06	+0.07

The agreement of the two sets of residuals is all that can be desired. As the residuals of the adopted positions are far smaller than the unavoidable errors of observation, a further correction of the elements would have no practical value. In this connection it may be stated that Mr. PALMER writes in a report, kindly furnished by Director CAMPBELL to aid me in determining the orbit, that "the point-image of June 30th is so faint that the measures are less accurate than on many very faint nebulae with no nucleus." Under these circumstances, no practical gain will result from a further reduction of the residuals.

With reference to the manner in which the trails were measured, Mr. PALMER states that "the intersection of the cross-wires was set on what seemed to be the points which would be occupied by the centers of two short exposures taken such that the first should begin at the same time as the long (trail) exposure and the last end at the same time as the long exposure." "The trail of July 2d had two good termini, but the others each had one very poor terminus." Let us, then, assume for a moment that the instants of the beginning and end of the trail-exposures were accurately known, that the trails were so distinct as to enable the observer to estimate accurately the positions on the trails of the centers of two short exposures as defined above, and that, in addition, all other conditions which might affect the accuracy of the measured coördinates were the same for the four trails. Then, by forming the mean of the beginning and end of the exposure time and of the coördinates of the termini for each trail, we should obtain positions of the middle of the trail which are free from error, provided the motion of the asteroid during the time of exposure is proportional to the time. An ephemeris which has been computed for the dates under consideration, shows this proportionality to be true within 0".01. Under these ideal conditions the residuals relative to the middle of the trail of the coördinates of the short exposure point

estimated at the beginning of the trail should be the same for the four trails, and, furthermore, the corresponding relative residuals for the ends should be numerically the same as the former, but of opposite sign. The algebraic sign of these relative residuals should be negative for the beginning of the trails, for both right ascension and declination.

In order to determine how nearly these ideal conditions are fulfilled in the present case, the positions of the asteroid were computed by VI for June 28.75, June 29.75, June 30.75, and July 2.75, Gr. M. T., the necessary solar coördinates for the same dates being interpolated from the *American Ephemeris*. From these positions and from the positions already computed by means of the constants to the equator for the four adopted observation-times, the coördinates of the asteroid were determined by proportional parts for the epochs of the beginning, middle, and end of the trails, and of the point-images, the results being as follows:

Date.	Point Measured.	Epoch.	app. α_0	app. α_c	$\cos \delta$ ($\alpha_0 - \alpha_c$)	app. δ_0	app. δ_c	$\delta_0 - \delta_c$
1900		Gr. M. T.	^h ^m ^s	^s				
June 28	(1) Beginning of trail.	June 28.747514	18 3 19.408	19.282	+1".75	-22° 28' 46".10	42.22	-3.88
	(2) Middle of trail.... = adopted position.	28.789181		17.542	+0.42	29 11.37	11.43	+0.06
	(3) End of trail	28.830848		15.676	-0.91	36.64	40.64	+4.00
June 29	(4) Beginning of trail.	June 29.760421	2 36.356	36.604	-3.43	40 35.97	34.86	-1.11
	(5) Middle of trail....	29.781255		35.857	+1.74	46.74	49.52	+2.78
	(6) End of trail	29.802088		35.358	+6.92	57.51	4.18	+6.67
June 30	(7) Beginning of trail.	June 30.772228	1 54.255	54.583	-4.53	52 28.91	27.66	-1.25
	(8) Middle of trail....	30.793061		53.695	-0.41	41.34	42.33	+0.99
	(9) End of trail	30.813895		53.135	+3.70	53.76	57.00	+3.24
	(10) Point image.....	30.829219		52.264	+0.37	53 8.29	7.77	-0.52
	(11) Adopted position.	30.817165		52.741	+0.11	52 59.31	59.29	-0.02
July 2	(12) Beginning of trail.	July 2.762156	0 34.214	34.280	-0.91	-23 15 45.20	49.38	+4.18
	(13) Middle of trail....	2.782990		33.759	+4.16	16 2.06	4.05	+1.99
	(14) End of trail	2.803823		33.304	+9.23	18.92	18.72	-0.20
	(15) Point image.....	2.808916		32.303	-1.76	23.22	22.31	-0.91
	(16) Adopted position.	2.800272		32.788	+0.19	16.16	16.22	+0.06

By subtracting the residuals for the middle of each trail from those for the termini, we obtain the following residuals of the termini relatively to the middle of the trails:

Trail.	$\cos \delta \Delta \alpha$		$\Delta \delta$	
	Beginning.	End.	Beginning.	End.
June 28	+1".3	-1".3	-3".9	+3".9
June 29	-5.2	+5.2	-3.9	+3.9
June 30	-4.1	+4.1	-2.2	+2.2
July 2	-5.1	+5.1	+2.2	-2.2

Three of the four residuals in each coördinate correspond to the ideal conditions remarkably well, but one in each coördinate has the wrong sign. The inconsistency, in this respect, of the declinations of the termini of the trail on July 2d can not affect the resulting orbit to any great extent, as a point-image is available for the same date, to which twice the weight of the middle of the trail has been assigned. But the inconsistency of the right ascensions of June 28th, for which date we have no point-image at our disposal, will affect the accuracy of the orbit which has been derived, unless, accidentally, both termini are given too long in right ascension by approximately the same amount.

The magnitude of the asteroid was estimated by Mr. PALMER to be between 15 and $16\frac{1}{2}$. Adopting 15.75, we find $g=16.1$ and $m_0=18.1$. The asteroid is, therefore, the faintest so far observed.

According to our elements, the next opposition will take place 1902, January 4^d.3, Gr. M. T. The magnitude of the asteroid at opposition will be 19.5 ± 0.75 . It is, therefore, extremely unlikely that the asteroid will be observed at this opposition. An ephemeris computed by Miss HOBE, to aid astronomers who have the necessary equipment in searching for this interesting object, has been published in *Astronomische Nachrichten*, No. 3752.

PRELIMINARY ELEMENTS OF COMET 1900 III.

BY R. H. CURTISS AND C. G. DALL,
BERKELEY ASTRONOMICAL DEPARTMENT.

~~Not Indexed~~

PRELIMINARY ELEMENTS OF COMET 1900 III.

BY R. H. CURTISS AND C. G. DALL.

In order to furnish an example for the application to comets of the short method of determining orbits from three observations, contained in Part 1, we have, at the suggestion and under the direction of Professor LEUSCHNER, derived preliminary elements of comet 1900 III from the first three observations secured by Assistant Astronomer AITKEN at the Lick Observatory.

The observations are published in the *Astronomical Journal* No. 489, and were reduced to the beginning of the year in the usual manner.

As the numerical operations in the case of comet and asteroid orbits are practically identical, and as the details of the computation are given in full in connection with the asteroid example in Part 2, it has been deemed sufficient, in the present case, to reproduce only the principal features of the numerical work.

The arrangement of the calculations is the same as in Part 2.

I.

From the *American Ephemeris and Nautical Almanac* for 1900:

Date, Gr. M. T., 1901, Dec.	24.602199	26.627974	28.619826
X'	+ 0.049167	+ 0.084519	+ 0.119168
Y'	— 0.901044	— 0.898757	— 0.895386
Z'	— 0.390889	— 0.389896	— 0.388435

$$m = -0.244052$$

X'_0	Y'_0	Z'_0
+ 1.013059	+ 0.082182	+ 0.035647

A	275° 22' 20".20	D	— 23° 21' 36".32
α	347 50 2 .40	$\log R$	9.992697
$(A - \alpha)$	— 72 27 42 .20		

The second observation was not corrected for the parallax corresponding to the unit distance.

II.

Date, Gr. M. T.	α (1900.0).	δ (1900.0).
1900 Dec. 24.602199	22 ^h 59 ^m 6 ^s .63	— 22° 44' 59".4
26.627974	23 11 20.16	— 22 58 20.2
28.619826	23 23 14.97	— 23 7 44.9
$\log \alpha'_0$ 0.182958	$\log (\tan \delta'_0)$ 9.051878 _n	
$\log \alpha''_0$ 9.595824 _n	$\log (\tan \delta''_0)$ 0.027986	

III.

$\log \kappa$ 0.805116 _n	$\log \lambda$ 0.044684 _n
ψ 65° 83	$\log \mu$ 9.129785
	z_1 0.916 (Oppolzer, Table XIIIa)

IV.

$$z_0 = 0.919233$$

V.

$\log \rho_0$ 9.956122	$\log \sigma'_0$ 9.1958	$\log \rho_1$ 9.9531
$\log \sigma_0$ 9.9202	$\log \sigma''_0$ 0.0532	$\log \rho_{\infty}$ 9.9595

With the foregoing values of ρ we obtain the following positions, corrected for parallax and aberration:

Date, Gr. M. T.	α (1900.0).	δ (1900.0).
1900 Dec. 24.597018	22 ^h 59 ^m 6 ^s .853	— 22° 44' 51".14
26.622757	23 11 20.456	— 22 58 12.16
28.614568	23 23 15.231	— 23 7 36.82

It will be seen later that another correction of the observations will not be necessary, the values of ρ , derived above, being sufficiently accurate for the determination of the final corrections for parallax and aberration.

$\log \alpha'_0$ 0.182972	$\log (\tan \delta'_0)$ 9.051928 _n	$\log \kappa$ 0.798948 _n
$\log \alpha''_0$ 9.604040 _n	$\log (\tan \delta''_0)$ 0.028506	$\log \lambda$ 0.046199 _n
z_0 0.920307	$\log \sigma_0$ 9.920986	$\log \sigma'_0$ 9.168237

As indicated in Part I, page 9, the second approximation of z_0 , on the basis of these correct positions, might have been avoided by applying the parallax correction to the middle place only and then proceeding at once, by VI, to the computa-

tion of the heliocentric coördinates and velocities of the middle place, and to the determination of the residuals of the first and third places. In determining these residuals the observations should be corrected by means of the values of ρ , and $\rho_{...}$ obtained by VI.

VI.

ξ_0	η_0	z_0
+ 0.814936	— 0.175671	— 0.353352
x_0	y_0	z_0
+ 0.730414	+ 0.723086	+ 0.036544
ξ'_0	η'_0	z'_0
+ 0.411712	+ 1.210885	— 0.156393
x'_0	y'_0	z'_0
— 0.601340	+ 1.128703	— 0.192040
$\log r_0$ 0.012181	$\log r'_0$ 9.555908	
$\log \theta_{...}$ 8.542164 _n	$\log f_i$ 9.999755	$\log g_i$ 8.54208 _n
$\log \theta_i$ 8.534829	$\log f_{...}$ 9.999768	$\log g_{...}$ 8.53475
ξ_i + 0.800124	η_i — 0.217690	z_i — 0.347675
$\xi_{...}$ — 0.828596	$\eta_{...}$ — 0.134022	$z_{...}$ — 0.358489
$(\alpha_i)_0$ 344° 46' 47".77	$(\delta_i)_0$ — 22° 44' 51".27	$(\log \rho_i)_0$ 9.9538
$(\alpha_{...})_0$ 350° 48' 43".97	$(\delta_{...})_0$ — 23° 7' 37".59	$(\log \rho_{...})_0$ 9.9603

A comparison of these values with those derived in V shows that in the present case further correction of the observations for parallax and aberration is unnecessary.

O — C	I.	III.
$\Delta\alpha$	— 4".97	+ 4".50
$\Delta\delta$	+ 0.13	+ 0.77

VII.

$$\log \cos \beta = 9.6893$$

	I.	III.		I.	III.
$\log f_i$	9.9542	9.9543	$\log A$	8.7428	8.7133 _n
$\log f_j$	9.2862 _n	9.2863 _n	$\log B$	8.5516	8.5667
$\log f_k$	9.5910 _n	9.5911 _n	$\log C$	8.5883 _n	8.5744
$\alpha_{...} - \alpha_i$	6° 1' 56".20		$\log \partial\alpha$	0.6616 _n	0.6168
$\delta_{...} - \delta_i$	— 0 22 46.32		$\log \partial\delta$	9.1139	9.8865

	$\log \partial \rho_0 \sin 1'' = 7.9198$	
$\log \partial x'_0 \sin 1''$	$\log \partial y'_0 \sin 1''$	$\log \partial z'_0 \sin 1''$
7.8388	8.0421	9.4286 _n

Corrected Geocentric Co-ordinates.

$\log \xi_0$	9.907071	$\log \eta_0$	9.341794 _n	$\log \zeta_0$	9.545090 _n
$\log \xi_{\infty}$	9.922371	$\log \eta_{\infty}$	9.131145 _n	$\log \zeta_{\infty}$	9.558498 _n
$(\alpha_0)_c$	344° 46' 42".65	$(\delta_0)_c$	- 22° 44' 51".36	$(\log \rho_0)_c$	9.958
$(\alpha_{\infty})_c$	350° 48' 48".42	$(\delta_{\infty})_c$	- 23° 7' 36".86	$(\log \rho_{\infty})_c$	9.964

O — C	I.	III.
$\cos \delta \Delta \alpha$	+ 0".1	0".0
$\Delta \delta$	+ 0".2	0".0

VIII.

$\log x$	$\log y$	$\log z$
9.867999	9.858220	8.522433
$\log x'$	$\log y'$	$\log z'$
9.774107 _n	0.056801	9.289419 _n

Elements.

M	3° 22' 34"
$\log a$	0.60780
φ	50° 12' 51"
μ	434".81

	Equator 1900.0.	Equator 1901.0.	Ecliptic 1901.0.
t	9° 42' 27"	9° 42' 43"	30° 0' 11"
Ω	235 13 38	235 15 31	196 5 39
ω	131 12 37	131 11 29	172 1 58

Constants to the Equator.

	1900.0.		1901.0.
$x = r$	$[9.99579] \sin (96^\circ 2' 58'' + \tau)$	$x = r$	$[9.99579] \sin (96^\circ 3' 42'' + \tau)$
$y = r$	$[9.99798] \sin (6^\circ 49' 25'' + \tau)$	$y = r$	$[9.99798] \sin (6^\circ 50' 11'' + \tau)$
$z = r$	$[9.22690] \sin (131^\circ 12' 37'' + \tau)$	$z = r$	$[9.22710] \sin (131^\circ 11' 29'' + \tau)$

The positions computed for the three dates of observation by means of the constants to the equator agree with those computed in VII, so that the calculation of the elements is fully checked.

The method as applied to this case seems to have accomplished all that can be desired—an exact representation of the given observations with less expendi-

ture of time than would have been required by using the ordinary methods of computing preliminary parabolic elements. As judged from published results based on longer arcs, the foregoing elements bring out the characteristic features of the orbit better than would a parabolic orbit computed from the same observations.

By comparison with an observation taken at the Lick Observatory, 1901, February 15th, about seven weeks after the mean epoch of the given observations, we find the following corrections to the ephemeris:

$$O - C \qquad \cos \delta \Delta\alpha = + 7''.3 \qquad \Delta\delta = - 55''$$

As the motion in the interval was about four hours in right ascension and eight degrees in declination, the method appears to be very satisfactory for ephemeris purposes.

TABLES FOR THE REDUCTION OF PHOTOGRAPHIC MEASURES

By BURT L. NEWKIRK,
BERKELEY ASTRONOMICAL DEPARTMENT.

Not Indexed

TABLES FOR THE REDUCTION OF PHOTOGRAPHIC MEASURES.*

By BURT L. NEWKIRK.

1905

The main obstacle to a wider application of the photographic method in the determination of celestial positions is the labor involved in the measurement and reduction of the star plates. Especially is this true in the case of plates made with the modern photographic telescopes which cover a field five degrees or more in radius. Special tables[†] have been prepared to facilitate the reduction of the measures made for the astrographic catalogue, and the Paris Observatory has recently issued a set of general tables[‡] for use in the reduction of the *Eros* plates. These are applicable in the reduction of plates covering a field of two degrees square within $55^{\circ} 20'$ of the equator. Professor HAROLD JACOBY[§] has just published tables for the reduction of plates covering the same area of the sky up to 75° of declination.

Before the appearance of Professor JACOBY's tables, work had been begun at this Observatory upon a set of tables designed for the same purpose, but of wider application and based on different formulas. They apply to the reduction of measures of stars whose right ascensions differ from that of the center of the plate by ten degrees or less, and whose declinations differ from that of the center of the plate by five degrees or less, and hold good for any declination. These tables are intended primarily for use in the reduction of measures of plates made with the two portrait lenses which form part of the equipment of the Students' Observatory, but astrophotographic lenses covering a field from five to ten degrees in

*These tables, with text, were completed and prepared for publication in May, 1905. The section on refraction was rewritten in June, 1906.—BURT L. NEWKIRK, Lick Observatory, March 23, 1907.

†*Bull. du Com. Int. Perm. pour l'exécution photographique de la Carte du Ciel*. Tome II, p. 303, Gauthier-Villars.

‡*Conférence Astrographique Internationale de Juillet*, 1900. Circ. No. 10.

§*Contributions from the Observatory of Columbia College*, No. 23.

diameter are rapidly becoming popular and it is hoped that the tables will find a wider sphere of usefulness than is afforded by the needs of this Observatory. (See general notes, p. 63.)

The process of reducing measured rectangular coördinates to intervals of right ascension and declination involves a preliminary determination of the constants of the plate by a comparison of the measured coördinates of catalogue stars found on the plate, with computed coördinates found with the help of the catalogue positions. The plate constants are used to correct the measured coördinates of stars whose positions are unknown, giving the so-called "standard coördinates"* which are to be reduced to intervals of right ascension and declination. Corrections for refraction must be applied either to the rectangular coördinates or to the intervals of right ascension and declination. Aside from the solution of equations of condition for the determination of plate constants and the correction of measured coördinates, the whole work of derivation of celestial positions from measured rectangular coördinates and known star places consists in the transformation of intervals of right ascension and declination into rectangular coördinates, and its converse, and the introduction of corrections for refraction. It is these transformations and the introduction of the correction for refraction that the present tables as well as those prepared by Professor JACOBY and by the Paris Observatory are designed to facilitate.

Of the various plans for the reduction of photographic measures, three have acquired especial prominence, namely: those of Professor TURNER, the late PROSPER HENRY, and Professor JACOBY. Professor JACOBY uses expressions for the rectangular coördinates in the form of series of ascending powers of the intervals of right ascension and declination, involving coefficients which depend only upon the coördinates of the center of the plate, and are, therefore, constant for any one plate. Analogous series express the intervals of right ascension and declination in terms of the measured coördinates. It is, therefore, necessary to compute four sets of coefficients for each plate, with the help of which the transformations are to be accomplished. A good deal can be said in favor of this method in the case of a plate covering only four square degrees of the sky within seventy-five degrees of the equator, especially if many stars are to be measured on a single plate. It was developed with a view to the reduction of the measures made for the astrographic catalogue, and to these cases it is entirely applicable, provided the region photographed is not too near the pole. The method is inadequate, however, in the case of plates made with the modern wide angle photographic lenses covering one hundred or more square degrees of the sky† unless the region photographed is near the celestial equator, and even in the case of plates covering a smaller area, the method fails if the region photographed is

* *Monthly Notices*, Vol. LIV, p. 13, par. 7.

† See *Monthly Notices*, Vol. LXIV, p. 608.

near the pole. The method loses its especial advantage and is cumbersome in cases where the positions of only one or two of the objects on a plate are required. A further objection to a wider application of Professor JACOBY's method is the lack of provision for any but the linear terms of the correction for refraction. It is necessary to take account of the terms of a higher order in the *Eros* work, even on plates covering a field of four square degrees, and they become appreciable where the standard of accuracy is less severe if the field is larger.

The formulas of M. HENRY have been thrown into a form convenient for the construction of general tables by M. LOEWY, and made the basis of his tables for the facilitation of the reduction of the *Eros* observations. Certain approximations have been introduced which would not be permissible if the tables were to be extended so as to suffice for the reduction of plates covering a field five degrees in radius.*

*M. HENRY's rigorous formulas are: (*Publ. Institut de France, Academie des Sciences, Bulletin du Comite international permanent pour l'exécution photographique de la carte du ciel*, Tome II, pp. 315, 318),

$$\begin{aligned} \tan x &= \xi \cos y; & \tan y &= \eta; \\ \sin x \sec \delta &= \sin (\alpha - \alpha_0); & \sin \delta &= \sin (\delta_0 + y) \cos x. \end{aligned}$$

x and y are two auxiliary quantities defined by M. HENRY on page 306 of the publication above cited. Otherwise, the notation conforms to that adopted in this paper. (Cf. page 52.) I have left out of these equations the scale value factors, and the quantities are all expressed in units of radius.

The formulas used by M. LOEWY in the construction of his tables are (*Conférence astrographique internationale de juillet*, No. 10, Gauthier-Villars):

$$\begin{aligned} x &= \xi - (\tan \xi - \xi + 2 \xi \sin^2 \frac{\eta}{2}) = \xi - d\xi \\ y &= \eta - (\tan \eta - \eta) = \eta - d\eta \\ \sin x &= \sin (\alpha - \alpha_0) \cos \delta \\ \sin \delta &= \sin (\delta_0 + y) \cos x \\ (\alpha - \alpha_0) \cos \delta &= x + t_a \\ \delta_0 + y &= \delta + t_d, \end{aligned}$$

where

$$\begin{aligned} t_a &= (\alpha - \alpha_0 - x \sec \delta) \cos \delta \\ t_d &= \frac{x \tan x \sin (\delta_0 + y - t_d)}{2 \cos (\delta_0 + y - \frac{t_d}{2})} \end{aligned}$$

$y = \eta - (\tan y - y)$ being the rigorous form, the neglected terms in the second equation are immediately seen to be:

$$\Delta d\eta = \tan \eta - \eta - (\tan y - y) = \frac{\eta^3 - y^3}{3} + 2 \frac{\eta^5 - y^5}{3 \cdot 5} + \text{etc.}$$

The neglected terms of the first equation are also found by subtracting the given value of $d\xi$ from the rigorous one. The rigorous relation being

$$x = \xi - [(\tan x - x) \sec y + x (\sec y - 1)],$$

we have

$$\Delta d\xi = (\tan x - x) \sec y + x (\sec y - 1) - [\tan \xi - \xi + \xi (1 - \cos \eta)].$$

Substituting

$$\sec y = 1 + \frac{y^2}{2} + \frac{5y^4}{2^3 \cdot 3} + \text{etc.},$$

$$\tan x - x = \frac{x^3}{3} + \frac{2x^5}{3 \cdot 5} + \text{etc.},$$

and corresponding values of $\cos \eta$ and $\tan \xi - \xi$, and neglecting terms higher than the 5th order,

$$\Delta d\xi = \frac{x^3 - \xi^3}{3} + \frac{(x y^2 - \xi \eta^2)}{2} + \frac{x^3 y^2}{2 \cdot 3} + \frac{5 x y^4 + \xi \eta^4}{24} + 2 \frac{x^5 - \xi^5}{3 \cdot 5} + \text{etc.}$$

When $\xi = \eta$ is equivalent to $+5^\circ$ the values of these terms are

$$\begin{aligned} \Delta d\eta &= 0''.3 \\ \Delta d\xi &= 1''.0 \end{aligned}$$

The introduction of a sufficient number of the neglected terms to permit of the use of these equations in the construction of more extended tables would considerably increase the numerical work involved in their application. Another disadvantage of these formulas as compared with those of Professor TURNER is that one of the two auxiliary quantities, requiring the construction of a table of double entry, can be dispensed with without increasing the extent or materially augmenting the difficulty of manipulation of the other three tables.

Professor TURNER's plan, making use of simple rigorous relations, is free from the objection to which Professor JACOBY's are open when applied to the reduction of plates of large area near the pole (for which, I repeat, they were not intended), and possess an advantage over M. LOEWY's adaptation of HENRY's plan, in the simplicity of the formulas involved and in the saving of one auxiliary quantity as stated above. I have, therefore, adopted Professor TURNER's formulas as the basis of these tables.

One of the essential points of difference between the plan for the reduction of photographic measures proposed by Professor TURNER and those proposed by HENRY, JACOBY, and others is in the number of the plate constants. TURNER recommends the use of six constants, whereas the others use only four. If only four constants are employed, it is necessary to compute a part of the linear terms of the refraction correction, which is unnecessary if the six-constant method be employed. In the tables designed to facilitate the computation of the refraction, I have included terms of higher orders and those terms of the first order which must be included if the four-constant solution is used. Any one computing the refraction corrections with the help of these tables, is, therefore, not committed to the use of the six-constant method.

Professor TURNER's fundamental relations are*:

$$\begin{aligned} (1) \quad & \eta = \tan (d - \delta_0) \\ (2) \quad & \tan d = \tan \delta \sec (\alpha - \alpha_0) \\ (3) \quad & \xi \sec d = \tan (\alpha - \alpha_0) \sec (d - \delta_0) \end{aligned}$$

where

α, δ = coördinates of a star,

ξ, η = rectangular coördinates of image of above, the unit of measurement being the focal length of the telescope,

α_0, δ_0 = coördinates of center of plate, *i. e.*, of the origin of rectangular coördinates,

d = the declination of the point of intersection of the hour circle passing through the center of the plate with a great circle perpendicular to it passing through the star.

The quantity $d - \delta_0$ is M. HENRY's auxiliary quantity y . The introduction of M. HENRY's second auxiliary quantity, x , is avoided by the choice of different trigonometric relations between the parts of the spherical triangle involved.

* *Monthly Notices*, Vol. LIV, p. 13.

To facilitate the application of these formulas, I have prepared three tables. If the tabulated quantities be represented by A , B , and C , the following relations hold (α_0 , δ_0 are supposed given in both cases):

Given α and δ ,	Given ξ and η ,
$d - \delta = B,$	$d - \delta_0 = k \eta - A,$
arg. $\left\{ \frac{(\alpha - \alpha_0)}{\delta} \right\}$	arg. $k \eta$
$k \eta = d - \delta_0 + A,$	$\alpha - \alpha_0 = k \xi \sec d - C,$
arg. $(d - \delta_0)$	arg. $\left\{ \frac{k \xi \sec d}{(d - \delta_0)} \right\}$
$k \xi \sec d = \alpha - \alpha_0 + C,$	$d - \delta = B,$
arg. $\left\{ \frac{(\alpha - \alpha_0)}{(d - \delta_0)} \right\}$	arg. $\left\{ \frac{(\alpha - \alpha_0)}{d} \right\}$
	$\delta - \delta_0 = d - \delta_0 - B$

in which all angles are expressed in degrees, ξ and η are expressed in units of the scale of the measuring apparatus, and k is a factor equal to the number of degrees in a radian divided by the focal length of the lens with which the plate was made, expressed in units of the scale of the measuring apparatus. $k \xi$ and $k \eta$ are, therefore, the measured coördinates, expressed in units of length equivalent to one degree of arc at the center of the plate. In the case of our 5-inch lens, the focal length is such that one unit of the scale of the measuring apparatus is equivalent to 0°.1. The factor k is, therefore, 0.1.

The three tables are based on TURNER'S fundamental formulas, numbers (1), (2), and (3), respectively. I insert here some special notes concerning each table with the modified forms of the fundamental relations which seemed best adapted to the computation of the tabulated quantities.

CONSTRUCTION OF TABLES. ARGUMENTS.

TABLE I.

If η be expressed in units of the scale of the measuring apparatus and k be taken as defined in the preceding paragraph, then:

$$k \eta = R \tan (d - \delta_0) = (d - \delta_0)^\circ + A,$$

where

$$R = 57^\circ.29578,$$

$$A = (d - \delta_0)^\circ \left\{ \frac{(d - \delta_0)^\circ}{3} + \frac{6}{5} \left[\frac{(d - \delta_0)^\circ}{3} \right]^2 + \text{etc.} \right\}$$

The quantity A is tabulated for each $0^\circ.1$ of the argument $d - \delta_0$ up to 3° and for every $0^\circ.01$ from 3° to 5° of the argument. The table may also be entered with the argument $k \eta = d - \delta_0 + A$. For values of the argument up to one degree, the quantity A is given in units of $0^\circ.000001 = 0''.0036$, but for larger values of the argument, the unit is ten times as large.

TABLE II.

Equation (2) yields the relation:

$$\tan (d - \delta) = \frac{\tan \delta}{1 + \tan^2 \delta \sec (\alpha - \alpha_0) [\sec (\alpha - \alpha_0) - 1]}.$$

Table II gives $(d - \delta)^\circ = B$ with the arguments $(\alpha - \alpha_0)^\circ$ and δ , when $0 < \delta < 45^\circ$ and $0 < (\alpha - \alpha_0) < 10^\circ$. It is evidently unnecessary to tabulate for negative values of δ or $(\alpha - \alpha_0)$. B is independent of the sign of $(\alpha - \alpha_0)$ and changes sign with δ . It is convenient to remember that d is always numerically greater than δ . The table may also be entered with the arguments $(\alpha - \alpha_0)^\circ$ and d , it being remembered that the value d , corresponding to any tabulated value B , is $\delta + B$, where δ is the argument corresponding to the tabulated quantity B . See example* p. 67.

This table suffices also for the cases in which

$$\begin{aligned} \delta &> 45^\circ \\ d &> 45^\circ. \end{aligned}$$

In equation (2), let

$$\begin{aligned} d &= 90^\circ - q \\ \delta &= 90^\circ - p, \end{aligned}$$

whence

$$p - q = d - \delta.$$

Formula (2) then becomes:

$$\tan p = \tan q \sec (\alpha - \alpha_0),$$

which is of the same form as (2).

$(\alpha - \alpha_0)$ and $d > 45^\circ$ being given, enter Table II with the arguments $(\alpha - \alpha_0)$ and $q = (90^\circ - d)$ and obtain $p - q = d - \delta$. It is to be noted that q would have to be found in the column of arguments at the left of the page. The addition of

*For example the quantity $B = 0.01608$ corresponds to the arguments $\alpha - \alpha_0 = 2^\circ.8$ and $d = 14. + .01608$. See table.

a column of arguments at the right of each page permits the use of d directly as an argument.

When δ and $(\alpha - \alpha_0)$ are given, $\delta > 45^\circ$, the arguments on the right are to be employed, it being remembered that the value δ , corresponding to any tabulated value B , is $d - B$, where d is the argument corresponding to the tabulated quantity B . See example p. 67.

Tabulated quantities are expressed in units of $0^\circ.000001 = 0''.0036$ for $0 < (\alpha - \alpha_0) < 2^\circ$. For larger values of $(\alpha - \alpha_0)$, the unit is ten times as large.

The following working directions for the use of Table II summarize the above:

- GIVEN $(\alpha - \alpha_0)$, δ :
1. $\delta < 45^\circ$ Argument δ at left of page.
 2. $\delta > 45^\circ$ Argument $(d - B) = \delta$, at right of page.
- GIVEN $(\alpha - \alpha_0)$, d :
3. $d < 45^\circ$ Argument $(\delta + B) = d$, at left of page.
 4. $d > 45^\circ$ Argument d at right of page.

See examples pp. 67-71.

TABLE III.

If in equation (3) we develop $\tan(\alpha - \alpha_0)$ and $\sec(d - \delta_0)$ into power series in $(\alpha - \alpha_0)$ and $(d - \delta_0)$ respectively, and perform the multiplication, neglecting terms higher than the seventh order in $(\alpha - \alpha_0)$ and those higher than the fifth order in $(d - \delta_0)$ we have:

$$k \xi \sec d = (\alpha - \alpha_0)^\circ + C,$$

where

$$C = (\alpha - \alpha_0)^\circ [L + M + N]$$

$$L = \frac{(\alpha - \alpha_0)^2}{3} + \frac{(d - \delta_0)^2}{2}$$

$$M = \frac{5}{6} \left[\frac{(d - \delta_0)^2}{2} \right]^2 + \frac{6}{5} \left[\frac{(\alpha - \alpha_0)^2}{3} \right]^2 + \frac{(\alpha - \alpha_0)^2}{3} \cdot \frac{(d - \delta_0)^2}{2}$$

$$N = \frac{17}{14} \cdot \frac{6}{5} \left[\frac{(\alpha - \alpha_0)^2}{3} \right]^3 \frac{(\alpha - \alpha_0)^2}{3} + \frac{6}{5} \left[\frac{(\alpha - \alpha_0)^2}{3} \right]^2 \frac{(d - \delta_0)^2}{2} + \frac{(\alpha - \alpha_0)^2}{3} \cdot \frac{5}{6} \left[\frac{(d - \delta_0)^2}{2} \right]^2$$

N amounts to less than $0''.1$ for $(\alpha - \alpha_0) = 10^\circ$, $(d - \delta_0) = 5^\circ$.

C is tabulated with the arguments $(\alpha - \alpha_0)$ and $(d - \delta_0)$ for

$$0 < (\alpha - \alpha_0) < 10^\circ$$

$$0 < (d - \delta_0) < 5^\circ$$

C changes sign with $(\alpha - \alpha_0)$ and is independent of the sign of $(d - \delta_0)$.

$k \xi \sec d$ is always numerically greater than $(\alpha - \alpha_0)$.

This table may be entered with the argument $k \xi \sec d$, it being remembered that the argument $(k \xi \sec d)$, corresponding to any tabulated value C , is $(\alpha - \alpha_0) + C$, where $(\alpha - \alpha_0)$ is the argument corresponding to the tabulated quantity C .* The quantity C is expressed in units of $0^\circ.000001 = 0''.0036$ for

$$0 < (\alpha - \alpha_0) < 2^\circ$$

$$0 < (d - \delta_0) < 1^\circ$$

For larger values of the arguments, the unit is ten times as large.

*The quantity $C = 0.05928$ corresponds to the arguments $k \xi \sec \delta = 7.9 + 0.05928$ and $d - \delta_0 = 2.7$. See example p. 68.

REFRACTION.

The correction of the measured coördinates for differential refraction has been the subject of considerable discussion. The larger part of the correction is a linear function of the measured coördinates, and is, therefore, provided for in Professor TURNER'S six-constant method of solution. Some authorities prefer the four-constant solution, with the necessity for computing some of the linear terms of the refraction correction. That part of the refraction which depends on higher powers of the measured coördinates* becomes quite appreciable in the case of plates made with the modern wide angle telescopes, where the field of good definition contains a hundred square degrees or more.

I have taken the formulas produced by Professor TURNER† as the basis of Tables IV–XI. They are:

$$(1) \quad \begin{aligned} \Delta x &= (X - x) t \\ \Delta y &= (Y - y) t \\ t &= \mu \frac{1 + x^2 + y^2}{1 + xX + yY}, \end{aligned}$$

where

$$\begin{aligned} \left. \begin{array}{l} X, Y \\ x, y \end{array} \right\} &= \text{Coördinates of zenith and star projected on "standard" plate.} \\ \left. \begin{array}{l} \Delta x \\ \Delta y \end{array} \right\} &= \text{Projection on plate of total displacement of star due to refraction.} \\ \mu &= \text{Constant of refraction.} \end{aligned}$$

It is necessary to insert some remarks regarding the details of the application of these relations before the construction of the tables is taken up. This is all the more desirable since certain difference of opinion‡ seems to prevail regarding this matter. The situation can perhaps be most easily made clear by outlining the following rigorously correct process.

Suppose the center of the plate (that is, the point on the plate where a radius of the projected sphere meets it perpendicularly) has been ascertained, plotted on a star chart by means of the neighboring stars on the plate, and its right ascension and declination (α_0, δ_0) read off accurately from the chart. Let $\Delta \alpha_0, \Delta \delta_0$ represent the components of the displacement of the point α_0, δ_0 due to refraction. The spherical coördinates of the "center of the plate" are then:†

$$\begin{aligned} \alpha_0 + \Delta \alpha_0 \\ \delta_0 + \Delta \delta_0 \end{aligned}$$

and these values must be used in computing X and Y according to the formulas given on p. 60.

*See WILSON. *Popular Astronomy*, Vol. XII, p. 159.

† *Monthly Notices*, Vol. LVII, p. 136.

‡Dissertation of WALTER ZURHELLEN. Bonn 1904, p. 33–48.

‡Cf. ÖSTEN BERGSTRAND. *Untersuchungen über das Doppelstern system 61 Cygni*. Upsala 1905.

The refraction corrections must first be computed for the catalogue stars so as to introduce the effect of refraction into the standard coördinates. For this purpose we should employ for x and y in the above formulas, the true standard coördinates on a plate tangent at the point $\alpha_0 + \Delta \alpha_0, \delta_0 + \Delta \delta_0$, and the appropriate refraction constant: $\mu = \alpha' \beta'^{A'} \gamma'^{\lambda'}$. (CHAUVENET, *Sph. Astr.*, Vol II, Table II, Col. B.) The corrections are the total displacements due to refraction. Applying these corrections we should have what may be called *the apparent standard coördinates on a plate tangent to the sphere at the point $\alpha_0 + \Delta \alpha_0, \delta_0 + \Delta \delta_0$* . The constants of the plate can then be determined in the ordinary manner by comparison of these computed with the measured coördinates.

The constants of the plate having been determined in this way and the apparent standard coördinates of the unknown stars having been derived with their help, the refraction corrections would be computed for these stars. This would be accomplished by means of TURNER'S formulas given above, using the values of X and Y previously employed, the "apparent standard coördinates" x and y and the appropriate refraction constant: $\mu = \alpha \beta^A \gamma^\lambda$ (CHAUVENET, Table II, Col. A). The true standard coördinates on a plate tangent at $\alpha_0 + \Delta \alpha_0, \delta_0 + \Delta \delta_0$ would be obtained by applying these corrections to the apparent standard coördinates.

It is to be noticed that the quantities x, y , defined above as "the coördinates of the star projected on the 'standard' plate," may designate either the true or the apparent coördinates, depending on which value of the constant μ is employed.

It is to be noted further that the constant of refraction μ is in every case the one corresponding to the mean or true zenith distance of the particular star for which the correction is being computed.

This process may be shortened by certain modifications, involving the omission of some terms of the refraction correction, and the use of tables. These modifications and approximations are indicated in the following paragraphs.

Since an important simplification is obtained by omitting certain large terms of the refraction correction which may be included in the plate constants, it is necessary to consider in this connection the process of determining these constants. The tables have been so constructed as to be available for the computation of the refraction corrections when the four-constant method is employed, and consequently are equally available for the six-constant method, and for cases in which further unknowns are introduced to provide for other corrections.

Let

x, y = True rectangular (standard) coördinates of a star on a plate tangent to the celestial sphere at the point $\alpha_0 + \Delta \alpha_0, \delta_0 + \Delta \delta_0$.

x', y' = Measured coördinates.

$\mu = \mu_0 + \Delta \mu = \alpha' \beta'^{A'} \gamma'^{\lambda'}$ = The constant of refraction for the star whose measured coördinates are x', y' .

$\mu_0 = \alpha' \beta'^{A'} \gamma'^{\lambda'}$ for point α_0, δ_0 .

$= \alpha \beta^A \gamma^\lambda$ for point $\alpha + \Delta \alpha_0, \delta + \Delta \delta_0$.

The four-constant formulas for the determination and correction of errors of orientation, scale value and assumed center are:

$$(2) \quad \begin{aligned} ax' + by' + c &= x - x' + \Delta x \\ -bx' + ay' + d &= y - y' + \Delta y, \end{aligned}$$

where

$$(3) \quad \begin{aligned} \Delta x &= \mu (X - x) \frac{1 + x^2 + y^2}{1 + Xx + Yy} \\ &= \mu (X - x - xX^2 - yXY + \delta x), \\ \Delta y &= \mu (Y - y) \frac{1 + x^2 + y^2}{1 + xX + yY} \\ &= \mu (Y - y - yY^2 - xXY + \delta y), \end{aligned}$$

the expressions in series of powers of x and y being obtained by developing the fraction $\left(\frac{1 + x^2 + y^2}{1 + Xx + Yy}\right)$ in powers of $(Xx + Yy)$ and multiplying out. δx and δy represent terms involving second and higher powers of x and y .

Adding $\mu_0 x'$ to both sides of the first of equations (2) and $\mu_0 y'$ to both sides of the second, substituting values of Δx and Δy from (3), putting $\mu = \mu_0 + \Delta\mu$, and substituting for $x - x'$ and $y - y'$ in the coefficients of μ , the values

$$(4) \quad \begin{aligned} x - x' &= ax' + by' + c - \Delta x \\ y - y' &= -bx' + ay' + d - \Delta y, \end{aligned}$$

we have

$$(5) \quad \begin{aligned} a'x' + b'y' + c' &= (x + r_x) - x' \\ -b'x' + a'y' + d' &= (y + r_y) - y', \end{aligned}$$

where the primed letters indicate new constants* and

$$(6) \quad \begin{aligned} r_x &= \mu (-xX^2 - yXY + \delta x) + \Delta\mu (X - x) + \mu_0 (\Delta x - \mu_0 X) \\ r_y &= \mu (-yY^2 - xXY + \delta y) + \Delta\mu (Y - y) + \mu_0 (\Delta y - \mu_0 Y). \end{aligned}$$

The values of the expressions:

$$(7) \quad \begin{aligned} &-xX^2 - yXY + \delta x \\ &-yY^2 - xXY + \delta y \end{aligned}$$

are obtained with the help of the tables. The last term of each of equations (6) is negligible, even in extreme cases.

Having computed the right-hand members of equations (5) and solved for the plate constants, we determine $x + r_x$ and $y + r_y$ for each of the unknown stars by means of the equations (5):

$$\begin{aligned} x + r_x &= x' + a'x' + b'y' + c' \\ y + r_y &= y' + b'x' + a'y' + d', \end{aligned}$$

and it remains to determine x and y by computing and applying the corrections r_x and r_y . This can be done most simply by means of the relations (6), using the quantities $(x + r_x)$ and $(y + r_y)$ as approximate values of the quantities x and y , and making a second approximation if necessary. If r_x and r_y are

$$*a' = a + \mu_0 a; \quad b' = b + \mu_0 b; \quad c' = c + \mu_0 (X + c) - \mu_0^2 \Delta x; \quad d' = d + \mu_0 (Y + d) - \mu_0^2 \Delta y.$$

computed in this way the values of μ corresponding to *true* zenith distance are to be employed.

When it becomes necessary to compute a number of values of μ for the same plate the approximate true zenith distances of points whose standard coördinates (referred to the center $\alpha_0 + \Delta\alpha_0, \delta_0 + \Delta\delta_0$) are x and y may be obtained with the help of the following relation:*

$$z = z_0 - \frac{Xx + Yy}{\tan z_0}$$

which gives a good approximation, provided the zenith distance is large. z_0 is the true zenith distance of the point $\alpha_0 + \Delta\alpha_0, \delta_0 + \Delta\delta_0$.

The process just outlined differs from the rigorous one in the transfer of the large terms $\mu_0 (X - x), \mu_0 (Y - y)$ from the refraction correction to the plate constants, which reduces the size of the computed corrections, and renders possible the further modification involving the use of the constant μ (eqs. 6) obtained with the argument true zenith distance in computing the corrections for the unknown, as well as for the fiducial stars. This principle may be carried further if a six-constant plate solution is employed, all of the linear terms of the refraction being omitted. The non-linear terms are given by the expressions

$$\begin{aligned} r_x + \mu_0 X (Xx + Yy) \\ r_y + \mu_0 Y (Xx + Yy). \end{aligned}$$

If the refractions are taken in this form, a second approximation will seldom if ever be necessary in computing the refractions for the unknown stars.

To put the relations (1) in a form suitable to the construction of tables I have developed the right hand members in a series of powers of the expression $(Xx + Yy)$.

$$\begin{aligned} \Delta x &= \mu [(X - x) + (X - x) \{D + (x^2 + y^2) (1 + D)\}] \\ \Delta y &= \mu [(Y - y) + (Y - y) \{D + (x^2 + y^2) (1 + D)\}], \end{aligned}$$

where

$$\begin{aligned} D &= -(Xx + Yy) + (Xx + Yy)^2 - (Xx + Yy)^3 + \text{etc.} \\ &= \frac{1}{1 + Xx + Yy} - 1. \end{aligned}$$

Subtracting the terms $\mu_0(X - x)$ and $\mu_0(Y - y)$, which, save for the negligible terms $\mu_0(\Delta x - \mu_0 X)$ and $\mu_0(\Delta y - \mu_0 Y)$, are included in the four plate constants, we have:

$$\begin{aligned} r_x &= \mu (X - x) [D + (x^2 + y^2) (1 + D)] + \Delta \mu (X - x) \\ r_y &= \mu (Y - y) [D + (x^2 + y^2) (1 + D)] + \Delta \mu (Y - y). \\ \mu &= \frac{1.0154}{3600} \alpha' \beta' \gamma' \quad \text{or} \quad \mu = \frac{1.0154}{3600} \alpha \beta \gamma. \quad (\text{Expressed in degrees}) \\ \log \frac{1.0154}{3600} &= 6.4503. \end{aligned}$$

1.0154 is the ratio of the photographic to the visual constant.† The focal length of

*See note p. 61.

†Bull. du Com. Int. Perm. T. I., p. 465.

the telescope is the unit for X, Y, x, y . The quantities r_x and r_y as obtained from the above formulas are expressed in decimals of a degree and are to be added algebraically to the computed standard coördinates in order to *introduce* the effect of refraction. The effect of refraction is eliminated from the measured coördinates by applying the above corrections with the opposite sign.

Tables IV and V give values of X and Y , respectively, as functions of the hour angle and declination of the center of the plate for

$$-45^\circ < (s - \alpha) < +45^\circ$$

$$-40^\circ < \delta < +90^\circ.$$

They are constructed for the latitude of Berkeley ($37^\circ 52'.4$) by means of the formulas*

$$\tan \Delta = \tan \varphi \sec (s - \alpha)$$

$$Y = \tan (\Delta - \delta)$$

$$X = \frac{\tan (s - \alpha)}{\cos (\Delta - \delta)} \cos \Delta,$$

where

$$\varphi = \text{Latitude of the Students' Observatory} = 37^\circ 52'.4,$$

$$s = \text{Sidereal time of exposure,}$$

$$\alpha, \delta = \text{Coördinates of center of plate.}$$

It is stated above that $\alpha_0 + \Delta \alpha_0$ and $\delta_0 + \Delta \delta_0$ are to be used as arguments in taking out values of X and Y . μ_0 is to be computed with the argument true zenith distance of the point α_0, δ_0 . X, Y and μ_0 may be conveniently computed as follows: Enter tables for X and Y with arguments α_0, δ_0 . Obtain Z from Table XI. Compute μ_0 .

$$\Delta \alpha_0 = \mu_0 X \sec \delta$$

$$\Delta \delta_0 = \mu_0 Y$$

Enter tables again for final values of X and Y with arguments $\alpha_0 + \Delta \alpha_0, \delta_0 + \Delta \delta_0$. In very extreme cases (see example p. 72) the zenith distance of the center of the plate and the coördinates X and Y must be computed by means of the rigorous formulas.

Tables VI and VII are to facilitate the computation of X and Y when the center of the plate is more than three hours from the meridian. Δ is independent of the sign of $(s - \alpha)$, always $\geq \varphi$, and $> 90^\circ$ when $(s - \alpha)$ is numerically greater than 90° . Y will therefore always be positive when $\delta < \varphi$ or $(s - \alpha)$ numerically $> 90^\circ$. $X \cos (\Delta - \delta)$ has the same sign as $(s - \alpha)$, and since $(\Delta - \delta)$ is always numerically less than 90° (Y would be ∞ and the center of the plate on the horizon for $\Delta - \delta = 90^\circ$) X has the same sign as the hour angle of the center of the plate, being positive when the center of the plate is west of the meridian.

* TURNER. *Monthly Notices*, Vol. LIV, p. 19.

ABERRATION.

TURNER'S formulas* for the introduction of the correction for differential aberration give only terms of the first power in the measured coördinates. The complete expressions may be obtained in a manner analogous to the one employed by him, *Monthly Notices*, Vol. LVII, p. 134, ff., in developing expressions for the refraction corrections. If W represent the apex of the Earth's way, S a star, and C the center of the sphere which is projected on the plate, the direction cosines, with respect to a rectangular coördinate system having its origin in C , of lines connecting C with S and W are proportional to $x, y, 1$, and $X, Y, 1$, respectively, x, y, X, Y being coördinates of the projections of the points S and W on the plate. We have then

$$\cos SCW = \frac{1 + xX + yY}{\sqrt{(1 + x^2 + y^2)(1 + X^2 + Y^2)}}$$

$$\sin SCW = \sqrt{\frac{(X-x)^2 + (Y-y)^2 + (xY - yX)^2}{(1 + x^2 + y^2)(1 + X^2 + Y^2)}}$$

The coördinates of S' which represents the position of the star S as displaced by aberration are

$$x + \Delta x = x + t(X - x)$$

$$y + \Delta y = y + t(Y - y).$$

Similarly,

$$\sin SCS' = t \sqrt{\frac{(X-x)^2 + (Y-y)^2 + (xY - yX)^2}{(1 + x^2 + y^2)[1 + (x + \Delta x)^2 + (y + \Delta y)^2]}}.$$

But,

$$\sin SCS' = \beta \sin SCW,$$

where β is the constant of aberration.

$$\text{Therefore } t = +\beta \sqrt{\frac{1 + (x + \Delta x)^2 + (y + \Delta y)^2}{1 + X^2 + Y^2}}$$

$$= \frac{+\beta}{\sqrt{1 + X^2 + Y^2}} [1 + \frac{1}{2}(x^2 + y^2) + \dots],$$

$$\text{and } \Delta x = \frac{+\beta}{\sqrt{1 + X^2 + Y^2}} [X - x + \frac{1}{2}(X - x)(x^2 + y^2) + \dots]$$

$$\Delta y = \frac{+\beta}{\sqrt{1 + X^2 + Y^2}} [Y - y + \frac{1}{2}(Y - y)(x^2 + y^2) + \dots].$$

The first term within the bracket in each case yields the displacement of the center of the plate due to aberration and should be omitted in computing differential aberration. The quantities $\frac{X}{\sqrt{1 + X^2 + Y^2}}$ and $\frac{Y}{\sqrt{1 + X^2 + Y^2}}$ can never exceed unity. The factor

$$\frac{1}{2} \beta (x^2 + y^2) = 20'' \left(\frac{5}{87}\right)^2 = 0''.15$$

when $x = y$ is equivalent to 5° . The terms of the first degree in x and y are provided for in either method of determination of the plate constants.

* *Monthly Notices*, Vol. LIV, p. 20.

Abbreviated tables for these transformations are given here. (See last page.) The multiplication of the measured coördinates by the factor k can be accomplished with the help of a Rechentafel, three places being sufficiently accurate for the preliminary scale value. A six-place number can be accurately multiplied by a three-place factor by entering the table twice with three digits of the multiplicand each time.

The number of decimals to which the tabulated quantities are given in Tables I–III makes it possible to reduce measures of stars within one degree of the center of the plate in declination and two degrees in right ascension, with an accuracy of about $0''.01$. (One unit of the last decimal place of the tabulated quantities is equivalent to $0''.0036$.) Throughout the remainder of the tables, one decimal place less is given. The error of interpolation due to second differences amounts to less than two units of the last place, except in parts of Table II. In a portion of this table, horizontal second differences are appreciable, and in another portion vertical second differences should be taken into account.* In each case, the second differences are nearly constant along a horizontal line across the page so that it is possible to use one approximate value for the whole row. One half this value has been tabulated in a column at the left of the page, headed " $\frac{h}{2}$ " or, " $\frac{v}{2}$ ". The formula to be used in the interpolation is:

$$f(a \pm nw) = f(a) \pm n[f'(a) \pm n\frac{f''(a)}{2}]$$

Where $f'(a)$ is the average of the tabulated first differences preceding and following the number from which the interpolation is made, and $\frac{f''(a)}{2}$ is the quantity given in the column at the left.

The tables of proportional parts for the interpolations are on the last page.

These tables have been computed to one decimal place more than is given, where this was necessary to insure the accuracy of all tabulated values within one unit of the last place. The checking has been done in part by duplicate computation, but mainly by comparison of differences. The methods of computation and checking are not such as to exclude in all cases errors of one unit in the last decimal place. This is evidently immaterial since, in Tables I–III one unit of the last place is equivalent to $0''.0036$, or $0''.036$, in portions of the tables where the accuracy aimed at is $0''.01$, and $0''.1$, respectively, and in the refraction tables a sufficient number of decimal places is given so that the last place of the computed refraction correction may be dropped without committing an error greater than the uncertainties of the value due to varying states of the atmosphere.

NOTE ON THE ASSUMED CENTER OF THE PLATE. In developing formulas for the reduction of measured rectangular coördinates to standard, and their transformation into intervals of right ascension and declination, it is generally assumed that the right ascension and declination of the point in the sky corresponding to the point at which the optic axis of the lens meets the plate are

*In a part of the table on page 81 both horizontal and vertical second differences are appreciable.

known, and that the plane of the plate is perpendicular to the optic axis. It is important to know how accurately these conditions must be fulfilled in the case of plates covering fields several degrees in diameter.

It is to be noted that errors introduced by the non-fulfillment of these two conditions can not be treated separately, if the following fundamental condition is fulfilled. It is supposed, namely, that the angle between the axis of two cones of light due to two stars, upon leaving the lens, is equal to the angle between the two bundles of parallel rays coming to the lens from the two stars. If this condition is fulfilled the impression on the plate is the central projection of a spherical surface, the center of projection being the optical center of the lens, the radius of the sphere being its focal length. If this is the case the optic axis is of no more importance than any other radius of the sphere except that the field of good definition is more symmetrically situated about it. If the plate is not perpendicular to the optic axis we should choose as the "center of the plate" the point where a radius of the sphere which is perpendicular to the plate meets it. The formulas for the transformation from rectangular coördinates to intervals of right ascension and declination are applicable to coördinates measured with this point as origin. It is sufficient to know the coördinates, α , δ , corresponding to the point where a radius of the sphere perpendicular to the plate meets it, this single condition replacing the two previously mentioned.

In the reduction of a set of measures this point is known only approximately and the coördinates, α , δ , of the "center of the plate" are taken somewhat arbitrarily. Professor JACOBV has developed expressions for the corrections to the measured coördinates due to error in the assumed value of the coördinates of the center of the plate. The terms of first degree in the measured coördinates are provided for along with the other linear corrections in the six-constant method of solution. The terms of second degree are:

$$\text{Corrections to the } x \text{ coördinate: } (x^2 \cos \delta) \Delta \alpha + xy \Delta \delta$$

$$\text{Corrections to the } y \text{ coördinate: } (xy \cos \delta) \Delta \alpha + y^2 \Delta \delta$$

where the coördinates of the assumed and true center of the plate are α , δ and $\alpha + \Delta \alpha$, $\delta + \Delta \delta$, respectively.

In the case of plates made with portrait lenses an error of one centimeter in locating the "center," as defined above, might easily occur unless special precautions were taken to secure a more accurate value. This would be equivalent to something like a degree of arc. To gain an idea of the error thus introduced into the computed coördinate of a catalogue star on the plate, take, as a special case, a star, declination zero, five degrees from the center of the plate along the x axis, and suppose $\cos \delta \Delta \alpha = 1^\circ$, $\Delta \delta = 0^\circ$.

Then

$$x^2 \cos \delta \Delta \alpha + xy \Delta \delta = 5 \cdot \frac{5}{57} \cdot \frac{1}{57} = 0.0076 = 27''.$$

For stars at a given distance from the center of the plate the effect of the second order terms is proportional to the error in the assumed values of the coördinates of the center.

It follows from the above that in reducing measured to standard coördinates over a field some degrees in diameter the right ascension and declination of the "center of the plate" (defined as the point of intersection of the plate with a perpendicular to it from the center of projection) must be known with considerable accuracy, or, failing in this, the relations between the measured and standard coördinates must be written:

$$\xi = ax + by + c + mx^2 + nxy$$

$$\eta = dx + ey + f + ny^2 + mxy,$$

where

$$m = \cos \delta \Delta \alpha,$$

$$n = \Delta \delta$$

The values of the coördinates of the true center of the plate are then:

$$\alpha + \Delta \alpha = \alpha + m \sec \delta$$

$$\delta + \Delta \delta = \delta + n.$$

A considerable proportion of the routine work involved in the preparation of these tables was done by computers at the expense of the Students' Observatory of the University of California. For this assistance, as well as for encouragement and valuable suggestions, I am indebted to the Director, Professor A. O. Leuschner.

SUMMARY OF DIRECTIONS FOR USE OF TABLES I, II, AND III.

- A. Given $\alpha_0, \delta_0, \alpha, \delta$. Required $k\xi, k\eta$.
 Express δ_0, δ and $(\alpha - \alpha_0)$ in decimals of a degree.
 If $\delta < 45^\circ$, with arguments, $(\alpha - \alpha_0)$ and δ , take B from Table II.
 If $\delta > 45^\circ$ enter Table II with argument $\delta = d - B$, d being found in the column at the right of the page. Then

$$d = \delta + B.$$
 With argument $(d - \delta_0)$ take A from Table I. Then

$$k\eta = (d - \delta_0) + A.$$
 With argument $(\alpha - \alpha_0)$ and $(d - \delta_0)$ take C from Table III. Then

$$k\xi \sec d = (\alpha - \alpha_0) + C.$$
- B. Given $\alpha_0, \delta_0, k\xi, k\eta$. Required $(\alpha - \alpha_0), \delta$.
 With argument $k\eta$ take A from Table I. Then

$$d - \delta_0 = k\eta - A.$$
 With arguments $(d - \delta_0)$ and $k\xi \sec d$, take C from Table III. Then

$$(\alpha - \alpha_0) = k\xi \sec d - C.$$
 If $d < 45^\circ$, with argument $(\alpha - \alpha_0)$ and $d = \delta + B$, take B from Table II.
 Then
$$\delta = d - B.$$
 If $d > 45^\circ$, take out B with argument $(\alpha - \alpha_0)$ and d in column at right.
- I insert here the details of the interpolation in the case under A when $\delta > 45^\circ$ and in the case under B when $d < 45^\circ$. Also in the use of Table III under B .

DETAILS OF INTERPOLATION.

TABLE II.

$(\alpha - \alpha_0) + 7^\circ.91210$		$(\alpha - \alpha_0) + 2^\circ.81400$	
$\delta + 78.88250$		$d + 14.15090$	
Arg. $\left\{ \begin{array}{l} (\alpha - \alpha_0) = 7^\circ.9 \\ \delta = 78.89721 \end{array} \right\}$		Arg. $\left\{ \begin{array}{l} (\alpha - \alpha_0) = 2^\circ.8 \\ d = 14.01608 \end{array} \right\}$	
$\frac{\partial B}{\partial \delta} (-0.01471) \dots \dots \dots + 13$		$\frac{\partial B}{\partial \delta} (+0.13482) \dots \dots \dots + 14$	
$\frac{\partial B}{\partial (\alpha - \alpha_0)} (+0.1210) \dots \dots \dots + 32$		$\frac{\partial B}{\partial (\alpha - \alpha_0)} (+0.140) \dots \dots \dots + 16$	
$B \dots \dots \dots 0.10324$		$B \dots \dots \dots + 0.01638$	

TABLE III.

$$\begin{array}{rcl}
 k \text{ \AA sec } d & 7.97150 & \\
 (d - \delta_0) & 2^{\circ}.68570 & \\
 \text{Arg. } \left\{ \begin{array}{l} k \text{ \AA sec } \delta = 7.95928 \\ (d - \delta_0) = 2.7 \end{array} \right\} & .05928 & \\
 \partial \frac{\partial C}{(k \text{ \AA sec } \delta)} (.1222) & & + 25 \\
 \partial \frac{\partial C}{(d - \delta_0)} (-.143) & & - 9 \\
 \hline
 C & & 0.05944
 \end{array}$$

EXAMPLES.

The following are the data of observation of

PLATE No. 46.

Exposed 1904, Nov. 11.

Sidereal time at beginning of exposure $\begin{matrix} \text{h} & \text{m} & \text{s} \\ 23 & 21 & 0 \end{matrix}$
 Sidereal time at end of exposure $\begin{matrix} & & \\ & & 0 \end{matrix} \begin{matrix} \text{h} & \text{m} & \text{s} \\ & 51 & 0 \end{matrix}$

Coördinates of center of plate $\left\{ \begin{array}{l} \alpha \quad 3^{\text{h}} \ 44^{\text{m}} \ 50^{\text{s}} \\ \delta + 10^{\circ} \ 28' \ 30'' \end{array} \right.$

Approximate scale value 1 mm = $0^{\circ}.1$.

Latitude of Berkeley = $37^{\circ} \ 52'$.

Star.			M. E. 1875.					
No.	AG No.	Mag.	α			δ		
			h	m	s	°	'	"
1	1062	7.4	3	32	19.94	+ 13	29	7.1
2	1136	7.6	3	47	35.13	+ 13	24	5.8
3	1183	8.5	3	56	5.36	+ 14	8	4.6
4	1345	8.6	3	33	2.62	+ 6	3	25.7
5	1431	8.1	3	46	29.93	+ 7	23	57.0
6	1511	8.3	4	0	9.95	+ 7	51	9.5

COMPUTATION OF $k\xi$ AND $k\eta$ FROM CATALOGUE POSITIONS.

$k = .1$

Star.	1	2	3	4	5	6
$(\alpha - \alpha)$ {	$-12^m 30^s.06$	$+ 2^m 45^s.13$	$+ 11^m 15^s.36$	$- 11^m 47^s.38$	$+ 1^m 39^s.93$	$+ 15^m 19^s.95$
	$- 3^{\circ}.1253$	$+ 0.6880$	$+ 2^{\circ}.8140$	$- 2^{\circ}.9474$	$+ 0.4164$	$+ 3^{\circ}.8331$
δ	$+ 13^{\circ}.4853$	$+ 13^{\circ}.4016$	$+ 14^{\circ}.1345$	$+ 6^{\circ}.0571$	$+ 7^{\circ}.3992$	$+ 7^{\circ}.8526$
B (Table II) $= d - \delta$	$+ 193$	$+ 9$	$+ 164$	$+ 80$	$+ 2$	$+ 174$
d	$+ 13 .5046$	$+ 13 .4025$	$+ 14 .1509$	$+ 6 .0571$	$+ 7 .3994$	$+ 7 .8700$
$d - \delta_0$	$+ 3 .0296$	$+ 2 .9275$	$+ 3 .6759$	$- 4 .4099$	$- 3 .0756$	$- 2 .6050$
A (Table I)	$+ 28$	$+ 25$	$+ 50$	$- 85$	$- 30$	$- 18$
C (Table III)	$- 74$	$+ 10$	$+ 81$	$- 114$	$+ 6$	$+ 97$
$\log k\xi \sec d$	0.49592	9.83820	0.45057	0.47111	9.62014	0.58465
$\log \cos d$	9.98782	9.98801	9.98662	9.99756	9.99637	9.99589
$\log k\xi$	0.48374	9.82621	0.43719	0.46827	9.61651	0.58054
$k\xi$	$- 3.0461$	$+ 0.6702$	$+ 2.7365$	$- 2.9422$	$+ 0.4135$	$+ 3.8066$
$k\eta$	$+ 3.0324$	$+ 2.9300$	$+ 3.6809$	$- 4.4184$	$- 3.0786$	$- 2.6068$

I have performed this transformation also by means of Professor JACOBV's formulas and obtain the same values to within two units of the last decimal place.

COMPUTATION OF THE CORRECTIONS FOR REFRACTION.

Table IV, $X = - 1.156$

Table V, $Y = + .943$

$\mu = 0.0164$

Star.	1	2	3	4	5	6
x { Table VIII	-0.0532	$+0.0117$	$+0.0478$	-0.0513	$+0.0072$	$+0.0667$
y {	$+ 528$	$+ 512$	$+ 642$	$- 771$	$- 537$	$- 455$
$x^2 + y^2$ Table IX	56	28	64	86	29	65
$\mu (X - x)$	$- 180$	$- 192$	$- 197$	$- 180$	$- 190$	$- 200$
$\mu (Y - y)$	$+ 146$	$+ 146$	$+ 144$	$- 167$	$+ 164$	$+ 162$
Xx	$+ 615$	$- 135$	$- 553$	$+ 593$	$- 83$	$- 771$
Yy	$+ 493$	$+ 478$	$+ 599$	$- 720$	$- 501$	$- 425$
Σ	$+ .1108$	$+ .0343$	$+ .0046$	$- .0127$	$- .0584$	$- .1196$
D	-0.0999	-0.0331	-0.0046	$+0.0129$	$+0.0620$	$+0.1358$
$(x^2 + y^2) (1 + D)$	$+ 5$	$+ 27$	$+ 64$	$+ 84$	$+ 28$	$+ 56$
r_x	$+0.0017$	$+0.0006$	-0.0000	-0.0004	-0.0012	-0.0028
r_y	$- 14$	$- 4$	0	$+ 4$	$+ 10$	$+ 23$

To illustrate the application of the tables in the converse transformation I have taken the values of $k\xi$ and $k\eta$ just computed and reproduced the values of $(\alpha - \alpha_0)$ and δ .

Star.	1	2	3	4	5	6
$k\xi$. . .	— 3.0461	+ 0.6702	+ 2.7365	— 2.9422	+ 0.4135	+ 3.8066
$k\eta$. . .	+ 3.0324	+ 2.9300	+ 3.6809	— 4.4184	— 3.0786	— 2.6068
A (Table I)	— 28	— 25	— 50	+ 85	+ 30	+ 18
$d - \delta_0$. .	+ 3°.0296	+ 2°.9275	+ 3°.6759	— 4°.4099	— 3°.0756	— 2°.6050
d	+ 13 .5046	+ 13 .4025	+ 14 .1509	+ 6 .0651	+ 7 .3994	+ 7 .8700
$\log \sec d$.	0.01218	0.01199	0.01338	0.00243	0.00363	0.00411
$\log k\xi \sec d$	0.49592	9.83820	0.45057	0.47111	9.62014	0.58465
$k\xi \sec d$.	— 3.1388	+ .6890	+ 2.8221	— 2.9588	+ 4.1700	+ 3.8428
C (Table III)	+ 74	— 10	— 81	+ 114	— 6	— 97
$(\alpha - \alpha_0)$.	— 3°.1253	+ 0°.6880	+ 2°.8140	— 2°.9474	+ 0°.4164	+ 3°.8331
B (Table II)	— 193	— 9	— 164	— 80	— 2	— 174
δ °	+ 13°.4853	+ 13°.4016	+ 14°.1345	+ 6°.0571	+ 7°.3992	+ 7°.8526

The transformation of intervals of right ascension and declination to rectangular coördinates, and its converse, for stars on another plate are inserted here to illustrate the use of the tables in the case of a plate taken near the pole.

DATA OF OBSERVATION.

Exposed 1905, April 11.

Sidereal time at beginning of exposure $10^h 30^m$

Sidereal time at end of exposure . . $10 50$

Coördinates of center of plate . . $\begin{cases} \alpha 14^h 38^m \\ \delta 76^\circ 12' \end{cases}$

Approximate scale value, 1 mm = $0''.1$

Latitude of Berkeley = $37^\circ 52'$.

Star.	1	2	3	4
$(\alpha - \alpha_0)$. . .	$-33^m 21^s.84$	$+31^m 38^s.90$	$-33^m 31^s.55$	$+30^m 40^s.97$
δ	$-8^\circ.3410$	$+7^\circ.9121$	$-8^\circ.3815$	$+7^\circ.6707$
B (Table II) . .	$+1167$	$+1032$	$+1502$	$+1278$
d	$+78.7998$	$+78.9857$	$+75.4653$	$+75.1846$
$d - \delta_0$	$+2.4998$	$+2.6857$	-0.8347	-1.1154
A (Table I) . .	$+44$	$+20$	-1	-1
C (Table III) . .	-673	$+594$	-612	$+476$
$\log k \xi \sec d$. .	0.92471	0.90154	0.92648	0.88752
$\log \cos d$. . .	9.28834	9.28115	9.39962	9.40774
$k \xi$	-1.6332	$+1.5230$	-2.1188	$+1.9736$
$k \eta$	$+2.5042$	$+2.6877$	-0.8348	-1.1155

Star.	1	2	3	4
$k \xi$	-1.6332	$+1.5230$	-2.1188	$+1.9736$
$k \eta$	$+2.5042$	$+2.6877$	-0.8348	-1.1155
A (Table I) $d - \delta_0$	-44	-20	$+1$	$+1$
d	$+78^\circ.7998$	$+78^\circ.9857$	$+75^\circ.4653$	$+75^\circ.1846$
$\log \sec d$. . .	0.71166	0.71885	0.60038	0.59226
$\log k \xi$	0.21305	0.18269	0.32610	0.29526
$k \xi \sec d$. . .	-8.4083	$+7.9715$	-8.4427	$+7.7183$
C (Table III) . .	$+673$	-594	$+612$	-476
$(\alpha - \alpha_0)$. . .	$-8^\circ.3410$	$+7^\circ.9121$	$-8^\circ.3815$	$+7^\circ.6707$
B (Table II) . .	-1167	-1032	-1502	-1278
δ	$+78^\circ.6831$	$+78^\circ.8825$	$+75^\circ.3151$	$+75^\circ.0568$

AS A TEST OF THE APPLICATION OF THE TABLES FOR REFRACTION to a very extreme case the following computation has been made. Five points on the celestial sphere were chosen arbitrarily, having the following coördinates:

	α	δ	Zenith Distance at Sid. t. 4 ^h
0	+15° 0' 0".0	—30° 0' 0".0	79° 50'.3
1	20 0 0.0	35 0 0.0	81 46.0
2	10 0 0.0	25 0 0.0	78 26.4
3	11 0 0.0	34 0 0.0	85 3.8
4	19 0 0.0	—29 0 0.0	77 5.4

Imagining stars to be situated at these points, their apparent spherical coördinates for the latitude of Berkeley and sidereal time four hours were computed, and from these were derived rectangular coördinates as they would have been measured on a plate taken at that hour angle. The so obtained fictitious plate measures were reduced, according to the plan outlined, pages 57–61, using the zero position as the center of the plate, supposed found by plotting on a star chart (see p. 56), and stars 1 and 2 as fiducial stars. The positions 3 and 4 are accurately reproduced.

The apparent spherical coördinates with the corresponding photographic refraction constants are:

	α	δ	μ
0	15° 3' 25".2	—29° 55' 41".7	0°.015613
1	20 3 58.1	34 54 33.0	0.015306
2	10 3 8.8	24 56 21.8	0.015756
3	11 7 2.2	33 52 10.1	0.014063
4	19 2 13.2	—28 56 29.0	0.015856

From these are deduced with the help of Tables I, II, and III (see example p. 69) the following fictitious measured coördinates, the center of the plate being taken at

$$\alpha = 15^{\circ} 3' 25''.2$$

$$\delta = -29^{\circ} 55' 41''.7$$

	x	y
1	+4.12937	—5.09725
2	—4.56308	+4.91709
3	—3.28197	—4.01049
4	+3.49146	+0.92819

Taking the same center of the plate and the true coördinates of stars 1 and 2, the standard coördinates are:

	ξ	η
1	+4.07076	—5.18620
2	—4.60838	+4.85411

The corrections introducing part of the effect of refraction into these standard coördinates are computed as follows:

α	15° 3'
δ	—29° 56'
s	60° 0
$s - \alpha$	44° 57'
Table VI \triangle	47 42
$\triangle - \delta$	77 38
Table VII $\log X \cos (\triangle - \delta)$	9.8272
$\log X \cos (\triangle - \delta)$	9.3308
* X	+3.136
Y	+4.560

	I	2		I	2
X	+3.1340		$\mu (X-x) []$	+0.01174	—0.00537
Y	+4.5571		$\Delta \mu (X-x)$	—0.00094	+0.00046
p. 72 $\left\{ \begin{array}{l} \mu \\ \Delta \mu \end{array} \right.$	+0.015306 —0.000307	+0.015756 +0.000143	r_x	+0.01080	—0.00491
Table VIII $\left\{ \begin{array}{l} x \\ y \end{array} \right.$	+0.07105 —0.09053	—0.08044 +0.08472	$\mu (Y-y) []$	+0.01782	—0.00748
Table IX, $x^2 + y^2$	+0.0132	+0.00136	$\Delta \mu (Y-y)$	—0.00143	+0.00064
$\mu (X-x)$	+0.04688	+0.05065	r_y	+0.01639	—0.00684
$\mu (Y-y)$	+0.07114	+0.07047	$x + r_x$	+4.08156	—4.61329
Xx	+0.2227	—0.2521	$x + r_x - x'$	— 4781	— 5021
Yy	—0.4124	+0.3861	$y + r_y$	—5.16981	+4.84727
Σ	—0.1897	+0.1340	$y + r_y - y'$	— 7256	— 6982
Table X, D	+0.2341	—0.1181			
$(x^2 + y^2) (1 + D)$	+0.0163	+0.0120			

The equations of condition are:

$$\begin{aligned} +4.129a' - 5.097b' + c' &= -4781 \\ -4.563 + 4.917 + c' &= -5021 \\ -5.097a' - 4.129b' + d' &= -7256 \\ +4.917 + 4.563 + d' &= -6982 \end{aligned}$$

From which:

$$\begin{aligned} a' &= +0.000275 \\ b' &= -0.000001 \\ c' &= -0.04895 \\ d' &= -0.07116 \end{aligned}$$

and for stars 3 and 4:

	3	4
$x + r_x$	—3.33182	+3.44347
$y + r_y$	—4.08275	+0.85729

* These values are not sufficiently accurate for this extreme case, and the values used in the following computation were derived by means of the rigorous relations. See p. 60.

The quantities r_x and r_y are computed for these stars as follows:

	3	4		3	4
X	+ 3.1340		$\Delta \mu (Y-y)$	- 0.00717	+ 0.00110
Y	+ 4.5571		1st approx. r_y	+ 0.06087	- 0.01339
μ	+ 0.014063	+ 0.015856			
$\Delta \mu$	- 0.001550	+ 0.000243	$x - r_x$	- 0.05889	+ 0.06026
x	- 0.05816	+ 0.06010	$y - r_y$	- 0.07228	+ 0.01519
y	- 0.07122	+ 0.01495	$(x^2 + y^2)$	87	38
$x^2 + y^2$	859	383	Nx	- 0.1846	+ 0.1888
$\mu (X-x)$	+ 0.04490	+ 0.04874	Yy	- 0.3294	+ 0.0692
$\mu (Y-y)$	+ 0.06509	+ 0.07203	Σ	- 0.5140	+ 0.2580
Nx	- 0.18227	+ 0.18835	D	+ 1.0575	- 0.2051
Yy	- 0.32455	+ 0.06814	$(x^2 + y^2) (1 + D)$	+ 179	+ 34
Σ	- 0.50682	+ 0.25649			
D	+ 1.0275	- 0.2042	$\mu (X-x) []$	+ 0.04828	- 0.00983
$(x^2 + y^2) (1 + D)$	+ 0.01742	+ 0.00305	$\Delta \mu (X-x)$	- 0.00495	+ 75
			2d approx. r_x	+ 0.04333	- 0.00908
$\mu (X-x) []$	+ 0.04691	- 0.00981			
$\Delta \mu (X-x)$	- 0.00495	+ 0.00075	$\mu (Y-y) []$	+ 0.07000	- 0.01453
1st approx. r_x	+ 0.04196	- 0.00906	$\Delta \mu (Y-y)$	- 0.00717	+ 110
			2d approx. r_y	+ 0.06283	- 0.01343
$\mu (Y-y) []$	+ 0.06801	- 0.01449			

Another approximation gives:

	3	4
r_x	+ 0.04340	- 0.00908
r_y	+ 0.06293	- 0.01343
x	- 3.37522	+ 3.45255
y	- 4.14568	+ 0.87072

Transforming these standard coördinates into intervals of right ascension and declination, the center of the plate being at

$$\begin{aligned} \alpha &= 15^\circ \quad 3' \quad 25''.2 \\ \delta &= 29 \quad 55 \quad 41.7, \end{aligned}$$

we reproduce the coördinates of the originally assumed points within a tenth of a second of arc:

	α	δ
3	10° 59' 59".9	- 34° 0' 0".1
4	19 0 0.0	- 28 59 59.9

TABLE I

GIVING

$$A = k \eta - (d - \delta_o)^c,$$

as function of $(d - \delta_o)^o$ or $k \eta$.

$$A = k\eta - (d - \delta_v)^0$$
$$0^{\circ}.00001 = 0''.036 = \text{unit.}$$

$d-\delta$	A	$k \eta$	$d-\delta$	A	$k \eta$	$d-\delta$	A	$k \eta$	$d-\delta$	A	$k \eta$	$d-\delta$	A	$k \eta$	$d-\delta$	A	$k \eta$	
0.0	.0	0.0	3.00	274	3.003	3.40	399	3.404	3.80	558	3.806	4.20	753	4.208	4.60	990	4.610	
.1	.0	.1	1	277 ³	013	41	403 ⁴	414	81	563 ⁵	816	21	759 ⁶	218	61	997 ⁷	620	
.2	.1	.2	2	280 ³	023	42	407 ⁴	424	82	567 ⁴	826	22	765 ⁶	228	62	1003 ⁶	630	
.3	.3	.3	3	283 ³	033	43	411 ⁴	434	83	571 ⁴	836	23	770 ⁵	238	63	1010 ⁷	640	
		.4		3 ³			3 ³			5 ⁵			5 ⁵			7 ⁷		
.4	.7	.6	4	286 ³	043	44	414 ³	444	84	576 ⁵	846	24	775 ⁶	248	64	1017 ⁶	650	
.5	1.3	.9	5	289 ²	053	45	417 ⁴	454	85	581 ⁴	856	25	781 ⁶	258	65	1023 ⁶	660	
.6	2.2	.6	6	291 ³	063	46	421 ⁴	464	86	585 ⁴	866	26	787 ⁵	268	66	1029 ⁷	670	
		1.3		3 ³			4 ⁴			4 ⁴			5 ⁵			7 ⁷		
.7	3.5	.7	7	294 ³	073	47	425 ⁴	474	87	589 ⁵	876	27	792 ⁵	278	67	1036 ⁷	680	
.8	5.2	1.7	8	297 ³	083	48	429 ³	484	88	594 ⁵	886	28	797 ⁶	288	68	1043 ⁶	690	
.9	7.4	2.2	9	300 ³	093	49	432 ⁴	494	89	599 ⁵	896	29	803 ⁶	298	69	1049 ⁷	700	
1.0	10.1	2.7	1.0	303 ³	3.103	3.50	436 ⁴	3.504	3.90	603 ⁴	3.906	4.30	809 ⁶	4.308	4.70	1056 ⁷	4.711	
		3		3 ³			4 ⁴			4 ⁴			6 ⁶			7 ⁷		
.1	13	4	.1	11	306 ³	113	51	440 ³	514	91	608 ⁵	916	31	815 ⁵	318	71	1063 ⁷	721
.2	17	5	.2	12	309 ³	123	52	443 ⁴	524	92	613 ⁴	926	32	820 ⁶	328	72	1070 ⁷	731
.3	22	6	.3	13	312 ³	133	53	447 ⁴	534	93	617 ⁵	936	33	826 ⁶	338	73	1077 ⁷	741
		8			3 ³		4 ⁴			5 ⁵			6 ⁶			7 ⁷		
.4	28	7	.4	14	315 ³	143	54	451 ⁴	545	94	622 ⁵	946	34	832 ⁵	348	74	1084 ⁷	751
.5	35	7	.5	15	318 ³	153	55	455 ⁴	555	95	627 ⁵	956	35	837 ⁶	358	75	1091 ⁷	761
.6	42	8	.6	16	321 ³	163	56	459 ⁴	565	96	632 ⁵	966	36	843 ⁶	368	76	1098 ⁷	771
		9			3 ³		4 ⁴			5 ⁵			6 ⁶			7 ⁷		
.7	50	9	.7	17	324 ³	173	57	463 ⁴	575	97	637 ⁴	976	37	849 ⁶	378	77	1105 ⁷	781
.8	59	11	.8	18	327 ³	183	58	467 ³	585	98	641 ⁵	986	38	855 ⁶	389	78	1112 ⁷	791
.9	70	11	.9	19	330 ³	193	59	470 ⁴	595	99	646 ⁵	996	39	861 ⁶	399	79	1119 ⁷	801
2.0	81	12	2.0	3.20	333 ⁴	3.203	3.60	474 ⁴	3.605	4.00	651 ⁵	4.007	4.40	867 ⁶	4.409	4.80	1126 ⁷	4.811
		14			4 ⁴		4 ⁴			5 ⁵			6 ⁶			7 ⁷		
.1	93	14	.1	21	337 ³	213	61	478 ⁴	615	1	656 ⁵	017	41	873 ⁶	419	81	1133 ⁷	821
.2	107	16	.2	22	340 ³	223	62	482 ⁴	625	2	661 ⁵	027	42	879 ⁶	429	82	1140 ⁷	831
.3	123	17	.3	23	343 ³	233	63	486 ⁴	635	3	666 ⁵	037	43	885 ⁶	439	83	1147 ⁷	841
		19			3 ³		4 ⁴			5 ⁵			6 ⁶			7 ⁷		
.4	140	19	.4	24	346 ³	243	64	490 ⁵	645	4	671 ⁵	047	44	891 ⁶	449	84	1154 ⁷	852
.5	159	20	.5	25	349 ³	253	65	495 ⁴	655	5	676 ⁵	057	45	897 ⁶	459	85	1161 ⁸	862
.6	179	21	.6	26	352 ³	264	66	499 ⁴	665	6	681 ⁵	067	46	903 ⁶	469	86	1169 ⁷	872
		23			3 ³		4 ⁴			5 ⁵			6 ⁶			7 ⁷		
.7	200	23	.7	27	355 ⁴	274	67	503 ⁴	675	7	686 ⁵	077	47	909 ⁶	479	87	1176 ⁷	882
.8	223	25	.802	28	359 ³	284	68	507 ⁴	685	8	691 ⁵	087	48	915 ⁶	489	88	1183 ⁷	892
.9	248	26	.902	29	362 ³	294	69	511 ⁴	695	9	696 ⁵	097	49	921 ⁶	499	89	1190 ⁷	902
3.0	274	26	3.003	3.30	365 ⁴	3.304	3.70	515 ⁴	3.705	4.10	701 ⁵	4.107	4.50	927 ⁶	4.509	4.90	1197 ⁷	4.912
				31	369 ³	314	71	519 ⁴	715	11	706 ⁵	117	51	933 ⁶	519	91	1205 ⁸	922
				32	372 ³	324	72	523 ⁴	725	12	711 ⁵	127	52	939 ⁶	529	92	1212 ⁷	932
				33	375 ³	334	73	528 ⁵	735	13	717 ⁶	137	53	945 ⁶	539	93	1220 ⁸	942
					4 ⁴		4 ⁴			5 ⁵			7 ⁷			7 ⁷		
				34	379 ⁴	344	74	532 ⁴	745	14	722 ⁵	147	54	952 ⁶	550	94	1227 ⁸	952
				35	383 ³	354	75	536 ⁵	755	15	727 ⁵	157	55	958 ⁷	560	95	1235 ⁷	962
				36	386 ³	364	76	541 ⁴	765	16	732 ⁵	167	56	965 ⁶	570	96	1242 ⁷	972
					3 ³		4 ⁴			5 ⁵			6 ⁶			7 ⁷		
				37	389 ⁴	374	77	545 ⁴	775	17	737 ⁶	177	57	971 ⁶	580	97	1249 ⁸	982
				38	393 ³	384	78	549 ⁴	785	18	743 ⁵	187	58	977 ⁷	590	98	1257 ⁸	993
				39	396 ³	394	79	553 ⁵	796	19	748 ⁵	197	59	984 ⁶	600	99	1265 ⁷	5.003
				3.40	399 ³	3.404	3.80	558 ⁵	3.806	4.20	753 ⁵	4.208	4.60	990 ⁶	4.610	5.00	1272 ⁷	5.013
$d-\delta$	A	$k \eta$	$d-\delta$	A	$k \eta$	$d-\delta$	A	$k \eta$	$d-\delta$	A	$k \eta$	$d-\delta$	A	$k \eta$	$d-\delta$	A	$k \eta$	

TABLE II

GIVING

$B = (d' - \delta)^\circ$, as function of $(\alpha - \alpha_0)$ and δ or d' .

TABLE II.

$$B = (d - \delta)^\circ.$$

 d is always numerically larger than δ .

$$0''.000001 = 0''.0036 = \text{unit.}$$

δ	$\frac{h}{2}$	$0^\circ.0$	hor. diff.	$0^\circ.1$	hor. diff.	$0^\circ.2$	hor. diff.	$0^\circ.3$	hor. diff.	$0^\circ.4$	hor. diff.	$0^\circ.5$	d
0	0	0	0	0	0	0	0	0	0	0	0	0	90
1	2		2	2 ₁	4	6 ₆	8	14 ₁₄	10	24 ₂₄	14	38 ₃₈	89
2	3		3	3 ₂	9	12 ₆	15	27 ₁₃	22	49 ₂₄	27	76 ₃₈	88
3	5		5	5 ₁	13	18 ₆	23	41 ₁₄	32	73 ₂₄	41	114 ₃₈	87
4	6		6	6 ₂	18	24 ₆	31	55 ₁₃	42	97 ₂₄	55	152 ₃₇	86
5	8		8	8 ₁	22	30 ₆	38	68 ₁₄	53	121 ₂₄	68	189 ₃₈	85
6	9		9	9 ₂	27	36 ₆	46	82 ₁₃	63	145 ₂₄	82	227 ₃₇	84
7	11		11	11 ₁	31	42 ₆	53	95 ₁₃	74	169 ₂₃	95	264 ₃₇	83
8	12		12	12 ₁	36	48 ₆	60	108 ₁₃	84	192 ₂₄	109	301 ₃₆	82
9	14		13	13 ₂	41	54 ₆	67	121 ₁₃	95	216 ₂₄	121	337 ₃₆	81
10	15		15	15 ₁	45	60 ₆	74	134 ₁₃	105	239 ₂₃	134	373 ₃₆	80
11	16		16	16 ₂	49	65 ₅	82	147 ₁₃	115	262 ₂₂	147	409 ₃₅	79
12	18		18	18 ₁	53	71 ₆	89	160 ₁₂	124	284 ₂₂	160	444 ₃₄	78
13	19		19	19 ₁	58	77 ₅	95	172 ₁₂	134	306 ₂₂	172	478 ₃₄	77
14	20		20	20 ₂	62	82 ₅	102	184 ₁₂	144	328 ₂₁	184	512 ₃₃	76
15	22		22	22 ₁	65	87 ₅	109	196 ₁₂	153	349 ₂₁	196	545 ₃₃	75
16	23		23	23 ₁	69	92 ₅	116	208 ₁₂	162	370 ₂₀	208	578 ₃₂	74
17	24		24	24 ₂	74	98 ₅	122	220 ₁₁	170	390 ₂₀	220	610 ₃₁	73
18	26		26	26 ₁	77	103 ₄	128	231 ₁₁	179	410 ₂₀	231	641 ₃₁	72
19	27		27	27 ₁	80	107 ₄	135	242 ₁₀	188	430 ₁₉	242	672 ₂₉	71
20	28		28	28 ₁	84	112 ₅	140	252 ₁₁	197	449 ₁₈	252	701 ₂₉	70
21	29		29	29 ₁	88	117 ₅	146	263 ₁₀	204	467 ₁₈	263	730 ₂₈	69
22	30		30	30 ₁	91	121 ₄	152	273 ₁₀	212	485 ₁₇	273	758 ₂₇	68
23	31		31	31 ₁	95	126 ₄	157	283 ₉	219	502 ₁₇	283	785 ₂₆	67
24	32		32	32 ₁	98	130 ₄	162	292 ₉	227	519 ₁₆	292	811 ₂₅	66
25	33		33	33 ₁	101	134 ₄	167	301 ₈	234	535 ₁₅	301	836 ₂₄	65
26	34		34	34 ₁	104	138 ₃	171	309 ₉	241	550 ₁₅	310	860 ₂₃	64
27	35		35	35 ₁	106	141 ₄	177	318 ₈	247	565 ₁₄	318	883 ₂₁	63
28	36		36	36 ₁	109	145 ₃	181	326 ₇	253	579 ₁₃	325	904 ₂₁	62
29	37		37	37 ₁	111	148 ₃	185	333 ₇	259	592 ₁₃	333	925 ₂₀	61
30	38		38	38 ₁	113	151 ₃	189	340 ₇	265	605 ₁₁	340	945 ₁₈	60
31	39		39	39 ₀	115	154 ₃	193	347 ₆	269	616 ₁₂	347	963 ₁₈	59
32	39		39	39 ₁	118	157 ₂	196	353 ₆	275	628 ₁₀	353	981 ₁₆	58
33	40		40	40 ₀	119	159 ₃	200	359 ₅	279	638 ₉	359	997 ₁₄	57
34	40		40	40 ₁	122	162 ₂	202	364 ₅	283	647 ₉	364	1011 ₁₄	56
35	41		41	41 ₁	123	164 ₂	205	369 ₅	287	656 ₈	369	1025 ₁₃	55
36	42		42	42 ₀	124	166 ₂	208	374 ₄	290	664 ₇	374	1038 ₁₁	54
37	42		42	42 ₁	126	168 ₁	210	378 ₃	293	671 ₆	378	1049 ₉	53
38	42		42	42 ₂	127	169 ₂	212	381 ₃	296	677 ₆	381	1058 ₉	52
39	43		43	43 ₀	128	171 ₁	213	384 ₃	299	683 ₅	384	1067 ₇	51
40	43		43	43 ₁	129	172 ₁	215	387 ₂	301	688 ₅	386	1074 ₆	50
41	43		43	43 ₂	130	173 ₀	216	389 ₂	302	691 ₄	389	1080 ₅	49
42	43		44	43 ₁	131	174 ₀	217	391 ₁	303	694 ₃	391	1085 ₃	48
43	44		44	44 ₀	130	174 ₁	218	392 ₀	304	696 ₂	392	1088 ₂	47
44	44		44	44 ₁	130	174 ₁	218	392 ₁	306	698 ₀	392	1090 ₁	46
45	44		44	44 ₂	131	175 ₀	218	393 ₁	305	698 ₀	393	1091 ₁	45
δ	$\frac{h}{2}$	$0^\circ.0$	hor. diff.	$0^\circ.1$	hor. diff.	$0^\circ.2$	hor. diff.	$0^\circ.3$	hor. diff.	$0^\circ.4$	hor. diff.	$0^\circ.5$	d
$\alpha - \alpha_0$													$\alpha - \alpha_0$

$$f(a \pm n w) = f(a) \pm n \left[f'(a) \pm n \frac{f''(a)}{2} \right], \quad \frac{f''(a)}{2} = \frac{h}{2}. \quad \text{See note on interpolation, p. 64.}$$

TABLE II.

$$B = (d - \delta)^\circ.$$

d is always numerically larger than δ .

$$0''.000001 = 0''.0036 = \text{unit.}$$

δ	$\frac{h}{2}$	$0^\circ.5$	hor. diff.	$0^\circ.6$	hor. diff.	$0^\circ.7$	hor. diff.	$0^\circ.8$	hor. diff.	$0^\circ.9$	hor. diff.	$1^\circ.0$	d
0	0	0		00	00	00	00	00	00	000	00	000	90
1	2	38 ³⁸	17	55 ⁵⁵	20	75 ⁷⁴	22	97 ⁹⁸	26	123 ¹²⁴	29	152 ¹⁵²	89
2	3	76 ³⁸	34	110 ⁵⁴	39	149 ⁷⁴	46	195 ⁹⁷	52	247 ¹²²	57	304 ¹⁵²	88
3	5	114 ³⁸	50	164 ⁵⁵	59	223 ⁷⁵	69	292 ⁹⁷	77	369 ¹²³	87	456 ¹⁵¹	87
4	6	152 ³⁷	67	219 ⁵⁴	79	298 ⁷³	91	389 ⁹⁶	103	492 ¹²²	115	607 ¹⁵¹	86
5	8	189 ³⁸	84	273 ⁵⁴	98	371 ⁷⁴	114	485 ⁹⁶	129	614 ¹²¹	144	758 ¹⁴⁹	85
6	9	227 ³⁷	100	327 ⁵³	118	445 ⁷²	136	581 ⁹⁵	154	735 ¹²⁰	172	907 ¹⁴⁹	84
7	11	264 ³⁷	116	380 ⁵³	137	517 ⁷²	159	676 ⁹⁴	179	855 ¹¹⁹	201	1056 ¹⁴⁷	83
8	12	301 ³⁶	132	433 ⁵²	156	589 ⁷²	181	770 ⁹³	204	974 ¹¹⁸	229	1203 ¹⁴⁵	82
9	14	337 ³⁶	148	485 ⁵²	176	661 ⁷⁰	202	863 ⁹²	229	1092 ¹¹⁷	256	1348 ¹⁴⁴	81
10	15	373 ³⁵	164	537 ⁵¹	194	731 ⁷⁰	224	955 ⁹¹	254	1209 ¹¹⁵	283	1492 ¹⁴³	80
11	16	409 ³⁵	179	588 ⁵¹	213	801 ⁶⁹	245	1046 ⁹⁰	278	1324 ¹¹⁴	311	1635 ¹⁴⁰	79
12	18	444 ³⁴	195	639 ⁵⁰	231	870 ⁶⁷	266	1136 ⁸⁸	302	1438 ¹¹²	337	1775 ¹³⁸	78
13	19	478 ³⁴	211	689 ⁴⁹	248	937 ⁶⁷	287	1224 ⁸⁷	326	1550 ¹⁰⁹	363	1913 ¹³⁶	77
14	20	512 ³³	226	738 ⁴⁷	266	1004 ⁶⁶	307	1311 ⁸⁵	348	1659 ¹⁰⁸	390	2049 ¹³³	76
15	22	545 ³³	240	785 ⁴⁷	284	1069 ⁶⁴	327	1396 ⁸⁴	371	1767 ¹⁰⁶	415	2182 ¹³⁰	75
16	23	578 ³²	254	832 ⁴⁶	301	1133 ⁶³	347	1480 ⁸²	393	1873 ¹⁰⁴	439	2312 ¹²⁸	74
17	24	610 ³¹	268	878 ⁴⁵	318	1196 ⁶¹	366	1562 ⁸⁰	415	1977 ¹⁰¹	463	2440 ¹²⁵	73
18	26	641 ³¹	282	923 ⁴⁴	334	1257 ⁵⁹	385	1642 ⁷⁷	436	2078 ⁹⁸	487	2565 ¹²²	72
19	27	672 ²⁹	295	967 ⁴³	349	1316 ⁵⁸	403	1719 ⁷⁶	457	2176 ⁹⁶	511	2687 ¹¹⁸	71
20	28	701 ²⁹	309	1010 ⁴¹	364	1374 ⁵⁷	421	1795 ⁷⁴	477	2272 ⁹³	533	2805 ¹¹⁵	70
21	29	730 ²⁷	321	1051 ⁴⁰	380	1431 ⁵⁴	438	1869 ⁷¹	496	2365 ⁹⁰	555	2920 ¹¹¹	69
22	30	758 ²⁷	333	1091 ³⁹	394	1485 ⁵³	455	1940 ⁶⁹	515	2455 ⁸⁸	576	3031 ¹⁰⁸	68
23	31	785 ²⁶	345	1130 ³⁷	408	1538 ⁵¹	471	2009 ⁶⁶	534	2543 ⁸⁴	596	3139 ¹⁰⁴	67
24	32	811 ²⁵	356	1167 ³⁶	422	1589 ⁴⁹	486	2075 ⁶⁴	552	2627 ⁸¹	616	3243 ¹⁰⁰	66
25	33	836 ²⁴	367	1203 ³⁵	435	1638 ⁴⁷	501	2139 ⁶²	569	2708 ⁷⁷	635	3343 ⁹⁶	65
26	34	860 ²³	378	1238 ³³	447	1685 ⁴⁵	516	2201 ⁵⁸	584	2785 ⁷⁴	654	3439 ⁹²	64
27	35	883 ²¹	388	1271 ³¹	459	1730 ⁴³	529	2259 ⁵⁶	600	2859 ⁷¹	672	3531 ⁸⁷	63
28	36	904 ²¹	398	1302 ³⁰	471	1773 ⁴⁰	542	2315 ⁵³	615	2930 ⁶⁷	688	3618 ⁸³	62
29	37	925 ²⁰	407	1332 ²⁸	481	1813 ³⁹	555	2368 ⁵¹	629	2997 ⁶⁴	704	3701 ⁷⁸	61
30	38	945 ¹⁸	415	1360 ²⁷	492	1852 ³⁶	567	2419 ⁴⁷	642	3061 ⁶⁰	718	3779 ⁷⁴	60
31	39	963 ¹⁸	424	1387 ²⁵	501	1888 ³⁴	578	2466 ⁴⁴	655	3121 ⁵⁶	732	3853 ⁶⁹	59
32	39	981 ¹⁶	431	1412 ²³	510	1922 ³¹	588	2510 ⁴¹	667	3177 ⁵²	745	3922 ⁶⁴	58
33	40	997 ¹⁴	438	1435 ²²	518	1953 ²⁹	598	2551 ³⁸	678	3229 ⁴⁸	757	3986 ⁶⁰	57
34	40	1011 ¹³	446	1457 ¹⁹	525	1982 ²⁷	607	2589 ³⁵	688	3277 ⁴⁴	769	4046 ⁵⁴	56
35	41	1025 ¹¹	451	1476 ¹⁸	533	2009 ²⁵	615	2624 ³²	697	3321 ⁴⁰	779	4100 ⁵⁰	55
36	42	1038 ¹¹	456	1494 ¹⁶	540	2034 ²²	622	2656 ²⁹	705	3361 ³⁶	789	4150 ⁴⁵	54
37	42	1049 ⁹	461	1510 ¹⁴	546	2056 ¹⁹	629	2685 ²⁵	712	3397 ³²	798	4195 ³⁹	53
38	42	1058 ⁹	466	1524 ¹³	551	2075 ¹⁶	635	2710 ²²	719	3429 ²⁸	805	4234 ³⁴	52
39	43	1067 ⁷	470	1537 ¹⁰	554	2091 ¹⁴	641	2732 ¹⁸	725	3457 ²⁴	811	4268 ²⁹	51
40	43	1074 ⁶	473	1547 ⁹	558	2105 ¹²	645	2750 ¹⁵	731	3481 ¹⁹	816	4297 ²⁴	50
41	43	1080 ⁵	476	1556 ⁶	561	2117 ⁹	648	2765 ¹²	735	3500 ¹⁵	821	4321 ¹⁹	49
42	43	1085 ³	477	1562 ⁵	564	2126 ⁷	651	2777 ⁹	738	3515 ¹¹	825	4340 ¹³	48
43	44	1088 ²	479	1567 ³	566	2133 ⁴	653	2786 ⁵	740	3526 ⁶	827	4353 ⁸	47
44	44	1090 ¹	480	1570 ¹	567	2137 ¹	654	2791 ¹	741	3532 ²	829	4361 ³	46
45	44	1091	480	1571	567	2138	654	2792	742	3534	830	4364	45

$$f(a \pm n w) = f(a) \pm n \left[f'(a) \pm n \frac{f''(a)}{2} \right], \quad \frac{f''(a)}{2} = \frac{h}{2}.$$

TABLE II.

$$B = (d - \delta)^\circ.$$

 d is always numerically larger than δ .

$$0''.000001 = 0''.0036 = \text{unit.}$$

δ	$\frac{h}{2}$	$1^\circ.0$	hor. diff.	$1^\circ.1$	hor. diff.	$1^\circ.2$	hor. diff.	$1^\circ.3$	hor. diff.	$1^\circ.4$	hor. diff.	$1^\circ.5$	d
0	0	000	00	000	00	000	00	000	00	000	00	000	90
1	2	152 ¹⁵²	32	184 ¹⁸⁴	35	219 ²¹⁹	38	257 ²⁵⁷	42	299 ²⁹⁹	44	343 ³⁴³	89
2	3	304 ¹⁵²	64	368 ¹⁸⁴	70	438 ²¹⁹	77	515 ²⁵⁶	82	597 ²⁹⁷	88	685 ³⁴²	88
3	5	456 ¹⁵¹	96	552 ¹⁸³	105	657 ²¹⁸	114	771 ²⁵⁶	123	894 ²⁹⁷	133	1027 ³⁴⁰	87
4	6	607 ¹⁵¹	128	735 ¹⁸²	140	875 ²¹⁶	152	1027 ²⁵⁴	164	1191 ²⁹⁵	176	1367 ³³⁸	86
5	8	758 ¹⁴⁹	159	917 ¹⁸¹	174	1091 ²¹⁶	190	1281 ²⁵³	205	1486 ²⁹³	219	1705 ³³⁷	85
6	9	907 ¹⁴⁹	191	1098 ¹⁷⁹	209	1307 ²¹³	227	1534 ²⁵⁰	245	1779 ²⁹¹	263	2042 ³³⁴	84
7	11	1056 ¹⁴⁷	221	1277 ¹⁷⁹	243	1520 ²¹²	264	1784 ²⁴⁹	286	2070 ²⁸⁸	306	2376 ³³¹	83
8	12	1203 ¹⁴⁵	253	1456 ¹⁷⁶	276	1732 ²¹⁰	301	2033 ²⁴⁶	325	2358 ²⁸⁶	349	2707 ³²⁸	82
9	14	1348 ¹⁴⁴	284	1632 ¹⁷⁴	310	1942 ²⁰⁷	337	2279 ²⁴⁴	365	2644 ²⁸²	391	3035 ³²⁴	81
10	15	1492 ¹⁴³	314	1806 ¹⁷²	343	2149 ²⁰⁵	374	2523 ²⁴⁰	403	2926 ²⁷⁹	433	3359 ³²⁰	80
11	16	1635 ¹⁴⁰	343	1978 ¹⁷⁰	376	2354 ²⁰²	409	2763 ²³⁷	442	3205 ²⁷⁵	474	3679 ³¹⁶	79
12	18	1775 ¹³⁸	373	2148 ¹⁶⁷	408	2556 ¹⁹⁹	444	3000 ²³⁴	480	3480 ²⁷⁰	515	3995 ³¹⁰	78
13	19	1913 ¹³⁶	402	2315 ¹⁶⁴	440	2755 ¹⁹⁵	479	3234 ²²⁹	516	3750 ²⁶⁶	555	4305 ³⁰⁵	77
14	21	2049 ¹³³	430	2479 ¹⁶¹	471	2950 ¹⁹²	513	3463 ²²⁵	553	4016 ²⁶¹	594	4610 ³⁰⁰	76
15	22	2182 ¹³⁰	458	2640 ¹⁵⁸	502	3142 ¹⁸⁸	546	3688 ²²¹	589	4277 ²⁵⁶	633	4910 ²⁹⁴	75
16	23	2312 ¹²⁸	486	2798 ¹⁵⁵	532	3330 ¹⁸⁴	579	3909 ²¹⁶	624	4533 ²⁵¹	671	5204 ²⁸⁸	74
17	24	2440 ¹²⁵	513	2953 ¹⁵¹	561	3514 ¹⁸⁰	611	4125 ²¹¹	659	4784 ²⁴⁵	708	5492 ²⁸¹	73
18	26	2565 ¹²²	539	3104 ¹⁴⁷	590	3694 ¹⁷⁵	642	4336 ²⁰⁵	693	5029 ²³⁸	744	5773 ²⁷³	72
19	27	2687 ¹¹⁸	564	3251 ¹⁴³	618	3869 ¹⁷¹	672	4541 ²⁰⁰	726	5267 ²³²	779	6046 ²⁶⁷	71
20	28	2805 ¹¹⁵	589	3394 ¹³⁹	646	4040 ¹⁶⁵	701	4741 ¹⁹⁴	758	5499 ²²⁵	814	6313 ²⁵⁸	70
21	29	2920 ¹¹¹	613	3533 ¹³⁵	672	4205 ¹⁶¹	730	4935 ¹⁸⁹	789	5724 ²¹⁸	847	6571 ²⁵¹	69
22	30	3031 ¹⁰⁸	637	3668 ¹³⁰	698	4366 ¹⁵⁵	758	5124 ¹⁸²	818	5942 ²¹²	880	6822 ²⁴³	68
23	31	3139 ¹⁰⁴	659	3798 ¹²⁶	723	4521 ¹⁴⁹	785	5306 ¹⁷⁵	848	6154 ²⁰³	911	7065 ²³³	67
24	32	3243 ¹⁰⁰	681	3924 ¹²¹	746	4670 ¹⁴⁴	811	5481 ¹⁶⁹	876	6357 ¹⁹⁶	941	7298 ²²⁵	66
25	33	3343 ⁹⁶	702	4045 ¹¹⁶	769	4814 ¹³⁸	836	5650 ¹⁶²	903	6553 ¹⁸⁸	970	7523 ²¹⁶	65
26	34	3439 ⁹²	722	4161 ¹¹¹	791	4952 ¹³²	860	5812 ¹⁵⁵	929	6741 ¹⁸⁰	998	7739 ²⁰⁶	64
27	35	3531 ⁸⁷	741	4272 ¹⁰⁶	812	5084 ¹²⁶	883	5967 ¹⁴⁸	954	6921 ¹⁷¹	1024	7945 ¹⁹⁶	63
28	36	3618 ⁸³	760	4378 ¹⁰⁰	832	5210 ¹¹⁹	905	6115 ¹⁴⁰	977	7092 ¹⁶²	1049	8141 ¹⁸⁷	62
29	37	3701 ⁷⁸	777	4478 ⁹⁵	851	5329 ¹¹³	926	6255 ¹³²	999	7254 ¹⁵⁴	1074	8328 ¹⁷⁶	61
30	38	3779 ⁷⁴	794	4573 ⁸⁹	869	5442 ¹⁰⁷	945	6387 ¹²⁵	1021	7408 ¹⁴⁵	1096	8504 ¹⁶⁷	60
31	38	3853 ⁶⁹	809	4662 ⁸⁴	887	5549 ¹⁰¹	963	6512 ¹¹⁷	1041	7553 ¹³⁵	1118	8671 ¹⁵⁵	59
32	39	3922 ⁶⁴	824	4746 ⁷⁸	902	5648 ⁹³	981	6629 ¹⁰⁹	1059	7688 ¹²⁶	1138	8826 ¹⁴⁵	58
33	40	3986 ⁶⁰	838	4824 ⁷²	917	5741 ⁸⁶	997	6738 ¹⁰⁰	1076	7814 ¹¹⁷	1157	8971 ¹³⁴	57
34	40	4046 ⁵⁴	850	4896 ⁶⁶	931	5827 ⁷⁸	1011	6838 ⁹²	1093	7931 ¹⁰⁷	1174	9105 ¹²³	56
35	41	4100 ⁵⁰	862	4962 ⁶⁰	943	5905 ⁷²	1025	6930 ⁸⁴	1108	8038 ⁹⁷	1190	9228 ¹¹¹	55
36	42	4150 ⁴⁵	872	5022 ⁵⁴	955	5977 ⁶⁴	1037	7014 ⁷⁵	1121	8135 ⁸⁷	1204	9339 ¹⁰⁰	54
37	42	4195 ³⁹	881	5076 ⁴⁷	965	6041 ⁵⁶	1048	7089 ⁶⁷	1133	8222 ⁷⁸	1217	9439 ⁸⁹	53
38	42	4234 ³⁴	889	5123 ⁴²	974	6097 ⁵⁰	1059	7156 ⁵⁸	1144	8300 ⁶⁷	1228	9528 ⁷⁷	52
39	43	4268 ²⁹	897	5165 ³⁵	982	6147 ⁴²	1067	7214 ⁴⁹	1153	8367 ⁵⁷	1238	9605 ⁶⁵	51
40	43	4297 ²⁴	903	5200 ²⁹	989	6189 ³⁴	1074	7263 ⁴⁰	1161	8424 ⁴⁶	1246	9670 ⁵⁴	50
41	43	4321 ¹⁹	908	5229 ²²	994	6223 ²⁷	1080	7303 ³²	1167	8470 ³⁷	1254	9724 ⁴¹	49
42	43	4340 ¹³	911	5251 ¹⁶	999	6250 ¹⁹	1085	7335 ²²	1172	8507 ²⁶	1258	9765 ³⁰	48
43	44	4353 ⁸	914	5267 ¹⁰	1002	6269 ¹¹	1088	7357 ¹³	1176	8533 ¹⁵	1262	9795 ¹⁸	47
44	44	4361 ³	916	5277 ³	1003	6280 ⁴	1090	7370 ⁵	1178	8548 ⁵	1265	9813 ⁶	46
45	44	4364	916	5280	1004	6284	1091	7375	1178	8553	1266	9819	45

$$f(a \pm n w) = f(a) \pm n \left[f'(a) \pm n \frac{f''(a)}{2} \right], \quad \frac{f''(a)}{2} = \frac{h}{2}.$$

TABLE II.

$$B = (d - \delta)^\circ.$$

d is always numerically larger than δ .

$$0''.000001 = 0''.0036 = \text{unit.}$$

δ	$\frac{h}{2}$	$\frac{v}{2}$	$1^\circ.5$	hor. diff.	$1^\circ.6$	hor. diff.	$1^\circ.7$	hor. diff.	$1^\circ.8$	hor. diff.	$1^\circ.9$	hor. diff.	$2^\circ.0$	d
0	0		000	00	000	00	000	00	000	00	000	00	000	90
1	2		343 ³⁴³	47	390 ³⁹⁰	50	440 ⁴⁴⁰	54	494 ⁴⁹⁴	56	550 ⁵⁵⁰	59	609 ⁶⁰⁹	89
2	3		685 ³⁴²	95	780 ³⁸⁸	100	880 ⁴³⁹	107	987 ⁴⁹¹	112	1099 ⁵⁴⁸	119	1218 ⁶⁰⁷	88
3	5		1027 ³⁴⁰	141	1168 ³⁸⁷	151	1319 ⁴³⁷	159	1478 ⁴⁹⁰	169	1647 ⁵⁴⁶	178	1825 ⁶⁰⁵	87
4	6		1367 ³³⁸	188	1555 ³⁸⁵	201	1756 ⁴³⁵	212	1968 ⁴⁸⁸	225	2193 ⁵⁴⁴	237	2430 ⁶⁰²	86
5	8		1705 ³³⁷	235	1940 ³⁸³	251	2191 ⁴³²	265	2456 ⁴⁸⁵	281	2737 ⁵⁴⁰	295	3032 ⁵⁹⁸	85
6	9		2042 ³³⁴	281	2323 ³⁸⁰	300	2623 ⁴²⁹	318	2941 ⁴⁸¹	336	3277 ⁵³⁶	353	3630 ⁵⁹⁴	84
7	11		2376 ³³¹	327	2703 ³⁷⁷	349	3052 ⁴²⁵	370	3422 ⁴⁷⁷	391	3813 ⁵³¹	411	4224 ⁵⁸⁹	83
8	12		2707 ³²⁸	373	3080 ³⁷³	397	3477 ⁴²¹	422	3899 ⁴⁷²	445	4344 ⁵²⁶	469	4813 ⁵⁸³	82
9	14	3	3035 ³²⁴	418	3453 ³⁶⁹	445	3898 ⁴¹⁷	473	4371 ⁴⁶⁶	499	4870 ⁵²⁰	526	5396 ⁵⁷⁶	81
10	15	3	3359 ³²⁰	463	3822 ³⁶⁴	493	4315 ⁴¹¹	522	4837 ⁴⁶¹	553	5390 ⁵¹³	582	5972 ⁵⁶⁹	80
11	16	3	3679 ³¹⁶	507	4186 ³⁵⁹	540	4726 ⁴⁰⁵	572	5298 ⁴⁵⁵	605	5903 ⁵⁰⁷	638	6541 ⁵⁶¹	79
12	18	3	3995 ³¹⁰	550	4545 ³⁵⁴	586	5131 ³⁹⁹	622	5753 ⁴⁴⁷	657	6410 ⁴⁹⁸	692	7102 ⁵⁵³	78
13	19	4	4305 ³⁰⁵	594	4899 ³⁴⁷	631	5530 ³⁹²	670	6200 ⁴⁴⁰	708	6908 ⁴⁹⁰	747	7655 ⁵⁴³	77
14	21	4	4610 ³⁰⁰	636	5246 ³⁴¹	676	5922 ³⁸⁶	718	6640 ⁴³²	758	7398 ⁴⁸¹	800	8198 ⁵³³	76
15	22	4	4910 ²⁹⁴	677	5587 ³³⁵	721	6308 ³⁷⁷	764	7072 ⁴²³	807	7879 ⁴⁷²	852	8731 ⁵²²	75
16	23	4	5204 ²⁸⁸	718	5922 ³²⁷	763	6685 ³⁶⁹	810	7495 ⁴¹⁴	856	8351 ⁴⁶¹	902	9253 ⁵¹¹	74
17	24	4	5492 ²⁸¹	757	6249 ³¹⁹	805	7054 ³⁶¹	855	7909 ⁴⁰⁴	903	8812 ⁴⁵¹	952	9764 ⁴⁹⁹	73
18	26	5	5773 ²⁷³	795	6568 ³¹²	847	7415 ³⁵²	898	8313 ³⁹⁴	950	9263 ⁴³⁹	1000	10263 ⁴⁸⁷	72
19	27	5	6046 ²⁶⁷	834	6880 ³⁰³	887	7767 ³⁴²	940	8707 ³⁸⁴	995	9702 ⁴²⁷	1048	10750 ⁴⁷⁴	71
20	28	5	6313 ²⁵⁸	870	7183 ²⁹⁴	926	8109 ³³²	982	9091 ³⁷²	1038	10129 ⁴¹⁵	1095	11224 ⁴⁶⁰	70
21	29	6	6571 ²⁵¹	906	7477 ²⁸⁵	964	8441 ³²²	1022	9463 ³⁶¹	1081	10544 ⁴⁰²	1140	11684 ⁴⁴⁵	69
22	30	6	6822 ²⁴³	940	7762 ²⁷⁶	1001	8763 ³¹¹	1061	9824 ³⁴⁹	1122	10946 ³⁸⁹	1183	12129 ⁴³¹	68
23	31	6	7065 ²³³	973	8038 ²⁶⁶	1036	9074 ³⁰⁰	1099	10173 ³³⁷	1162	11335 ³⁷⁵	1225	12560 ⁴¹⁶	67
24	32	6	7298 ²²⁵	1006	8304 ²⁵⁶	1070	9374 ²⁸⁹	1136	10510 ³²⁴	1200	11710 ³⁶¹	1266	12976 ³⁹⁹	66
25	33	6	7523 ²¹⁶	1037	8560 ²⁴⁵	1103	9663 ²⁷⁷	1171	10834 ³¹⁰	1237	12071 ³⁴⁶	1304	13375 ³⁸⁴	65
26	34	7	7739 ²⁰⁶	1066	8805 ²³⁵	1135	9940 ²⁶⁵	1204	11144 ²⁹⁷	1273	12417 ³³¹	1342	13759 ³⁶⁶	64
27	35	7	7945 ¹⁹⁶	1095	9040 ²²³	1165	10205 ²⁵³	1236	11441 ²⁸³	1307	12748 ³¹⁵	1377	14125 ³⁵⁰	63
28	36	7	8141 ¹⁸⁷	1122	9263 ²¹³	1195	10458 ²³⁹	1266	11724 ²⁶⁹	1339	13063 ³⁰⁰	1412	14475 ³³²	62
29	37	7	8328 ¹⁷⁶	1148	9476 ²⁰⁰	1221	10697 ²²⁷	1296	11993 ²⁵⁴	1370	13363 ²⁸³	1444	14807 ³¹⁴	61
30	38	7	8504 ¹⁶⁷	1172	9676 ¹⁸⁹	1248	10924 ²¹³	1323	12247 ²³⁹	1399	13646 ²⁶⁶	1475	15121 ²⁹⁵	60
31	38	8	8671 ¹⁵⁵	1194	9865 ¹⁷⁷	1272	11137 ²⁰⁰	1349	12486 ²²⁴	1426	13912 ²⁵⁰	1504	15416 ²⁷⁶	59
32	39	8	8826 ¹⁴⁵	1216	10042 ¹⁶⁵	1295	11337 ¹⁸⁶	1373	12710 ²⁰⁹	1452	14162 ²³²	1530	15692 ²⁵⁸	58
33	40	8	8971 ¹³⁴	1236	10207 ¹⁵²	1316	11523 ¹⁷²	1396	12919 ¹⁹³	1475	14394 ²¹⁵	1556	15950 ²³⁸	57
34	40	8	9105 ¹²³	1254	10359 ¹⁴⁰	1336	11695 ¹⁵⁸	1417	13112 ¹⁷⁶	1497	14609 ¹⁹⁷	1579	16188 ²¹⁸	56
35	41	8	9228 ¹¹¹	1271	10499 ¹²⁷	1354	11853 ¹⁴³	1435	13288 ¹⁶¹	1518	14806 ¹⁷⁹	1600	16406 ¹⁹⁸	55
36	42	8	9339 ¹⁰⁰	1287	10626 ¹¹⁴	1370	11996 ¹²⁹	1453	13449 ¹⁴⁴	1536	14985 ¹⁶¹	1619	16604 ¹⁷⁸	54
37	42	8	9439 ⁸⁹	1301	10740 ¹⁰¹	1385	12125 ¹¹³	1468	13593 ¹²⁸	1553	15146 ¹⁴²	1636	16782 ¹⁵⁷	53
38	42	8	9528 ⁷⁷	1313	10841 ⁸⁷	1397	12238 ⁹⁹	1483	13721 ¹¹¹	1567	15288 ¹²³	1651	16939 ¹³⁷	52
39	43	8	9605 ⁶⁵	1323	10928 ⁷⁵	1409	12337 ⁸⁴	1495	13832 ⁹⁴	1579	15411 ¹⁰⁵	1665	17076 ¹¹⁶	51
40	43	8	9670 ⁵⁴	1333	11003 ⁶¹	1418	12421 ⁶⁹	1505	13926 ⁷⁷	1590	15516 ⁸⁶	1676	17192 ⁹⁶	50
41	43	8	9724 ⁴¹	1340	11064 ⁴⁷	1426	12490 ⁵³	1513	14003 ⁶⁰	1599	15602 ⁶⁷	1686	17288 ⁷⁴	49
42	43	8	9765 ³⁰	1346	11111 ³⁴	1432	12543 ³⁹	1520	14063 ⁴³	1606	15669 ⁴⁸	1693	17362 ⁵³	48
43	44	8	9795 ¹⁸	1350	11145 ²⁰	1437	12582 ²³	1524	14106 ²⁵	1611	15717 ²⁸	1698	17415 ³¹	47
44	44	8	9813 ⁶	1352	11165 ⁷	1440	12605 ⁷	1526	14131 ⁹	1614	15745 ¹⁰	1701	17446 ¹¹	46
45	44	8	9819	1353	11172	1440	12612	1528	14140	1615	15755	1702	17457	45

$$f(a \pm n w) = f(a) \pm n \left[f'(a) \pm n \frac{f''(a)}{2} \right], \quad \frac{f''(a)}{2} = \frac{h}{2} \text{ or } \frac{v}{2}$$

TABLE II.

$$B = (d - \delta)^\circ.$$

 d is always numerically larger than δ .

$$0''.00001 = 0''.036 = \text{unit.}$$

δ	$a - a_0$	$2^\circ.0$	hor. diff.	$2^\circ.1$	hor. diff.	$2^\circ.2$	hor. diff.	$2^\circ.3$	hor. diff.	$2^\circ.4$	hor. diff.	$2^\circ.5$	$a - a_0$	d
0	00	0	00	0	00	0	00	0	00	0	00	0	90	
1	61	6	67	7	74	7	81	7	88	7	95	7	89	
2	122	12	134	13	147	14	161	14	175	15	190	15	88	
3	182	19	201	20	221	20	241	22	263	22	285	22	87	
4	243	25	268	26	294	27	321	29	350	30	380	30	86	
5	303	31	334	33	367	34	401	36	437	37	474	37	85	
6	363	37	400	39	439	41	480	43	523	44	567	44	84	
7	422	44	466	45	511	48	559	49	608	52	660	52	83	
8	481	50	531	51	582	55	637	56	693	59	752	59	82	
9	540	55	595	58	653	61	714	63	777	66	843	66	81	
10	597	61	658	65	723	67	790	70	860	73	933	73	80	
11	654	67	721	71	792	73	865	77	942	80	1022	80	79	
12	710	73	783	77	860	79	939	84	1023	87	1110	87	78	
13	766	78	844	82	926	87	1013	90	1103	93	1196	93	77	
14	820	84	904	88	992	92	1084	97	1181	100	1281	100	76	
15	873	90	963	94	1057	98	1155	103	1258	107	1365	107	75	
16	925	95	1020	100	1120	104	1224	109	1333	113	1446	113	74	
17	976	101	1077	105	1182	110	1292	114	1406	120	1526	120	73	
18	1026	106	1132	110	1242	116	1358	120	1478	126	1604	126	72	
19	1075	110	1185	116	1301	121	1422	126	1548	132	1680	132	71	
20	1122	115	1237	121	1358	127	1485	132	1617	137	1754	137	70	
21	1168	120	1288	126	1414	131	1545	138	1683	143	1826	143	69	
22	1213	124	1337	131	1468	136	1604	143	1747	149	1896	149	68	
23	1256	129	1385	135	1520	141	1661	148	1809	154	1963	154	67	
24	1298	133	1431	139	1570	146	1716	153	1869	159	2028	159	66	
25	1338	137	1475	144	1619	150	1769	157	1926	164	2090	164	65	
26	1376	141	1517	148	1665	155	1820	162	1982	168	2150	168	64	
27	1412	145	1557	152	1709	159	1868	166	2034	174	2208	174	63	
28	1448	148	1596	156	1752	163	1915	170	2085	177	2262	177	62	
29	1481	152	1633	159	1792	166	1958	175	2133	181	2314	181	61	
30	1512	155	1667	163	1830	170	2000	178	2178	185	2363	185	60	
31	1542	158	1700	165	1865	174	2039	181	2220	189	2409	189	59	
32	1569	161	1730	169	1899	177	2076	184	2260	192	2452	192	58	
33	1595	164	1759	171	1930	180	2110	187	2297	196	2493	196	57	
34	1619	166	1785	174	1959	182	2141	190	2331	199	2530	199	56	
35	1641	168	1809	176	1985	185	2170	193	2363	201	2564	201	55	
36	1660	171	1831	178	2009	187	2196	195	2391	204	2595	204	54	
37	1678	172	1850	181	2031	189	2220	197	2417	206	2623	206	53	
38	1694	174	1868	182	2050	190	2240	200	2440	207	2647	207	52	
39	1708	175	1883	183	2066	193	2259	200	2459	210	2669	210	51	
40	1719	177	1896	184	2080	194	2274	202	2476	211	2687	211	50	
41	1729	177	1906	186	2092	195	2287	203	2490	212	2702	212	49	
42	1736	178	1914	187	2101	195	2296	204	2500	213	2713	213	48	
43	1742	178	1920	187	2107	196	2303	205	2508	213	2721	213	47	
44	1745	179	1924	187	2111	196	2307	206	2513	213	2726	213	46	
45	1746	179	1925	187	2112	197	2309	205	2514	214	2728	214	45	

$$f(a \pm nw) = f(a) \pm n \left[f'(a) \pm n \frac{f''(a)}{2} \right], \quad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$$B = (d - \delta)^\circ.$$

 d is always numerically larger than δ .

$$0''.00001 = 0''.036 = \text{unit.}$$

δ	$2^\circ.5$	hor. diff.	$2^\circ.6$	hor. diff.	$2^\circ.7$	hor. diff.	$2^\circ.8$	hor. diff.	$2^\circ.9$	hor. diff.	$3^\circ.0$	d
0	00	0	000	0	000	0	000	0	000	0	000	90
1	95 ⁹⁵	8	103 ¹⁰³	8	111 ¹¹¹	9	120 ¹²⁰	8	128 ¹²⁸	9	137 ¹³⁷	89
2	190 ⁹⁵	16	206 ¹⁰³	16	222 ¹¹¹	17	239 ¹¹⁹	17	256 ¹²⁸	18	274 ¹³⁷	88
3	285 ⁹⁵	24	309 ¹⁰²	24	333 ¹¹⁰	25	358 ¹¹⁹	26	384 ¹²⁷	27	411 ¹³⁶	87
4	380 ⁹⁴	31	411 ¹⁰²	32	443 ¹¹⁰	34	477 ¹¹⁸	34	511 ¹²⁷	36	547 ¹³⁶	86
5	474 ⁹³	39	513 ¹⁰¹	40	553 ¹⁰⁹	42	595 ¹¹⁷	43	638 ¹²⁶	45	683 ¹³⁴	85
6	567 ⁹³	47	614 ¹⁰⁰	48	662 ¹⁰⁸	50	712 ¹¹⁶	52	764 ¹²⁵	53	817 ¹³⁴	84
7	660 ⁹²	54	714 ¹⁰⁰	56	770 ¹⁰⁸	58	828 ¹¹⁶	61	889 ¹²⁴	62	951 ¹³³	83
8	752 ⁹¹	62	814 ⁹⁸	64	878 ¹⁰⁶	66	944 ¹¹⁴	69	1013 ¹²²	71	1084 ¹³¹	82
9	843 ⁹⁰	69	912 ⁹⁸	72	984 ¹⁰⁵	74	1058 ¹¹³	77	1135 ¹²¹	80	1215 ¹³⁰	81
10	933 ⁸⁹	77	1010 ⁹⁶	79	1089 ¹⁰⁴	82	1171 ¹¹²	85	1256 ¹²⁰	87	1345 ¹²⁸	80
11	1022 ⁸⁸	84	1106 ⁹⁵	87	1193 ¹⁰²	90	1283 ¹¹⁰	93	1376 ¹¹⁸	97	1473 ¹²⁶	79
12	1110 ⁸⁶	91	1201 ⁹³	97	1295 ¹⁰¹	98	1393 ¹⁰⁸	101	1494 ¹¹⁶	105	1599 ¹²⁴	78
13	1196 ⁸⁵	98	1294 ⁹²	102	1396 ⁹⁹	105	1501 ¹⁰⁷	109	1610 ¹¹⁴	113	1723 ¹²³	77
14	1281 ⁸⁴	105	1386 ⁹⁰	109	1495 ⁹⁷	113	1608 ¹⁰⁴	116	1724 ¹¹³	122	1846 ¹²⁰	76
15	1365 ⁸¹	111	1476 ⁸⁸	116	1592 ⁹⁵	120	1712 ¹⁰²	125	1837 ¹⁰⁹	129	1966 ¹¹⁷	75
16	1446 ⁸⁰	118	1564 ⁸⁷	123	1687 ⁹³	127	1814 ¹⁰¹	132	1946 ¹⁰⁸	137	2083 ¹¹⁵	74
17	1526 ⁷⁸	125	1651 ⁸⁴	129	1780 ⁹¹	135	1915 ⁹⁸	139	2054 ¹⁰⁵	144	2198 ¹¹³	73
18	1604 ⁷⁶	131	1735 ⁸²	136	1871 ⁸⁹	142	2013 ⁹⁵	146	2159 ¹⁰²	152	2311 ¹⁰⁹	72
19	1680 ⁷⁴	137	1817 ⁸⁰	143	1960 ⁸⁶	148	2108 ⁹³	153	2261 ¹⁰⁰	159	2420 ¹⁰⁷	71
20	1754 ⁷³	143	1897 ⁷⁸	149	2046 ⁸⁴	155	2201 ⁹⁰	160	2361 ⁹⁷	166	2527 ¹⁰³	70
21	1826 ⁷⁰	149	1975 ⁷⁶	155	2130 ⁸¹	161	2291 ⁸⁷	167	2458 ⁹³	172	2630 ¹⁰⁰	69
22	1896 ⁶⁷	155	2051 ⁷²	160	2211 ⁷⁹	167	2378 ⁸⁵	173	2551 ⁹¹	179	2730 ⁹⁷	68
23	1963 ⁶⁵	160	2123 ⁷¹	167	2290 ⁷⁶	173	2463 ⁸¹	179	2642 ⁸⁷	185	2827 ⁹⁴	67
24	2028 ⁶²	166	2194 ⁶⁷	172	2366 ⁷³	178	2544 ⁷⁹	185	2729 ⁸⁴	192	2921 ⁹⁰	66
25	2090 ⁶⁰	171	2261 ⁶⁵	178	2439 ⁶⁹	184	2623 ⁷⁵	190	2813 ⁸¹	198	3011 ⁸⁶	65
26	2150 ⁵⁸	176	2326 ⁶²	182	2508 ⁶⁷	190	2698 ⁷²	196	2894 ⁷⁷	203	3097 ⁸³	64
27	2208 ⁵⁴	180	2388 ⁵⁹	187	2575 ⁶⁴	195	2770 ⁶⁸	201	2971 ⁷⁴	209	3180 ⁷⁸	63
28	2262 ⁵²	185	2447 ⁵⁶	192	2639 ⁶⁰	199	2838 ⁶⁵	207	3045 ⁶⁹	213	3258 ⁷⁵	62
29	2314 ⁴⁹	189	2503 ⁵³	196	2699 ⁵⁸	204	2903 ⁶²	211	3114 ⁶⁶	219	3333 ⁷¹	61
30	2363 ⁴⁶	193	2556 ⁵⁰	201	2757 ⁵⁵	208	2965 ⁵⁸	215	3180 ⁶²	224	3404 ⁶⁶	60
31	2409 ⁴³	197	2606 ⁴⁷	204	2810 ⁵¹	212	3023 ⁵⁴	219	3242 ⁵⁹	228	3470 ⁶²	59
32	2452 ⁴¹	201	2653 ⁴³	208	2861 ⁴⁷	216	3077 ⁵⁰	224	3301 ⁵⁴	231	3532 ⁵⁸	58
33	2493 ³⁷	203	2696 ⁴⁰	212	2908 ⁴³	219	3127 ⁴⁷	228	3355 ⁵⁰	235	3590 ⁵⁴	57
34	2530 ³⁴	206	2736 ³⁷	215	2951 ⁴⁰	223	3174 ⁴³	231	3405 ⁴⁶	239	3644 ⁴⁹	56
35	2564 ³¹	209	2773 ³⁴	218	2991 ³⁶	226	3217 ³⁸	234	3451 ⁴¹	242	3693 ⁴⁴	55
36	2595 ²⁸	212	2807 ³⁰	220	3027 ³²	228	3255 ³⁵	237	3492 ³⁸	245	3737 ⁴⁰	54
37	2623 ²⁴	214	2837 ²⁷	222	3059 ²⁹	231	3290 ³¹	240	3530 ³³	247	3777 ³⁶	53
38	2647 ²²	217	2864 ²³	224	3088 ²⁵	233	3321 ²⁷	242	3563 ²⁸	250	3813 ³⁰	52
39	2669 ¹⁸	218	2887 ¹⁹	226	3113 ²¹	235	3348 ²³	243	3591 ²⁵	252	3843 ²⁷	51
40	2687 ¹⁵	219	2906 ¹⁶	228	3134 ¹⁷	237	3371 ¹⁸	245	3616 ²⁰	254	3870 ²¹	50
41	2702 ¹¹	220	2922 ¹³	229	3151 ¹⁴	238	3389 ¹⁵	247	3636 ¹⁵	255	3891 ¹⁷	49
42	2713 ⁸	222	2935 ⁹	230	3165 ¹¹	239	3404 ¹⁰	247	3651 ¹¹	257	3908 ¹¹	48
43	2721 ⁵	223	2944 ⁵	230	3174 ⁶	240	3414 ⁶	248	3662 ⁷	257	3919 ⁷	47
44	2726 ²	223	2949 ²	231	3180 ²	240	3420 ²	249	3669 ²	257	3926 ³	46
45	2728	223	2951	231	3182	240	3422	249	3671	258	3929	45

$$f(a \pm nw) = f(a) \pm n \left[f'(a) \pm n \frac{f''(a)}{2} \right], \quad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$$B = (d' - \delta)^2.$$

 d is always numerically larger than δ .

$$0''.00001 = 0''.036 = \text{unit.}$$

$\frac{a-a_0}{\delta}$	$\frac{v}{2}$	3°.0	hor. diff.	3°.1	hor. diff.	3°.2	hor. diff.	3°.3	hor. diff.	3°.4	hor. diff.	3°.5	$\frac{a-a_0}{d}$
0		000	0	000	00	000	00	000	00	000	00	000	90
1		137 ¹³⁷	9	146 ¹⁴⁶	10	156 ¹⁵⁶	10	166 ¹⁶⁶	10	176 ¹⁷⁶	11	187 ¹⁸⁷	89
2		274 ¹³⁷	19	293 ¹⁴⁷	19	312 ¹⁵⁶	20	332 ¹⁶⁶	20	352 ¹⁷⁶	21	373 ¹⁸⁶	88
3		411 ¹³⁷	28	439 ¹⁴⁶	29	468 ¹⁵⁵	29	497 ¹⁶⁵	31	528 ¹⁷⁵	32	560 ¹⁸⁵	87
4		547 ¹³⁶	37	584 ¹⁴⁵	39	623 ¹⁵⁴	39	662 ¹⁶⁴	41	703 ¹⁷⁴	42	745 ¹⁸⁵	86
5		683 ¹³⁴	46	729 ¹⁴⁴	48	777 ¹⁵³	49	826 ¹⁶³	51	877 ¹⁷³	53	930 ¹⁸³	85
6		817 ¹³⁴	56	873 ¹⁴³	57	930 ¹⁵²	59	989 ¹⁶²	61	1050 ¹⁷²	63	1113 ¹⁸²	84
7		951 ¹³³	65	1016 ¹⁴¹	66	1082 ¹⁵¹	69	1151 ¹⁶¹	71	1222 ¹⁷⁰	73	1295 ¹⁸¹	83
8		1084 ¹³¹	73	1157 ¹⁴⁰	76	1233 ¹⁴⁹	79	1312 ¹⁵⁸	80	1392 ¹⁶⁹	84	1476 ¹⁷⁸	82
9		1215 ¹³⁰	82	1297 ¹³⁹	85	1382 ¹⁴⁸	88	1470 ¹⁵⁷	91	1561 ¹⁶⁷	93	1654 ¹⁷⁷	81
10		1345 ¹²⁸	91	1436 ¹³⁷	94	1530 ¹⁴⁶	97	1627 ¹⁵⁵	101	1728 ¹⁶⁴	103	1831 ¹⁷⁴	80
11		1473 ¹²⁶	100	1573 ¹³⁵	103	1676 ¹⁴⁴	106	1782 ¹⁵³	110	1892 ¹⁶²	113	2005 ¹⁷²	79
12		1599 ¹²⁴	109	1708 ¹³²	112	1820 ¹⁴¹	115	1935 ¹⁵¹	119	2054 ¹⁶⁰	123	2177 ¹⁷⁰	78
13		1723 ¹²³	117	1840 ¹³¹	121	1961 ¹³⁹	125	2086 ¹⁴⁸	128	2214 ¹⁵⁷	133	2347 ¹⁶⁶	77
14		1846 ¹²⁰	125	1971 ¹²⁸	129	2100 ¹³⁷	134	2234 ¹⁴⁵	137	2371 ¹⁵⁴	142	2513 ¹⁶³	76
15		1966 ¹¹⁷	133	2099 ¹²⁶	138	2237 ¹³⁴	142	2379 ¹⁴²	146	2525 ¹⁵²	151	2676 ¹⁶¹	75
16		2083 ¹¹⁵	142	2225 ¹²²	146	2371 ¹³⁰	150	2521 ¹³⁹	156	2677 ¹⁴⁷	160	2837 ¹⁵⁶	74
17		2198 ¹¹³	149	2347 ¹²⁰	154	2501 ¹²⁸	159	2660 ¹³⁶	164	2824 ¹⁴⁵	169	2993 ¹⁵³	73
18		2311 ¹⁰⁹	156	2467 ¹¹⁷	162	2629 ¹²⁵	167	2796 ¹³³	173	2969 ¹⁴⁰	177	3146 ¹⁴⁹	72
19		2420 ¹⁰⁷	164	2584 ¹¹⁴	170	2754 ¹²¹	175	2929 ¹²⁹	180	3109 ¹³⁷	186	3295 ¹⁴⁵	71
20		2527 ¹⁰³	171	2698 ¹¹¹	177	2875 ¹¹⁸	183	3058 ¹²⁵	188	3246 ¹³³	194	3440 ¹⁴¹	70
21		2630 ¹⁰⁰	179	2809 ¹⁰⁷	184	2993 ¹¹⁴	190	3183 ¹²²	196	3379 ¹²⁹	202	3581 ¹³⁷	69
22		2730 ⁹⁷	186	2916 ¹⁰³	191	3107 ¹¹¹	198	3305 ¹¹⁷	203	3508 ¹²⁵	210	3718 ¹³²	68
23		2827 ⁹⁴	192	3019 ¹⁰⁰	199	3218 ¹⁰⁶	204	3422 ¹¹³	211	3633 ¹²⁰	217	3850 ¹²⁷	67
24		2921 ⁹⁰	198	3119 ⁹⁶	205	3324 ¹⁰²	211	3535 ¹⁰⁹	218	3753 ¹¹⁵	224	3977 ¹²³	66
25		3011 ⁸⁶	204	3215 ⁹²	211	3426 ⁹⁸	218	3644 ¹⁰⁴	224	3868 ¹¹¹	232	4100 ¹¹⁷	65
26		3097 ⁸³	210	3307 ⁸⁸	217	3524 ⁹⁴	224	3748 ¹⁰⁰	231	3979 ¹⁰⁶	238	4217 ¹¹²	64
27		3180 ⁷⁸	215	3395 ⁸⁴	223	3618 ⁹⁰	230	3848 ⁹⁵	237	4085 ¹⁰¹	244	4329 ¹⁰⁷	63
28		3258 ⁷⁵	221	3479 ⁸⁰	229	3708 ⁸⁵	235	3943 ⁹¹	243	4186 ⁹⁶	250	4436 ¹⁰²	62
29		3333 ⁷¹	226	3559 ⁷⁵	234	3793 ⁸⁰	241	4034 ⁸⁵	248	4282 ⁹¹	256	4538 ⁹⁶	61
30		3404 ⁶⁶	230	3634 ⁷¹	239	3873 ⁷⁶	246	4119 ⁸¹	254	4373 ⁸⁵	261	4634 ⁹¹	60
31		3470 ⁶²	235	3705 ⁶⁷	244	3949 ⁷⁰	251	4200 ⁷⁵	258	4458 ⁸⁰	267	4725 ⁸⁴	59
32		3532 ⁵⁸	240	3772 ⁶²	247	4019 ⁶⁶	256	4275 ⁷⁰	263	4538 ⁷⁴	271	4809 ⁷⁹	58
33		3590 ⁵⁴	244	3834 ⁵⁷	251	4085 ⁶¹	260	4345 ⁶⁵	267	4612 ⁶⁹	276	4888 ⁷³	57
34		3644 ⁴⁹	247	3891 ⁵²	255	4146 ⁵⁶	264	4410 ⁵⁹	271	4681 ⁶³	280	4961 ⁶⁷	56
35		3693 ⁴⁴	250	3943 ⁴⁸	259	4202 ⁵¹	267	4469 ⁵⁴	275	4744 ⁵⁷	284	5028 ⁶⁰	55
36		3737 ⁴⁰	254	3991 ⁴³	262	4253 ⁴⁵	270	4523 ⁴⁸	278	4801 ⁵²	287	5088 ⁵⁵	54
37	3	3777 ³⁶	257	4034 ³⁷	264	4298 ⁴⁰	273	4571 ⁴³	282	4853 ⁴⁵	290	5143 ⁴⁸	53
38	3	3813 ³⁰	258	4071 ³³	267	4338 ³⁵	276	4614 ³⁷	284	4898 ⁴⁰	293	5191 ⁴²	52
39	3	3843 ²⁷	261	4104 ²⁸	269	4373 ³⁰	278	4651 ³²	287	4938 ³³	295	5233 ³⁵	51
40	3	3870 ²¹	262	4132 ²³	271	4403 ²⁴	280	4683 ²⁶	288	4971 ²⁷	297	5268 ²⁹	50
41	3	3891 ¹⁷	264	4155 ¹⁸	272	4427 ¹⁹	282	4709 ²⁰	289	4998 ²²	299	5297 ²³	49
42	3	3908 ¹¹	265	4173 ¹²	273	4446 ¹⁴	283	4729 ¹⁴	291	5020 ¹⁵	300	5320 ¹⁶	48
43	3	3919 ⁷	266	4185 ⁸	275	4460 ⁸	283	4743 ⁹	292	5035 ⁹	301	5336 ⁹	47
44	3	3926 ³	267	4193 ²	275	4468 ²	284	4752 ²	292	5044 ³	301	5345 ³	46
45	3	3929	266	4195	275	4470	284	4754	293	5047	301	5348	45

$$f(a \pm nv) = f(a) \pm n \left[f'(a) \pm n \frac{f''(a)}{2} \right], \quad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$$B = (d - \delta)^\circ.$$

d is always numerically larger than δ .

$$0''.00001 = 0''.036 = \text{unit.}$$

δ	$\frac{v}{2}$	3°.5	hor. diff.	3°.6	hor. diff.	3°.7	hor. diff.	3°.8	hor. diff.	3°.9	hor. diff.	4°.0	d
0		000	00	000	00	000	00	000	00	000	00	000	90
1		187 ¹⁸⁷	11	198 ¹⁹⁸	11	209 ²⁰⁹	11	220 ²²⁰	12	232 ²³²	12	244 ²⁴⁴	89
2		373 ¹⁸⁶	22	395 ¹⁹⁷	22	417 ²⁰⁸	23	440 ²²⁰	24	464 ²³¹	24	488 ²⁴³	88
3		560 ¹⁸⁵	32	592 ¹⁹⁶	33	625 ²⁰⁸	35	660 ²¹⁹	35	695 ²³¹	36	731 ²⁴³	87
4		745 ¹⁸⁵	43	788 ¹⁹⁶	45	833 ²⁰⁶	46	879 ²¹⁷	47	926 ²²⁹	48	974 ²⁴¹	86
5		930 ¹⁸³	54	984 ¹⁹⁴	55	1039 ²⁰⁵	57	1096 ²¹⁶	59	1155 ²²⁸	60	1215 ²³⁹	85
6		1113 ¹⁸²	65	1178 ¹⁹²	66	1244 ²⁰⁴	68	1312 ²¹⁵	71	1383 ²²⁶	71	1454 ²³⁸	84
7		1295 ¹⁸¹	75	1370 ¹⁹¹	78	1448 ²⁰¹	79	1527 ²¹³	82	1609 ²²⁴	83	1692 ²³⁶	83
8		1476 ¹⁷⁸	85	1561 ¹⁸⁹	88	1649 ²⁰⁰	91	1740 ²¹¹	93	1833 ²²²	95	1928 ²³⁴	82
9		1654 ¹⁷⁷	96	1750 ¹⁸⁷	99	1849 ¹⁹⁸	102	1951 ²⁰⁸	104	2055 ²¹⁹	107	2162 ²³⁰	81
10		1831 ¹⁷⁴	106	1937 ¹⁸⁵	110	2047 ¹⁹⁴	112	2159 ²⁰⁵	115	2274 ²¹⁷	118	2392 ²²⁸	80
11		2005 ¹⁷²	117	2122 ¹⁸²	119	2241 ¹⁹³	123	2364 ²⁰³	127	2491 ²¹³	129	2620 ²²⁵	79
12		2177 ¹⁷⁰	127	2304 ¹⁷⁹	130	2434 ¹⁸⁹	133	2567 ²⁰⁰	137	2704 ²¹¹	141	2845 ²²¹	78
13		2347 ¹⁶⁶	136	2483 ¹⁷⁶	140	2623 ¹⁸⁶	144	2767 ¹⁹⁶	148	2915 ²⁰⁶	151	3066 ²¹⁸	77
14		2513 ¹⁶³	146	2659 ¹⁷³	150	2809 ¹⁸³	154	2963 ¹⁹³	158	3121 ²⁰³	163	3284 ²¹³	76
15		2676 ¹⁶¹	156	2832 ¹⁶⁹	160	2992 ¹⁷⁹	164	3156 ¹⁸⁸	168	3324 ¹⁹⁹	173	3497 ²⁰⁹	75
16		2837 ¹⁵⁶	164	3001 ¹⁶⁶	170	3171 ¹⁷⁵	173	3344 ¹⁸⁵	179	3523 ¹⁹⁵	183	3706 ²⁰⁵	74
17		2993 ¹⁵³	174	3167 ¹⁶²	179	3346 ¹⁷¹	183	3529 ¹⁸¹	189	3718 ¹⁹⁰	193	3911 ²⁰⁰	73
18		3146 ¹⁴⁹	183	3329 ¹⁵⁸	188	3517 ¹⁶⁶	193	3710 ¹⁷⁵	198	3908 ¹⁸⁵	203	4111 ¹⁹⁵	72
19		3295 ¹⁴⁵	192	3487 ¹⁵³	196	3683 ¹⁶²	202	3885 ¹⁷¹	208	4093 ¹⁸⁰	213	4306 ¹⁸⁹	71
20		3440 ¹⁴¹	200	3640 ¹⁴⁹	205	3845 ¹⁵⁸	211	4056 ¹⁶⁷	217	4273 ¹⁷⁵	222	4495 ¹⁸⁴	70
21		3581 ¹³⁷	208	3789 ¹⁴⁵	214	4003 ¹⁵³	220	4223 ¹⁶¹	225	4448 ¹⁷⁰	231	4679 ¹⁷⁹	69
22		3718 ¹³²	216	3934 ¹³⁹	222	4156 ¹⁴⁷	228	4384 ¹⁵⁵	234	4618 ¹⁶⁴	240	4858 ¹⁷²	68
23		3850 ¹²⁷	223	4073 ¹³⁵	230	4303 ¹⁴²	236	4539 ¹⁵⁰	243	4782 ¹⁵⁸	248	5030 ¹⁶⁷	67
24	3	3977 ¹²³	231	4208 ¹³⁰	237	4445 ¹³⁷	244	4689 ¹⁴⁵	251	4940 ¹⁵²	257	5197 ¹⁶⁰	66
25	3	4100 ¹¹⁷	238	4338 ¹²⁴	244	4582 ¹³¹	252	4834 ¹³⁸	258	5092 ¹⁴⁵	265	5357 ¹⁵³	65
26	3	4217 ¹¹²	245	4462 ¹¹⁹	251	4713 ¹²⁶	259	4972 ¹³²	265	5237 ¹⁴⁰	273	5510 ¹⁴⁷	64
27	3	4329 ¹⁰⁷	252	4581 ¹¹³	258	4839 ¹²⁰	265	5104 ¹²⁷	273	5377 ¹³³	280	5657 ¹³⁹	63
28	3	4436 ¹⁰²	258	4694 ¹⁰⁷	265	4959 ¹¹³	272	5231 ¹¹⁹	279	5510 ¹²⁶	286	5796 ¹³³	62
29	3	4538 ⁹⁶	263	4801 ¹⁰²	271	5072 ¹⁰⁸	278	5350 ¹¹⁴	286	5636 ¹¹⁹	293	5929 ¹²⁶	61
30	3	4634 ⁹¹	269	4903 ⁹⁶	277	5180 ¹⁰¹	284	5464 ¹⁰⁶	291	5755 ¹¹³	300	6055 ¹¹⁸	60
31	3	4725 ⁸⁴	274	4999 ⁸⁹	282	5281 ⁹⁴	289	5570 ¹⁰⁰	298	5868 ¹⁰⁵	305	6173 ¹¹⁰	59
32	3	4809 ⁷⁹	279	5088 ⁸⁴	287	5375 ⁸⁸	295	5670 ⁹³	303	5973 ⁹⁷	310	6283 ¹⁰³	58
33	3	4888 ⁷³	284	5172 ⁷⁷	291	5463 ⁸²	300	5763 ⁸⁶	307	6070 ⁹¹	316	6386 ⁹⁵	57
34	3	4961 ⁶⁷	288	5249 ⁷⁰	296	5545 ⁷⁴	304	5849 ⁷⁸	312	6161 ⁸³	320	6481 ⁸⁷	56
35	3	5028 ⁶⁰	291	5319 ⁶⁴	300	5619 ⁶⁸	308	5927 ⁷²	317	6244 ⁷⁵	324	6568 ⁸⁰	55
36	3	5088 ⁵⁵	295	5383 ⁵⁸	304	5687 ⁶¹	312	5999 ⁶⁴	320	6319 ⁶⁸	329	6648 ⁷¹	54
37	3	5143 ⁴⁸	298	5441 ⁵¹	307	5748 ⁵⁴	315	6063 ⁵⁷	324	6387 ⁵⁹	332	6719 ⁶²	53
38	3	5191 ⁴²	301	5492 ⁴⁴	310	5802 ⁴⁶	318	6120 ⁴⁹	326	6446 ⁵²	335	6781 ⁵⁵	52
39	3	5233 ³⁵	303	5536 ³⁸	312	5848 ³⁹	321	6169 ⁴²	329	6498 ⁴⁴	338	6836 ⁴⁶	51
40	3	5268 ²⁹	306	5574 ³⁰	314	5888 ³²	323	6211 ³⁴	331	6542 ³⁶	340	6882 ³⁸	50
41	3	5297 ²³	307	5604 ²⁴	316	5920 ²⁶	325	6245 ²⁷	333	6578 ²⁸	342	6920 ³⁰	49
42	4	5320 ¹⁶	308	5628 ¹⁷	318	5946 ¹⁷	326	6272 ¹⁸	334	6606 ²⁰	344	6950 ²¹	48
43	4	5336 ⁹	309	5645 ¹⁰	318	5963 ¹¹	327	6290 ¹²	336	6626 ¹²	345	6971 ¹²	47
44	4	5345 ³	310	5655 ⁴	319	5974 ⁴	328	6302 ³	336	6638 ⁴	345	6983 ⁴	46
45	4	5348 ³	311	5659 ⁴	319	5978 ⁴	327	6305 ³	337	6642 ⁴	345	6987 ⁴	45

$$f(a \pm nw) = f(a) \pm n \left[f'(a) \pm n \frac{f''(a)}{2} \right], \quad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$$B = (d - \delta)^\circ.$$

 d is always numerically larger than δ .

$$0''.00001 = 0''.036 = \text{unit.}$$

δ	$\frac{v}{2}$	4°.0	hor. diff.	4°.1	hor. diff.	4°.2	hor. diff.	4°.3	hor. diff.	4°.4	hor. diff.	4°.5	δ
0		000	00	000	00	000	00	000	00	000	00	000	90
1		244 ²⁴⁴	12	256 ²⁵⁶	13	269 ²⁶⁹	13	282 ²⁸²	14	296 ²⁹⁶	13	309 ³⁰⁹	89
2		488 ²⁴³	25	513 ²⁵⁵	25	538 ²⁶⁸	26	564 ²⁸¹	27	591 ²⁹⁴	27	618 ³⁰⁸	88
3		731 ²⁴³	37	768 ²⁵⁵	38	806 ²⁶⁸	39	845 ²⁸⁰	40	885 ²⁹⁴	41	926 ³⁰⁷	87
4		974 ²⁴¹	49	1023 ²⁵³	51	1074 ²⁶⁵	51	1125 ²⁷⁹	54	1179 ²⁹¹	54	1233 ³⁰⁵	86
5		1215 ²³⁹	61	1276 ²⁵²	63	1339 ²⁶⁵	65	1404 ²⁷⁷	66	1470 ²⁹¹	68	1538 ³⁰⁴	85
6		1454 ²³⁸	74	1528 ²⁵⁰	76	1604 ²⁶²	77	1681 ²⁷⁵	80	1761 ²⁸⁸	81	1842 ³⁰¹	84
7		1692 ²³⁶	86	1778 ²⁴⁸	88	1866 ²⁶⁰	90	1956 ²⁷³	93	2049 ²⁸⁵	94	2143 ²⁹⁹	83
8		1928 ²³⁴	98	2026 ²⁴⁵	100	2126 ²⁵⁸	103	2229 ²⁷⁰	105	2334 ²⁸³	108	2442 ²⁹⁵	82
9		2162 ²³⁰	109	2271 ²⁴³	113	2384 ²⁵⁴	115	2499 ²⁶⁷	118	2617 ²⁷⁹	120	2737 ²⁹³	81
10		2392 ²²⁸	122	2514 ²³⁹	124	2638 ²⁵²	128	2766 ²⁶³	130	2896 ²⁷⁶	134	3030 ²⁸⁸	80
11		2620 ²²⁵	133	2753 ²³⁶	137	2890 ²⁴⁷	139	3029 ²⁶⁰	143	3172 ²⁷²	146	3318 ²⁸⁵	79
12		2845 ²²¹	144	2989 ²³³	148	3137 ²⁴⁴	152	3289 ²⁵⁶	155	3444 ²⁶⁸	159	3603 ²⁸⁰	78
13		3066 ²¹⁸	156	3222 ²²⁸	159	3381 ²⁴⁰	164	3545 ²⁵¹	167	3712 ²⁶³	171	3883 ²⁷⁵	77
14		3284 ²¹³	166	3450 ²²⁵	171	3621 ²³⁵	175	3796 ²⁴⁷	179	3975 ²⁵⁸	183	4158 ²⁷¹	76
15		3497 ²⁰⁹	178	3675 ²¹⁹	181	3856 ²³¹	187	4043 ²⁴²	190	4233 ²⁵³	196	4429 ²⁶⁴	75
16	3	3706 ²⁰⁵	188	3894 ²¹⁵	193	4087 ²²⁶	198	4285 ²³⁶	201	4486 ²⁴⁸	207	4693 ²⁵⁹	74
17	3	3911 ²⁰⁰	198	4109 ²¹⁰	204	4313 ²²⁰	208	4521 ²³¹	213	4734 ²⁴²	218	4952 ²⁵⁴	73
18	3	4111 ¹⁹⁵	208	4319 ²⁰⁵	214	4533 ²¹⁵	219	4752 ²²⁵	224	4976 ²³⁶	230	5206 ²⁴⁶	72
19	3	4306 ¹⁸⁹	218	4524 ¹⁹⁹	224	4748 ²⁰⁹	229	4977 ²¹⁹	235	5212 ²³⁰	240	5452 ²⁴⁰	71
20	3	4495 ¹⁸⁴	228	4723 ¹⁹⁴	234	4957 ²⁰³	239	5196 ²¹³	246	5442 ²²²	250	5692 ²³³	70
21	3	4679 ¹⁷⁹	238	4917 ¹⁸⁷	243	5160 ¹⁹⁷	249	5409 ²⁰⁶	255	5664 ²¹⁶	261	5925 ²²⁶	69
22	4	4858 ¹⁷²	246	5104 ¹⁸¹	253	5357 ¹⁹⁰	258	5615 ²⁰⁰	265	5880 ²⁰⁹	271	6151 ²¹⁹	68
23	4	5030 ¹⁶⁷	255	5285 ¹⁷⁵	262	5547 ¹⁸³	268	5815 ¹⁹²	274	6089 ²⁰¹	281	6370 ²¹⁰	67
24	4	5197 ¹⁶⁰	263	5460 ¹⁶⁸	270	5730 ¹⁷⁷	277	6007 ¹⁸⁵	283	6290 ¹⁹⁴	290	6580 ²⁰²	66
25	4	5357 ¹⁵³	271	5628 ¹⁶¹	279	5907 ¹⁶⁹	285	6192 ¹⁷⁷	292	6484 ¹⁸⁵	298	6782 ¹⁹⁵	65
26	4	5510 ¹⁴⁷	279	5789 ¹⁵⁴	287	6076 ¹⁶²	293	6369 ¹⁷⁰	300	6669 ¹⁷⁸	308	6977 ¹⁸⁵	64
27	4	5657 ¹³⁹	286	5943 ¹⁴⁷	295	6238 ¹⁵⁴	301	6539 ¹⁶¹	308	6847 ¹⁶⁹	315	7162 ¹⁷⁷	63
28	4	5796 ¹³³	294	6090 ¹⁴⁰	302	6392 ¹⁴⁶	308	6700 ¹⁵⁴	316	7016 ¹⁶¹	323	7339 ¹⁶⁸	62
29	4	5929 ¹²⁶	301	6230 ¹³²	308	6538 ¹³⁸	316	6854 ¹⁴⁵	323	7177 ¹⁵¹	330	7507 ¹⁵⁹	61
30	4	6055 ¹¹⁸	307	6362 ¹²⁴	314	6676 ¹³⁰	323	6999 ¹³⁶	329	7328 ¹⁴³	338	7666 ¹⁴⁹	60
31	4	6173 ¹¹⁰	313	6486 ¹¹⁶	320	6806 ¹²²	329	7135 ¹²⁸	336	7471 ¹³⁴	344	7815 ¹⁴⁰	59
32	4	6283 ¹⁰³	319	6602 ¹⁰⁸	326	6928 ¹¹⁴	335	7263 ¹¹⁹	342	7605 ¹²⁵	350	7955 ¹³⁰	58
33	5	6386 ⁹⁵	324	6710 ¹⁰⁰	332	7042 ¹⁰⁵	340	7382 ¹⁰⁹	348	7730 ¹¹⁵	355	8085 ¹²¹	57
34	5	6481 ⁸⁷	329	6810 ⁹¹	337	7147 ⁹⁶	344	7491 ¹⁰¹	354	7845 ¹⁰⁵	361	8206 ¹¹⁰	56
35	5	6568 ⁸⁰	333	6901 ⁸³	342	7243 ⁹⁷	349	7592 ⁹²	358	7950 ⁹⁶	366	8316 ¹⁰⁰	55
36	5	6648 ⁷¹	336	6984 ⁷⁵	346	7330 ⁹⁸	354	7684 ⁸²	362	8046 ⁸⁶	370	8416 ⁹⁰	54
37	5	6719 ⁶²	340	7059 ⁶⁶	349	7408 ⁶⁹	358	7766 ⁷²	366	8132 ⁷⁶	374	8506 ⁷⁹	53
38	5	6781 ⁵⁵	344	7125 ⁵⁸	352	7477 ⁶¹	361	7838 ⁶³	370	8208 ⁶⁶	377	8585 ⁶⁹	52
39	5	6836 ⁴⁶	347	7183 ⁴⁸	355	7538 ⁵¹	363	7901 ⁵⁴	373	8274 ⁵⁶	380	8654 ⁵⁹	51
40	5	6882 ³⁸	349	7231 ⁴⁰	358	7589 ⁴¹	366	7955 ⁴³	375	8330 ⁴⁵	383	8713 ⁴⁸	50
41	5	6920 ³⁰	351	7271 ³¹	359	7630 ³³	368	7998 ³⁴	377	8375 ³⁶	386	8761 ³⁷	49
42	5	6950 ²¹	352	7302 ²²	361	7663 ²³	369	8032 ²⁵	379	8411 ²⁵	387	8798 ²⁶	48
43	5	6971 ¹²	353	7324 ¹³	362	7686 ¹⁴	371	8057 ¹⁴	379	8436 ¹⁵	388	8824 ¹⁶	47
44	5	6983 ⁴	354	7337 ⁴	363	7700 ⁴	371	8071 ⁴	380	8451 ⁵	389	8840 ⁵	46
45	5	6987	354	7341	363	7704	371	8075	381	8456	389	8845	45

$$f(a \pm nw) = f(a) \pm n \left[f'(a) \pm n \frac{f''(a)}{2} \right], \quad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$$B = (d - \delta)^\circ.$$

d is always numerically larger than δ .

$$0''.00001 = 0''.036 = \text{unit.}$$

δ	$\frac{v}{2}$	4°.5	hor. diff.	4°.6	hor. diff.	4°.7	hor. diff.	4°.8	hor. diff.	4°.9	hor. diff.	5°.0	d
0		000	00	000	00	000	00	000	00	000	00	000	90
1		309 ³⁰⁹	14	323 ³²³	14	337 ³³⁷	15	352 ³⁵²	15	367 ³⁶⁷	15	382 ³⁸²	89
2		618 ³⁰⁹	28	646 ³²³	28	674 ³³⁷	29	703 ³⁵¹	30	733 ³⁶⁵	30	763 ³⁸¹	88
3		926 ³⁰⁸	42	968 ³²²	42	1010 ³³⁶	44	1054 ³⁴⁹	44	1098 ³⁶⁴	46	1144 ³⁷⁹	87
4		1233 ³⁰⁷	55	1288 ³²⁰	57	1345 ³³³	58	1403 ³⁴⁸	59	1462 ³⁶³	61	1523 ³⁷⁷	86
5		1538 ³⁰⁵	70	1608 ³¹⁷	70	1678 ³³²	73	1751 ³⁴⁵	74	1825 ³⁶⁰	75	1900 ³⁷⁵	85
6		1842 ³⁰⁴	83	1925 ³¹⁵	85	2010 ³²⁸	86	2096 ³⁴³	89	2185 ³⁵⁷	90	2275 ³⁷²	84
7		2143 ²⁹⁹	97	2240 ³¹²	98	2338 ³²⁶	101	2439 ³⁴⁰	103	2542 ³⁵⁴	105	2647 ³⁶⁹	83
8		2442 ²⁹⁵	110	2552 ³⁰⁹	112	2664 ³²³	115	2779 ³³⁶	117	2896 ³⁵¹	120	3016 ³⁶⁵	82
9		2737 ²⁹³	124	2861 ³⁰⁵	126	2987 ³¹⁹	128	3115 ³³³	132	3247 ³⁴⁷	134	3381 ³⁶¹	81
10		3030 ²⁸⁸	136	3166 ³⁰²	140	3306 ³¹⁵	142	3448 ³²⁹	146	3594 ³⁴²	148	3742 ³⁵⁷	80
11	3	3318 ²⁸⁵	150	3468 ²⁹⁷	153	3621 ³¹⁰	156	3777 ³²³	159	3936 ³³⁸	163	4099 ³⁵¹	79
12	3	3603 ²⁸⁰	162	3765 ²⁹³	166	3931 ³⁰⁵	169	4100 ³¹⁹	174	4274 ³³²	176	4450 ³⁴⁶	78
13	3	3883 ²⁷⁵	175	4058 ²⁸⁸	178	4236 ³⁰¹	183	4419 ³¹⁴	187	4606 ³²⁶	190	4796 ³⁴⁰	77
14	3	4158 ²⁷¹	188	4346 ²⁸²	191	4537 ²⁹⁵	196	4733 ³⁰⁷	199	4932 ³²¹	204	5136 ³³⁴	76
15	3	4429 ²⁶⁴	199	4628 ²⁷⁷	204	4832 ²⁸⁹	208	5040 ³⁰²	213	5253 ³¹⁴	217	5470 ³²⁷	75
16	3	4693 ²⁵⁹	212	4905 ²⁷¹	216	5121 ²⁸³	221	5342 ²⁹⁵	225	5567 ³⁰⁷	230	5797 ³²⁰	74
17	3	4952 ²⁵⁴	224	5176 ²⁶⁴	228	5404 ²⁷⁶	233	5637 ²⁸⁸	237	5874 ³⁰¹	243	6117 ³¹³	73
18	4	5206 ²⁴⁶	234	5440 ²⁵⁸	240	5680 ²⁶⁹	245	5925 ²⁸⁰	250	6175 ²⁹²	255	6430 ³⁰⁴	72
19	4	5452 ²⁴⁰	246	5698 ²⁵¹	251	5949 ²⁶²	256	6205 ²⁷⁴	262	6467 ²⁸⁵	267	6734 ²⁹⁷	71
20	4	5692 ²³³	257	5949 ²⁴³	262	6211 ²⁵⁴	268	6479 ²⁶⁵	273	6752 ²⁷⁶	279	7031 ²⁸⁸	70
21	4	5925 ²²⁶	267	6192 ²³⁶	273	6465 ²⁴⁶	279	6744 ²⁵⁷	284	7028 ²⁶⁸	291	7319 ²⁷⁹	69
22	4	6151 ²¹⁹	277	6428 ²²⁸	283	6711 ²³⁹	290	7001 ²⁴⁸	295	7296 ²⁵⁹	302	7598 ²⁶⁹	68
23	4	6370 ²¹⁰	286	6656 ²²⁰	294	6950 ²²⁹	299	7249 ²⁴⁰	306	7555 ²⁵⁰	312	7867 ²⁶⁰	67
24	5	6580 ²⁰²	296	6876 ²¹²	303	7179 ²²¹	310	7489 ²³⁰	316	7805 ²⁴⁰	322	8127 ²⁵⁰	66
25	5	6782 ¹⁹⁵	306	7088 ²⁰³	312	7400 ²¹²	319	7719 ²²¹	326	8045 ²³⁰	332	8377 ²⁴⁰	65
26	5	6977 ¹⁸⁵	314	7291 ¹⁹⁴	321	7612 ²⁰²	328	7940 ²¹¹	335	8275 ²²⁰	342	8617 ²²⁹	64
27	5	7162 ¹⁷⁷	323	7485 ¹⁸⁵	329	7814 ¹⁹³	337	8151 ²⁰²	344	8495 ²¹⁰	351	8846 ²¹⁹	63
28	5	7339 ¹⁶⁸	331	7670 ¹⁷⁵	337	8007 ¹⁸⁴	346	8353 ¹⁹¹	352	8705 ¹⁹⁹	360	9065 ²⁰⁷	62
29	5	7507 ¹⁵⁹	338	7845 ¹⁶⁶	346	8191 ¹⁷³	353	8544 ¹⁸⁰	360	8904 ¹⁸⁸	368	9272 ¹⁹⁶	61
30	5	7666 ¹⁴⁹	345	8011 ¹⁵⁶	353	8354 ¹⁶³	360	8724 ¹⁷⁰	368	9092 ¹⁷⁷	376	9468 ¹⁸⁴	60
31	5	7815 ¹⁴⁰	352	8167 ¹⁴⁶	360	8527 ¹⁵²	367	8894 ¹⁵⁹	375	9269 ¹⁶⁶	383	9652 ¹⁷³	59
32	5	7955 ¹³⁰	358	8313 ¹³⁶	366	8679 ¹⁴²	374	9053 ¹⁴⁸	382	9435 ¹⁵⁵	390	9825 ¹⁶¹	58
33	6	8085 ¹²¹	364	8449 ¹²⁶	372	8821 ¹³²	380	9201 ¹³⁷	389	9590 ¹⁴²	396	9986 ¹⁴⁸	57
34	6	8206 ¹¹⁰	369	8575 ¹¹⁵	378	8953 ¹²⁰	385	9338 ¹²⁶	394	9732 ¹³¹	402	10134 ¹³⁶	56
35	6	8316 ¹⁰⁰	374	8690 ¹⁰⁵	383	9073 ¹⁰⁹	391	9464 ¹¹⁴	399	9863 ¹¹⁸	407	10270 ¹²⁴	55
36	6	8416 ⁹⁰	379	8795 ⁹⁴	387	9182 ⁹⁸	396	9578 ¹⁰²	403	9981 ¹⁰⁷	413	10394 ¹¹¹	54
37	6	8506 ⁷⁹	383	8889 ⁸³	391	9280 ⁸⁷	400	9680 ⁹⁰	408	10088 ⁹⁴	417	10505 ⁹⁸	53
38	6	8585 ⁶⁹	387	8972 ⁷²	395	9367 ⁷⁵	403	9770 ⁷⁹	412	10182 ⁸²	421	10603 ⁸⁵	52
39	6	8654 ⁵⁹	390	9044 ⁶¹	398	9442 ⁶⁴	407	9849 ⁶⁶	415	10264 ⁶⁹	424	10688 ⁷²	51
40	6	8713 ⁴⁸	392	9105 ⁵⁰	401	9506 ⁵²	409	9915 ⁵⁵	418	10333 ⁵⁷	427	10760 ⁵⁹	50
41	6	8761 ³⁷	394	9155 ³⁹	403	9558 ⁴⁰	412	9970 ⁴²	420	10390 ⁴⁴	429	10819 ⁴⁵	49
42	6	8798 ²⁶	396	9194 ²⁷	404	9598 ²⁹	414	10012 ³⁰	422	10434 ³¹	430	10864 ³³	48
43	6	8824 ¹⁶	397	9221 ¹⁷	406	9627 ¹⁷	415	10042 ¹⁸	423	10465 ¹⁹	432	10897 ¹⁹	47
44	6	8840 ⁵	398	9238 ⁵	406	9644 ⁶	416	10060 ⁵	424	10484 ⁵	432	10916 ⁶	46
45	6	8845	398	9243	407	9650	415	10065	424	10489	433	10922	45

$$f(a \pm nw) = f(a) \pm n \left[f'(a) \pm n \frac{f''(a)}{2} \right], \quad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$$B = (d - \delta)^\circ.$$

d is always numerically larger than δ .

$$0.00001 = 0.036 = \text{unit.}$$

δ	$\frac{v}{2}$	$5^\circ.0$	hor. diff.	$5^\circ.1$	hor. diff.	$5^\circ.2$	hor. diff.	$5^\circ.3$	hor. diff.	$5^\circ.4$	hor. diff.	$5^\circ.5$	δ
0		000	00	000	00	000	00	000	00	000	00	000	90
1		382 ³⁸²	15	397 ³⁹⁷	16	413 ⁴¹³	16	429 ⁴²⁹	17	446 ⁴⁴⁶	16	462 ⁴⁶²	89
2		763 ³⁸¹	31	794 ³⁹⁶	32	826 ⁴¹¹	32	858 ⁴²⁸	33	891 ⁴⁴⁴	33	924 ⁴⁶¹	88
3		1144 ³⁷⁹	46	1190 ³⁹⁵	47	1237 ⁴¹¹	49	1286 ⁴²⁶	49	1335 ⁴⁴²	50	1385 ⁴⁵⁹	87
4		1523 ³⁷⁷	62	1585 ³⁹²	63	1648 ⁴⁰⁸	64	1712 ⁴²⁴	65	1777 ⁴⁴¹	67	1844 ⁴⁵⁷	86
5		1900 ³⁷⁵	77	1977 ³⁹⁰	79	2056 ⁴⁰⁵	80	2136 ⁴²¹	82	2218 ⁴³⁷	83	2301 ⁴⁵⁴	85
6		2275 ³⁷²	92	2367 ³⁸⁸	94	2461 ⁴⁰³	96	2557 ⁴¹⁹	98	2655 ⁴³⁴	100	2755 ⁴⁵⁰	84
7		2647 ³⁶⁹	108	2755 ³⁸³	109	2864 ³⁹⁹	112	2976 ⁴¹⁴	113	3089 ⁴³¹	116	3205 ⁴⁴⁷	83
8		3016 ³⁶⁵	122	3138 ³⁸⁰	125	3263 ³⁹⁵	127	3390 ⁴¹¹	130	3520 ⁴²⁶	132	3652 ⁴⁴²	82
9		3381 ³⁶¹	137	3518 ³⁷⁶	140	3658 ³⁹¹	143	3801 ⁴⁰⁶	145	3946 ⁴²¹	148	4094 ⁴³⁷	81
10	3	3742 ³⁵⁷	152	3894 ³⁷¹	155	4049 ³⁸⁵	158	4207 ⁴⁰⁰	160	4367 ⁴¹⁶	164	4531 ⁴³²	80
11	3	4099 ³⁵¹	166	4265 ³⁶⁶	169	4434 ³⁸¹	173	4607 ³⁹⁵	176	4783 ⁴¹⁰	180	4963 ⁴²⁵	79
12	3	4450 ³⁴⁶	181	4631 ³⁶⁰	184	4815 ³⁷⁴	187	5002 ³⁸⁹	191	5193 ⁴⁰⁴	195	5388 ⁴¹⁹	78
13	3	4796 ³⁴⁰	195	4991 ³⁵⁴	198	5189 ³⁶⁸	202	5391 ³⁸²	206	5597 ³⁹⁷	210	5807 ⁴¹²	77
14	3	5136 ³³⁴	209	5345 ³⁴⁷	212	5557 ³⁶¹	216	5773 ³⁷⁶	221	5994 ³⁹⁰	225	6219 ⁴⁰⁴	76
15	4	5470 ³²⁷	222	5692 ³⁴⁰	226	5918 ³⁵⁴	231	6149 ³⁶⁷	235	6384 ³⁸¹	239	6623 ³⁹⁷	75
16	4	5797 ³²⁰	235	6032 ³³³	240	6272 ³⁴⁶	244	6516 ³⁶⁰	249	6765 ³⁷⁴	255	7020 ³⁸⁷	74
17	4	6117 ³¹³	248	6365 ³²⁵	253	6618 ³³⁸	258	6876 ³⁵¹	263	7139 ³⁶⁵	268	7407 ³⁷⁸	73
18	4	6430 ³⁰⁴	260	6690 ³¹⁷	266	6956 ³³⁰	271	7227 ³⁴³	277	7504 ³⁵⁵	281	7785 ³⁶⁹	72
19	4	6734 ²⁹⁷	273	7007 ³⁰⁹	279	7286 ³²⁰	284	7570 ³³³	289	7859 ³⁴⁶	295	8154 ³⁵⁹	71
20	5	7031 ²⁸⁸	285	7316 ²⁹⁹	290	7606 ³¹²	297	7903 ³²³	302	8205 ³³⁶	308	8513 ³⁴⁸	70
21	5	7319 ²⁷⁹	296	7615 ²⁹⁰	303	7918 ³⁰¹	308	8226 ³¹⁴	315	8541 ³²⁵	320	8861 ³³⁷	69
22	5	7598 ²⁶⁹	307	7905 ²⁸¹	314	8219 ²⁹²	321	8540 ³⁰³	326	8866 ³¹⁴	332	9198 ³²⁷	68
23	5	7867 ²⁶⁰	319	8186 ²⁷⁰	325	8511 ²⁸¹	332	8843 ²⁹²	337	9180 ³⁰⁴	345	9525 ³¹⁴	67
24	5	8127 ²⁵⁰	329	8456 ²⁶¹	336	8792 ²⁷¹	343	9135 ²⁸¹	349	9484 ²⁹¹	355	9839 ³⁰³	66
25	5	8377 ²⁴⁰	340	8717 ²⁴⁹	346	9063 ²⁵⁹	353	9416 ²⁶⁹	359	9775 ²⁸⁰	367	10142 ²⁹⁰	65
26	5	8617 ²²⁹	349	8966 ²³⁹	356	9322 ²⁴⁸	363	9685 ²⁵⁸	370	10055 ²⁶⁸	377	10432 ²⁷⁸	64
27	6	8846 ²¹⁹	359	9205 ²²⁷	365	9570 ²³⁶	373	9943 ²⁴⁵	380	10323 ²⁵⁴	387	10710 ²⁶⁴	63
28	6	9065 ²⁰⁷	367	9432 ²¹⁵	374	9806 ²²⁴	382	10188 ²³³	389	10577 ²⁴²	397	10974 ²⁵¹	62
29	6	9272 ¹⁹⁶	375	9647 ²⁰⁴	383	10030 ²¹²	391	10421 ²²⁰	398	10819 ²²⁹	406	11225 ²³⁷	61
30	6	9468 ¹⁸⁴	383	9851 ¹⁹²	391	10242 ²⁰⁰	399	10641 ²⁰⁸	407	11048 ²¹⁵	414	11462 ²²³	60
31	6	9652 ¹⁷³	391	10043 ¹⁸⁰	399	10442 ¹⁸⁷	407	10849 ¹⁹³	414	11263 ²⁰¹	422	11685 ²⁰⁹	59
32	6	9825 ¹⁶¹	398	10223 ¹⁶⁷	406	10629 ¹⁷³	413	11042 ¹⁸¹	422	11464 ¹⁸⁸	430	11894 ¹⁹⁴	58
33	7	9986 ¹⁴⁸	404	10390 ¹⁵⁴	412	10802 ¹⁶¹	421	11223 ¹⁶⁷	429	11652 ¹⁷³	436	12088 ¹⁸⁰	57
34	7	10134 ¹³⁶	410	10544 ¹⁴²	419	10963 ¹⁴⁷	427	11390 ¹⁵³	435	11825 ¹⁵⁸	443	12268 ¹⁶⁴	56
35	7	10270 ¹²⁴	416	10686 ¹²⁸	424	11110 ¹³⁴	433	11543 ¹³⁸	440	11983 ¹⁴⁴	449	12432 ¹⁵⁰	55
36	7	10394 ¹¹¹	420	10814 ¹¹⁶	430	11244 ¹²⁰	437	11681 ¹²⁵	446	12127 ¹²⁹	455	12582 ¹³⁴	54
37	7	10505 ⁹⁸	425	10930 ¹⁰²	434	11364 ¹⁰⁶	442	11806 ¹¹⁰	450	12256 ¹¹⁵	460	12716 ¹¹⁸	53
38	7	10603 ⁸⁵	429	11032 ⁸⁸	438	11470 ⁹²	446	11916 ⁹⁵	455	12371 ⁹⁹	463	12834 ¹⁰³	52
39	7	10688 ⁷²	432	11120 ⁷⁵	442	11562 ⁷⁷	449	12011 ⁸¹	459	12470 ⁸⁴	467	12937 ⁸⁷	51
40	7	10760 ⁵⁹	435	11195 ⁶²	444	11639 ⁶⁴	453	12092 ⁶⁶	462	12554 ⁶⁸	470	13024 ⁷¹	50
41	7	10819 ⁴⁵	438	11257 ⁴⁷	446	11703 ⁵⁰	455	12158 ⁵²	464	12622 ⁵⁴	473	13095 ⁵⁵	49
42	7	10864 ³³	440	11304 ³⁴	449	11753 ³⁵	457	12210 ³⁶	466	12676 ³⁷	474	13150 ³⁹	48
43	7	10897 ¹⁹	441	11338 ²⁰	450	11788 ²¹	458	12246 ²²	467	12713 ²³	476	13189 ²⁴	47
44	8	10916 ⁶	442	11358 ⁶	451	11809 ⁶	459	12268 ⁶	468	12736 ⁶	477	13213 ⁷	46
45	8	10922	442	11364	451	11815	459	12274	468	12742	478	13220	45

$$f(a \pm nw) = f(a) \pm n \left[f'(a) \pm n \frac{f''(a)}{2} \right], \quad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$$B = (d - \delta)^\circ.$$

d is always numerically larger than δ .

$$0''.00001 = 0''.036 = \text{unit.}$$

δ	$\frac{v}{2}$	5°.5	hor. diff.	5°.6	hor. diff.	5°.7	hor. diff.	5°.8	hor. diff.	5°.9	hor. diff.	6°.0	d
0		000	00	000	00	000	00	000	00	000	00	000	90
1		462 ⁴⁶²	17	479 ⁴⁷⁹	18	497 ⁴⁹⁷	17	514 ⁵¹⁴	18	532 ⁵³²	19	551 ⁵⁵¹	89
2		924 ⁴⁶¹	34	958 ⁴⁷⁸	35	993 ⁴⁹⁶	35	1028 ⁵¹³	36	1064 ⁵³¹	37	1101 ⁵⁴⁸	88
3		1385 ⁴⁵⁹	51	1436 ⁴⁷⁶	52	1488 ⁴⁹³	53	1541 ⁵¹¹	54	1595 ⁵²⁸	54	1649 ⁵⁴⁷	87
4		1844 ⁴⁵⁷	68	1912 ⁴⁷⁴	69	1981 ⁴⁹¹	71	2052 ⁵⁰⁸	71	2123 ⁵²⁶	73	2196 ⁵⁴⁴	86
5		2301 ⁴⁵⁴	85	2386 ⁴⁷⁰	86	2472 ⁴⁸⁸	88	2560 ⁵⁰⁵	89	2649 ⁵²³	91	2740 ⁵⁴¹	85
6		2755 ⁴⁵⁰	101	2856 ⁴⁶⁷	104	2960 ⁴⁸⁴	105	3065 ⁵⁰¹	107	3172 ⁵¹⁹	109	3281 ⁵³⁶	84
7		3205 ⁴⁴⁷	118	3323 ⁴⁶³	121	3444 ⁴⁷⁹	122	3566 ⁴⁹⁷	125	3691 ⁵¹⁴	126	3817 ⁵³²	83
8	3	3652 ⁴⁴²	134	3786 ⁴⁵⁹	137	3923 ⁴⁷⁵	140	4063 ⁴⁹²	142	4205 ⁵⁰⁹	144	4349 ⁵²⁷	82
9	3	4094 ⁴³⁷	151	4245 ⁴⁵³	153	4398 ⁴⁷⁰	157	4555 ⁴⁸⁶	159	4714 ⁵⁰³	162	4876 ⁵²⁰	81
10	3	4531 ⁴³²	167	4698 ⁴⁴⁸	170	4868 ⁴⁶⁴	173	5041 ⁴⁸⁰	176	5217 ⁴⁹⁷	179	5396 ⁵¹⁴	80
11	4	4963 ⁴²⁵	183	5146 ⁴⁴¹	186	5332 ⁴⁵⁷	189	5521 ⁴⁷⁴	193	5714 ⁴⁹⁰	196	5910 ⁵⁰⁷	79
12	4	5388 ⁴¹⁹	199	5587 ⁴³⁴	202	5789 ⁴⁵⁰	206	5995 ⁴⁶⁵	209	6204 ⁴⁸²	213	6417 ⁴⁹⁹	78
13	4	5807 ⁴¹²	214	6021 ⁴²⁷	218	6239 ⁴⁴²	221	6460 ⁴⁵⁸	226	6686 ⁴⁷⁴	230	6916 ⁴⁹⁰	77
14	4	6219 ⁴⁰⁴	229	6448 ⁴¹⁹	233	6681 ⁴³⁴	237	6918 ⁴⁵⁰	242	7160 ⁴⁶⁶	246	7406 ⁴⁸¹	76
15	5	6623 ³⁹⁷	244	6867 ⁴¹¹	248	7115 ⁴²⁶	253	7368 ⁴⁴¹	258	7626 ⁴⁵⁵	261	7887 ⁴⁷²	75
16	5	7020 ³⁸⁷	258	7278 ⁴⁰¹	263	7541 ⁴¹⁶	268	7809 ⁴³¹	272	8081 ⁴⁴⁶	278	8359 ⁴⁶¹	74
17	5	7407 ³⁷⁸	272	7679 ³⁹³	278	7957 ⁴⁰⁶	283	8240 ⁴²¹	287	8527 ⁴³⁶	293	8820 ⁴⁵⁰	73
18	5	7785 ³⁶⁹	287	8072 ³⁸²	291	8363 ³⁹⁷	298	8661 ⁴¹⁰	302	8963 ⁴²⁴	307	9270 ⁴⁴⁰	72
19	5	8154 ³⁵⁹	300	8454 ³⁷²	306	8760 ³⁸⁵	311	9071 ³⁹⁹	316	9387 ⁴¹³	323	9710 ⁴²⁷	71
20	6	8513 ³⁴⁸	313	8826 ³⁶¹	319	9145 ³⁷⁴	325	9470 ³⁸⁷	330	9800 ⁴⁰¹	337	10137 ⁴¹⁴	70
21	6	8861 ³³⁷	326	9187 ³⁵⁰	332	9519 ³⁶³	338	9857 ³⁷⁶	344	10201 ³⁸⁹	350	10551 ⁴⁰²	69
22	6	9198 ³²⁷	339	9537 ³³⁸	345	9882 ³⁵⁰	351	10233 ³⁶³	357	10590 ³⁷⁵	363	10953 ³⁸⁹	68
23	6	9525 ³¹⁴	350	9875 ³²⁷	357	10232 ³³⁸	364	10596 ³⁵⁰	369	10965 ³⁶³	377	11342 ³⁷⁴	67
24	7	9839 ³⁰³	363	10202 ³¹³	368	10570 ³²⁵	376	10946 ³³⁶	382	11328 ³⁴⁸	388	11716 ³⁶⁰	66
25	7	10142 ²⁹⁰	373	10515 ³⁰¹	380	10895 ³¹²	387	11282 ³²³	394	11676 ³³⁴	400	12076 ³⁴⁶	65
26	7	10432 ²⁷⁸	384	10816 ²⁸⁸	391	11207 ²⁹⁸	398	11605 ³⁰⁸	405	12010 ³¹⁹	412	12422 ³³⁰	64
27	7	10710 ²⁶⁴	394	11104 ²⁷⁴	401	11505 ²⁸⁴	408	11913 ²⁹⁴	416	12329 ³⁰⁴	423	12752 ³¹⁴	63
28	7	10974 ²⁵¹	404	11378 ²⁶⁰	411	11789 ²⁶⁹	418	12207 ²⁷⁹	426	12633 ²⁸⁹	433	13066 ²⁹⁹	62
29	7	11225 ²³⁷	413	11638 ²⁴⁶	420	12058 ²⁵⁵	428	12486 ²⁶⁴	436	12922 ²⁷³	443	13365 ²⁸²	61
30	8	11462 ²²³	422	11884 ²³¹	429	12313 ²⁴⁰	437	12750 ²⁴⁸	445	13195 ²⁵⁷	452	13647 ²⁶⁶	60
31	8	11685 ²⁰⁹	430	12115 ²¹⁶	438	12553 ²²⁴	445	12998 ²³²	454	13452 ²⁴⁰	461	13913 ²⁴⁸	59
32	8	11894 ¹⁹⁴	437	12331 ²⁰²	446	12777 ²⁰⁹	453	13230 ²¹⁶	462	13692 ²²³	469	14161 ²³¹	58
33	8	12088 ¹⁸⁰	445	12533 ¹⁸⁶	453	12986 ¹⁹²	460	13446 ²⁰⁰	469	13915 ²⁰⁷	477	14392 ²¹⁴	57
34	8	12268 ¹⁶⁴	451	12719 ¹⁷¹	459	13178 ¹⁷⁷	468	13646 ¹⁸³	476	14122 ¹⁸⁹	484	14606 ¹⁹⁶	56
35	8	12432 ¹⁵⁰	458	12890 ¹⁵⁴	465	13355 ¹⁶⁰	474	13829 ¹⁶⁶	482	14311 ¹⁷²	491	14802 ¹⁷⁷	55
36	8	12582 ¹³⁴	462	13044 ¹³⁹	471	13515 ¹⁴⁴	480	13995 ¹⁴⁹	488	14483 ¹⁵⁴	496	14979 ¹⁵⁹	54
37	8	12716 ¹¹⁸	467	13183 ¹²³	476	13659 ¹²⁸	485	14144 ¹³²	493	14637 ¹³⁶	501	15138 ¹⁴¹	53
38	8	12834 ¹⁰³	472	13306 ¹⁰⁷	481	13787 ¹¹⁰	489	14276 ¹¹⁴	497	14773 ¹¹⁸	506	15279 ¹⁴³	52
39	8	12937 ⁸⁷	476	13413 ⁹⁰	484	13897 ⁹³	493	14390 ⁹⁷	501	14891 ¹⁰⁰	511	15402 ¹⁰³	51
40	8	13024 ⁷¹	479	13503 ⁷⁴	487	13990 ⁷⁷	497	14487 ⁸¹	504	14991 ⁸²	514	15505 ⁸⁵	50
41	8	13095 ⁵⁵	482	13577 ⁵⁷	490	14067 ⁵⁹	499	14566 ⁶¹	507	15073 ⁶³	517	15590 ⁶⁵	49
42	8	13150 ³⁹	484	13634 ⁴⁰	492	14126 ⁴²	501	14627 ⁴³	509	15136 ⁴⁵	519	15655 ⁴⁶	48
43	8	13189 ²⁴	485	13674 ²⁴	494	14168 ²⁵	502	14670 ²⁶	511	15181 ²⁷	520	15701 ²⁸	47
44	9	13213 ⁷	485	13698 ⁷	495	14193 ⁷	503	14696 ⁷	512	15208 ⁸	521	15729 ⁸	46
45	9	13220 ⁷	485	13705 ⁷	495	14200 ⁷	503	14703 ⁷	513	15216 ⁸	521	15737 ⁸	45
δ	$\frac{v}{2}$	5°.5	hor. diff.	5°.6	hor. diff.	5°.7	hor. diff.	5°.8	hor. diff.	5°.9	hor. diff.	6°.0	d

$$f(a \pm nw) = f(a) \pm n \left[f'(a) \pm n \frac{f''(a)}{2} \right], \quad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$$B = (d - \delta)^\circ.$$

 d is always numerically larger than δ .

 $0^{\circ}.00001 = 0^{\circ}.036 = \text{unit}.$

δ	$\frac{v}{2}$	6°.0	hor. diff.	6°.1	hor. diff.	6°.2	hor. diff.	6°.3	hor. diff.	6°.4	hor. diff.	6°.5	d
0		000	00	000	00	000	00	000	00	000	00	000	90
1		551 ⁵⁵¹	18	569 ⁵⁶⁹	19	588 ⁵⁸⁸	19	607 ⁶⁰⁷	20	627 ⁶²⁷	20	647 ⁶⁴⁷	89
2		1101 ⁵⁵⁰	37	1138 ⁵⁶⁹	38	1176 ⁵⁸⁸	38	1214 ⁶⁰⁷	39	1253 ⁶²⁷	40	1293 ⁶⁴⁷	88
3		1649 ⁵⁴⁸	56	1705 ⁵⁶⁷	57	1762 ⁵⁸⁶	57	1819 ⁶⁰⁵	59	1878 ⁶²⁵	59	1937 ⁶⁴⁴	87
4		2196 ⁵⁴⁷	74	2270 ⁵⁶⁵	76	2346 ⁵⁸⁴	76	2422 ⁶⁰³	78	2500 ⁶²²	79	2579 ⁶⁴²	86
5		2740 ⁵⁴⁴	93	2833 ⁵⁶³	94	2927 ⁵⁸¹	95	3022 ⁶⁰⁰	98	3120 ⁶²⁰	98	3218 ⁶³⁹	85
6		3281 ⁵⁴¹	111	3392 ⁵⁵⁹	112	3504 ⁵⁷⁷	115	3619 ⁵⁹⁷	116	3735 ⁶¹⁵	118	3853 ⁶³⁵	84
7	3	3817 ⁵³⁶	129	3946 ⁵⁵⁴	131	4077 ⁵⁷³	134	4211 ⁵⁹²	135	4346 ⁶¹¹	138	4484 ⁶³¹	83
8	3	4349 ⁵³²	147	4496 ⁵⁵⁰	149	4645 ⁵⁶⁸	152	4797 ⁵⁸⁶	154	4951 ⁶⁰⁵	157	5108 ⁶²⁴	82
9	4	4876 ⁵²⁷	164	5040 ⁵⁴⁴	168	5208 ⁵⁶³	170	5378 ⁵⁸¹	173	5551 ⁶⁰⁰	176	5727 ⁶¹⁹	81
10	4	5396 ⁵²⁰	182	5578 ⁵³⁸	186	5764 ⁵⁵⁶	188	5952 ⁵⁷⁴	192	6144 ⁵⁹³	194	6338 ⁶¹¹	80
11	4	5910 ⁵¹⁴	200	6110 ⁵³²	203	6313 ⁵⁴⁹	206	6519 ⁵⁶⁷	210	6729 ⁵⁸⁵	213	6942 ⁶⁰⁴	79
12	4	6417 ⁵⁰⁷	217	6634 ⁵²⁴	220	6854 ⁵⁴¹	224	7078 ⁵⁵⁹	227	7305 ⁵⁷⁶	232	7537 ⁵⁹⁵	78
13	4	6916 ⁴⁹⁹	233	7149 ⁵¹⁵	238	7387 ⁵³³	241	7628 ⁵⁵⁰	245	7873 ⁵⁶⁸	249	8122 ⁵⁸⁵	77
14	5	7406 ⁴⁹⁰	250	7656 ⁵⁰⁷	254	7910 ⁵²³	259	8169 ⁵⁴¹	262	8431 ⁵⁵⁸	267	8698 ⁵⁷⁶	76
15	5	7887 ⁴⁸¹	267	8154 ⁴⁹⁸	270	8424 ⁵¹⁴	276	8700 ⁵³¹	279	8979 ⁵⁴⁸	284	9263 ⁵⁶⁵	75
16	6	8359 ⁴⁷²	282	8641 ⁴⁸⁷	287	8928 ⁵⁰⁴	292	9220 ⁵²⁰	296	9516 ⁵³⁷	301	9817 ⁵⁵⁴	74
17	6	8820 ⁴⁶¹	298	9118 ⁴⁷⁷	302	9420 ⁴⁹²	308	9728 ⁵⁰⁸	313	10041 ⁵²⁵	318	10359 ⁵⁴²	73
18	6	9270 ⁴⁵⁰	313	9583 ⁴⁶⁵	319	9902 ⁴⁸²	323	10225 ⁴⁹⁷	329	10554 ⁵¹³	334	10888 ⁵²⁹	72
19	7	9710 ⁴⁴⁰	327	10037 ⁴⁵⁴	334	10371 ⁴⁶⁹	338	10709 ⁴⁸⁴	345	11054 ⁵⁰⁰	349	11403 ⁵¹⁵	71
20	7	10137 ⁴²⁷	342	10479 ⁴⁴²	348	10827 ⁴⁵⁶	353	11180 ⁴⁷¹	360	11540 ⁴⁸⁶	365	11905 ⁵⁰²	70
21	7	10551 ⁴¹⁴	357	10908 ⁴²⁹	362	11270 ⁴⁴³	368	11638 ⁴⁵⁸	374	12012 ⁴⁷²	380	12392 ⁴⁸⁷	69
22	7	10953 ⁴⁰²	370	11323 ⁴¹⁵	377	11699 ⁴²⁹	382	12081 ⁴⁴³	388	12469 ⁴⁵⁷	395	12864 ⁴⁷²	68
23	8	11342 ³⁸⁹	382	11724 ⁴⁰¹	390	12114 ⁴¹⁵	395	12509 ⁴²⁸	402	12911 ⁴⁴²	409	13320 ⁴⁵⁶	67
24	8	11716 ³⁷⁴	395	12111 ³⁸⁷	402	12513 ³⁹⁹	409	12922 ⁴¹³	415	13337 ⁴²⁶	422	13759 ⁴³⁹	66
25	8	12076 ³⁶⁰	408	12484 ³⁷³	414	12898 ³⁸⁵	421	13319 ³⁹⁷	428	13747 ⁴¹⁰	435	14182 ⁴²³	65
26	8	12422 ³⁴⁶	419	12841 ³⁵⁷	426	13267 ³⁶⁹	433	13700 ³⁸¹	440	14140 ³⁹³	448	14588 ⁴⁰⁶	64
27	9	12752 ³³⁰	430	13182 ³⁴¹	438	13620 ³⁵³	444	14064 ³⁶⁴	452	14516 ³⁷⁶	459	14975 ³⁸⁷	63
28	9	13066 ³¹⁴	441	13507 ³²⁵	448	13955 ³³⁵	456	14411 ³⁴⁷	463	14874 ³⁵⁸	470	15344 ³⁶⁹	62
29	9	13365 ²⁹⁹	451	13816 ³⁰⁹	458	14274 ³¹⁹	466	14740 ³²⁹	474	15214 ³⁴⁰	481	15695 ³⁵¹	61
30	9	13647 ²⁸²	460	14107 ²⁹¹	468	14575 ³⁰¹	476	15051 ³¹¹	484	15535 ³²¹	491	16026 ³³¹	60
31	9	13913 ²⁶⁶	469	14382 ²⁷⁵	477	14859 ²⁸⁴	485	15344 ²⁹³	493	15837 ³⁰²	500	16337 ³¹¹	59
32	10	14161 ²⁴⁸	478	14639 ²⁵⁷	485	15124 ²⁶⁵	494	15618 ²⁷⁴	502	16120 ²⁸³	509	16629 ²⁹²	58
33	10	14392 ²³¹	486	14878 ²³⁹	493	15371 ²⁴⁷	502	15873 ²⁵⁵	509	16382 ²⁶²	518	16900 ²⁷¹	57
34	10	14606 ²¹⁴	492	15098 ²²⁰	501	15599 ²²⁸	509	16108 ²³⁵	517	16625 ²⁴³	525	17150 ²⁵⁰	56
35	10	14802 ¹⁹⁶	499	15301 ²⁰³	507	15808 ²⁰⁹	516	16324 ²¹⁶	524	16848 ²²³	532	17380 ²³⁰	55
36	10	14979 ¹⁷⁷	505	15484 ¹⁸³	514	15998 ¹⁹⁰	521	16519 ¹⁹⁵	531	17050 ²⁰²	538	17588 ²⁰⁸	54
37	10	15138 ¹⁵⁹	511	15649 ¹⁶⁵	519	16168 ¹⁷⁰	527	16695 ¹⁷⁶	536	17231 ¹⁸¹	544	17775 ¹⁸⁷	53
38	10	15279 ¹⁴¹	515	15794 ¹⁴⁵	524	16318 ¹⁵⁰	532	16850 ¹⁵⁵	541	17391 ¹⁶⁰	549	17940 ¹⁶⁵	52
39	10	15402 ¹²³	518	15920 ¹²⁶	528	16448 ¹³⁰	537	16985 ¹³⁵	545	17530 ¹³⁹	553	18083 ¹⁴³	51
40	10	15505 ¹⁰³	522	16027 ¹⁰⁷	532	16559 ¹¹¹	540	17099 ¹¹⁴	548	17647 ¹¹⁷	557	18204 ¹²¹	50
41	10	15590 ⁸⁵	525	16115 ⁸⁸	534	16649 ⁹⁰	543	17192 ⁹³	551	17743 ⁹⁶	560	18303 ⁹⁹	49
42	10	15655 ⁶⁵	527	16182 ⁶⁷	536	16718 ⁶⁹	545	17263 ⁷¹	554	17817 ⁷⁴	563	18380 ⁷⁷	48
43	11	15701 ⁴⁶	529	16230 ⁴⁸	538	16768 ⁵⁰	546	17314 ⁵¹	556	17870 ⁵³	564	18434 ⁵⁵	47
44	11	15729 ²⁸	529	16258 ²⁸	539	16797 ²⁹	547	17344 ³⁰	557	17901 ³¹	565	18466 ³²	46
45	11	15737 ⁸	530	16267 ⁹	538	16805 ⁸	548	17353 ⁹	557	17910 ⁹	565	18475 ⁹	45
δ	$\frac{v}{2}$	6°.0	hor. diff.	6°.1	hor. diff.	6°.2	hor. diff.	6°.3	hor. diff.	6°.4	hor. diff.	6°.5	d
$\alpha - \alpha_0$													$\alpha - \alpha_0$

$$f(a \pm nw) = f(a) \pm n \left[f'(a) \pm n \frac{f''(a)}{2} \right], \quad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$$B = (d - \delta)^\circ.$$

d is always numerically larger than δ .

$$0''.00001 = 0''.036 = \text{unit.}$$

δ	$\frac{v}{2}$	6°.5	hor. diff.	6°.6	hor. diff.	6°.7	hor. diff.	6°.8	hor. diff.	6°.9	hor. diff.	7°.0	d
0		000	00	000	00	000	00	000	00	000	00	000	90
1		647 ⁶⁴⁷	20	667 ⁶⁶⁷	20	687 ⁶⁸⁷	21	708 ⁷⁰⁸	21	729 ⁷²⁹	22	751 ⁷⁵¹	89
2		1293 ⁶⁴⁶	40	1333 ⁶⁶⁶	41	1374 ⁶⁸⁷	42	1416 ⁷⁰⁸	42	1458 ⁷²⁹	43	1501 ⁷⁵⁰	88
3		1937 ⁶⁴⁴	61	1998 ⁶⁶⁵	61	2059 ⁶⁸⁸	63	2122 ⁷⁰⁹	63	2185 ⁷²⁹	64	2249 ⁷⁴⁸	87
4		2579 ⁶⁴²	81	2660 ⁶⁶²	82	2742 ⁶⁸³	83	2825 ⁷⁰³	84	2909 ⁷²⁴	85	2994 ⁷⁴⁵	86
5		3218 ⁶³⁹	101	3319 ⁶⁵⁹	102	3421 ⁶⁷⁹	103	3524 ⁶⁹⁹	105	3629 ⁷²⁰	107	3736 ⁷⁴²	85
6		3853 ⁶³⁵	120	3973 ⁶⁵⁴	122	4095 ⁶⁷⁴	124	4219 ⁶⁹⁵	126	4345 ⁷¹⁶	128	4473 ⁷³⁷	84
7	3	4484 ⁶³¹	139	4623 ⁶⁵⁰	142	4765 ⁶⁷⁰	144	4909 ⁶⁹⁰	147	5056 ⁷¹¹	148	5204 ⁷³¹	83
8	3	5108 ⁶²⁴	160	5268 ⁶⁴⁵	161	5429 ⁶⁶⁴	164	5593 ⁶⁸⁴	167	5760 ⁷⁰⁴	169	5929 ⁷²⁵	82
9	4	5727 ⁶¹⁹	178	5905 ⁶³⁷	181	6086 ⁶⁵⁷	185	6271 ⁶⁷⁸	186	6457 ⁶⁹⁷	190	6647 ⁷¹⁸	81
10	4	6338 ⁶¹¹	198	6536 ⁶³¹	200	6736 ⁶⁵⁰	204	6940 ⁶⁶⁹	207	7147 ⁶⁹⁰	209	7356 ⁷⁰⁹	80
11	5	6942 ⁶⁰⁴	216	7158 ⁶²²	220	7378 ⁶⁴²	223	7601 ⁶⁶¹	226	7827 ⁶⁸⁰	230	8057 ⁷⁰¹	79
12	5	7537 ⁵⁹⁵	235	7772 ⁶¹⁴	238	8010 ⁶³²	242	8252 ⁶⁵¹	246	8498 ⁶⁷¹	250	8748 ⁶⁹¹	78
13	5	8122 ⁵⁸⁵	254	8376 ⁶⁰⁴	257	8633 ⁶²³	261	8894 ⁶⁴²	265	9159 ⁶⁶¹	269	9428 ⁶⁸⁰	77
14	6	8698 ⁵⁷⁶	271	8969 ⁵⁹³	276	9245 ⁶¹²	279	9524 ⁶³⁰	284	9808 ⁶⁴⁹	288	10096 ⁶⁶⁸	76
15	6	9263 ⁵⁶⁵	289	9552 ⁵⁸³	293	9845 ⁶⁰⁰	298	10143 ⁶¹⁹	302	10445 ⁶³⁷	307	10752 ⁶⁵⁶	75
16	6	9817 ⁵⁵⁴	306	10123 ⁵⁷¹	310	10433 ⁵⁸⁸	316	10749 ⁶⁰⁶	320	11069 ⁶²⁴	325	11394 ⁶⁴²	74
17	7	10359 ⁵⁴²	323	10682 ⁵⁵⁹	327	11009 ⁵⁷⁶	333	11342 ⁵⁹³	338	11680 ⁶¹¹	343	12023 ⁶²⁹	73
18	7	10888 ⁵²⁹	339	11227 ⁵⁴⁵	345	11572 ⁵⁶³	349	11921 ⁵⁸¹	355	12276 ⁵⁹⁹	361	12637 ⁶¹⁴	72
19	8	11403 ⁵¹⁵	356	11759 ⁵³²	360	12119 ⁵⁵¹	367	12486 ⁵⁶⁹	372	12858 ⁵⁸⁷	377	13235 ⁵⁹⁸	71
20	8	11905 ⁵⁰²	371	12276 ⁵¹⁷	376	12654 ⁵³³	383	13035 ⁵⁵¹	389	13424 ⁵⁶⁹	393	13817 ⁵⁸²	70
21	8	12392 ⁴⁸⁷	386	12778 ⁵⁰²	392	13170 ⁵¹⁸	398	13568 ⁵³³	404	13972 ⁵⁴⁸	410	14382 ⁵⁶⁵	69
22	8	12864 ⁴⁷²	400	13264 ⁴⁸⁶	407	13671 ⁵⁰¹	413	14084 ⁵¹⁶	420	14504 ⁵³²	425	14929 ⁵⁴⁷	68
23	9	13320 ⁴⁵⁶	414	13734 ⁴⁷⁰	422	14156 ⁴⁸⁵	428	14584 ⁵⁰⁰	434	15018 ⁵¹⁴	440	15458 ⁵²⁹	67
24	9	13759 ⁴³⁹	429	14188 ⁴⁵⁴	435	14623 ⁴⁶⁷	442	15065 ⁴⁸¹	448	15513 ⁴⁹⁵	455	15968 ⁵¹⁰	66
25	9	14182 ⁴²³	442	14624 ⁴³⁶	448	15072 ⁴⁴⁹	455	15527 ⁴⁶²	463	15990 ⁴⁷⁷	469	16459 ⁴⁹¹	65
26	10	14588 ⁴⁰⁶	454	15042 ⁴¹⁸	461	15503 ⁴³¹	468	15971 ⁴⁴⁴	476	16447 ⁴⁵⁷	482	16929 ⁴⁷⁰	64
27	10	14975 ³⁸⁷	466	15441 ³⁹⁹	474	15915 ⁴¹²	480	16395 ⁴²⁴	488	16883 ⁴³⁶	495	17378 ⁴⁴⁹	63
28	10	15344 ³⁶⁹	478	15822 ³⁸¹	485	16307 ³⁹²	492	16799 ⁴⁰⁴	500	17299 ⁴¹⁶	507	17806 ⁴²⁸	62
29	11	15695 ³⁵¹	488	16183 ³⁶¹	496	16679 ³⁷²	504	17183 ³⁸⁴	511	17694 ³⁹⁵	519	18213 ⁴⁰⁷	61
30	11	16026 ³³¹	498	16524 ³⁴¹	507	17031 ³⁵²	514	17545 ³⁶²	522	18067 ³⁷³	530	18597 ³⁸⁴	60
31	11	16337 ³¹¹	509	16846 ³²²	516	17362 ³³¹	524	17886 ³⁴¹	532	18418 ³⁵¹	540	18958 ³⁶¹	59
32	11	16629 ²⁹²	517	17146 ³⁰⁰	526	17672 ³¹⁰	533	18205 ³¹⁹	542	18747 ³²⁹	549	19296 ³³⁸	58
33	11	16900 ²⁷¹	526	17426 ²⁸⁰	534	17960 ²⁸⁸	542	18502 ²⁹⁷	550	19052 ³⁰⁵	558	19610 ³¹⁴	57
34	11	17150 ²⁵⁰	534	17684 ²⁵⁸	542	18226 ²⁶⁶	550	18776 ²⁷⁴	558	19334 ²⁸²	567	19901 ²⁹¹	56
35	11	17380 ²³⁰	541	17921 ²³⁷	549	18470 ²⁴⁴	557	19027 ²⁵¹	566	19593 ²⁵⁹	574	20167 ²⁶⁶	55
36	11	17588 ²⁰⁸	545	18135 ²¹⁴	556	18691 ²²¹	564	19255 ²²⁸	572	19827 ²³⁴	581	20408 ²⁴¹	54
37	12	17775 ¹⁸⁷	553	18328 ¹⁹³	561	18889 ¹⁹⁸	570	19459 ²⁰⁴	579	20038 ²¹¹	586	20624 ²¹⁶	53
38	12	17940 ¹⁶⁵	558	18498 ¹⁷⁰	567	19065 ¹⁷⁶	575	19640 ¹⁸¹	583	20223 ¹⁸⁵	593	20816 ¹⁹²	52
39	12	18083 ¹⁴³	563	18646 ¹⁴⁸	571	19217 ¹⁵²	579	19796 ¹⁵⁶	589	20385 ¹⁶²	597	20982 ¹⁶⁶	51
40	12	18204 ¹²¹	566	18770 ¹²⁴	575	19345 ¹²⁸	584	19929 ¹³³	592	20521 ¹³⁶	601	21122 ¹⁴⁰	50
41	12	18303 ⁹⁹	569	18872 ¹⁰²	578	19450 ¹⁰⁵	587	20037 ¹⁰⁸	595	20632 ¹¹¹	604	21236 ¹¹⁴	49
42	12	18380 ⁷⁷	571	18951 ⁷⁹	580	19531 ⁸¹	589	20120 ⁸³	598	20718 ⁸⁶	606	21324 ⁸⁸	48
43	12	18434 ⁵⁴	573	19007 ⁵⁶	582	19589 ⁵⁸	590	20179 ⁵⁹	600	20779 ⁶¹	608	21387 ⁶³	47
44	12	18466 ³²	574	19040 ³³	582	19622 ³⁵	592	20214 ³⁵	600	20814 ³⁵	609	21423 ³⁶	46
45	12	18475 ⁹	574	19049 ⁹	583	19632 ¹⁰	592	20224 ¹⁰	600	20824 ¹⁰	610	21434 ¹¹	45
δ	$\frac{v}{2}$	6°.5	hor. diff.	6°.6	hor. diff.	6°.7	hor. diff.	6°.8	hor. diff.	6°.9	hor. diff.	7°.0	d

$$f(a \pm nv) = f(a) \pm n \left[f'(a) \pm n \frac{f''(a)}{2} \right], \quad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$$B = (d - \delta)^\circ.$$

 d is always numerically larger than δ .

 $0''.00001 = 0''.036 = \text{unit.}$

δ	$\frac{v}{2}$	$7^\circ.0$	hor. diff.	$7^\circ.1$	hor. diff.	$7^\circ.2$	hor. diff.	$7^\circ.3$	hor. diff.	$7^\circ.4$	hor. diff.	$7^\circ.5$	d
0		000	00	000	00	000	00	000	00	000	00	000	90
1		751 ⁷⁵¹	22	773 ⁷⁷³	22	795 ⁷⁹⁵	22	817 ⁸¹⁷	23	840 ⁸⁴⁰	23	863 ⁸⁶³	89
2		1501 ⁷⁵⁰	43	1544 ⁷⁷¹	44	1588 ⁷⁹³	45	1633 ⁸¹⁴	45	1678 ⁸³⁸	46	1724 ⁸⁶⁰	88
3		2249 ⁷⁴⁸	65	2314 ⁷⁷⁰	66	2380 ⁷⁹²	67	2447 ⁸¹¹	68	2515 ⁸³⁴	69	2584 ⁸⁵⁶	87
4		2994 ⁷⁴⁵	87	3081 ⁷⁶⁷	88	3169 ⁷⁸⁹	89	3258 ⁸⁰⁷	91	3349 ⁸²⁹	91	3440 ⁸⁵²	86
5		3736 ⁷⁴²	108	3844 ⁷⁶³	110	3954 ⁷⁸⁵	111	4065 ⁸⁰²	113	4178 ⁸²⁴	114	4292 ⁸⁴⁷	85
6	3	4473 ⁷³⁷	129	4602 ⁷⁵⁸	132	4734 ⁷⁸⁰	133	4867 ⁷⁹⁶	135	5002 ⁸¹⁸	137	5139 ⁸⁴¹	84
7	4	5204 ⁷³¹	151	5355 ⁷⁵³	153	5508 ⁷⁷⁴	155	5663 ⁷⁹⁶	157	5820 ⁸¹¹	160	5980 ⁸³³	83
8	4	5929 ⁷²⁵	172	6101 ⁷⁴⁶	174	6275 ⁷⁶⁷	177	6452 ⁷⁸¹	179	6631 ⁸⁰³	182	6813 ⁸²⁴	82
9	4	6647 ⁷¹⁸	192	6839 ⁷³⁸	196	7035 ⁷⁶⁰	198	7233 ⁷⁷²	201	7434 ⁷⁹³	203	7637 ⁸¹⁶	81
10	5	7356 ⁷⁰⁹	214	7570 ⁷³¹	216	7786 ⁷⁵¹	219	8005 ⁷⁷²	222	8227 ⁷⁹³	226	8453 ⁸¹⁶	80
11	5	8057 ⁷⁰¹	233	8290 ⁷²⁰	237	8527 ⁷⁴¹	240	8767 ⁷⁶²	244	9011 ⁷⁸⁴	247	9258 ⁸⁰⁵	79
12	6	8748 ⁶⁹¹	253	9001 ⁷¹¹	257	9258 ⁷³¹	261	9519 ⁷⁵²	264	9783 ⁷⁷²	268	10051 ⁷⁹³	78
13	6	9428 ⁶⁸⁰	273	9701 ⁷⁰⁰	277	9978 ⁷²⁰	280	10258 ⁷³⁹	285	10543 ⁷⁶⁰	289	10832 ⁷⁸¹	77
14	7	10096 ⁶⁶⁸	292	10388 ⁶⁸⁷	297	10685 ⁷⁰⁷	301	10986 ⁷²⁸	305	11291 ⁷⁴⁸	309	11600 ⁷⁶⁸	76
15	7	10752 ⁶⁵⁶	311	11063 ⁶⁷⁵	316	11379 ⁶⁹⁴	320	11699 ⁷¹³	325	12024 ⁷³³	329	12353 ⁷⁵³	75
16	7	11394 ⁶⁴²	330	11724 ⁶⁶¹	335	12059 ⁶⁸⁰	339	12398 ⁷⁰³	344	12742 ⁷¹⁸	349	13091 ⁷³⁸	74
17	8	12023 ⁶²⁹	348	12371 ⁶⁴⁷	353	12724 ⁶⁶⁵	358	13082 ⁶⁸⁴	363	13445 ⁷⁰³	368	13813 ⁷²²	73
18	8	12637 ⁶¹⁴	365	13002 ⁶³¹	371	13373 ⁶⁴⁹	377	13750 ⁶⁶⁸	382	14132 ⁶⁸⁷	387	14519 ⁷⁰⁶	72
19	9	13235 ⁵⁹⁸	383	13618 ⁶¹⁶	389	14007 ⁶³⁴	394	14401 ⁶⁵¹	399	14800 ⁶⁶⁸	406	15206 ⁶⁸⁷	71
20	9	13817 ⁵⁸²	400	14217 ⁵⁹⁹	406	14623 ⁶¹⁶	411	15034 ⁶³³	417	15451 ⁶⁵¹	423	15874 ⁶⁶⁹	70
21	9	14382 ⁵⁶⁵	416	14798 ⁵⁸¹	422	15220 ⁵⁹⁷	428	15648 ⁶¹⁴	435	16083 ⁶³²	440	16523 ⁶⁴⁹	69
22	10	14929 ⁵⁴⁷	432	15361 ⁵⁶³	438	15799 ⁵⁷⁹	445	16244 ⁵⁹⁶	451	16695 ⁶¹²	457	17152 ⁶²⁹	68
23	10	15458 ⁵²⁹	447	15905 ⁵⁴⁴	454	16359 ⁵⁶⁰	460	16819 ⁵⁷⁵	467	17286 ⁵⁹¹	473	17759 ⁶⁰⁷	67
24	11	15968 ⁵¹⁰	462	16430 ⁵²⁵	469	16899 ⁵⁴⁰	475	17374 ⁵⁵⁵	482	17856 ⁵⁷⁰	489	18345 ⁵⁸⁶	66
25	11	16459 ⁴⁹¹	476	16935 ⁵⁰⁵	483	17418 ⁵¹⁹	490	17908 ⁵³⁴	496	18404 ⁵⁴⁸	504	18908 ⁵⁶³	65
26	11	16929 ⁴⁷⁰	490	17419 ⁴⁸⁴	496	17915 ⁴⁹⁷	504	18419 ⁵¹¹	511	18930 ⁵²⁶	518	19448 ⁵⁴⁰	64
27	12	17378 ⁴⁴⁹	503	17881 ⁴⁶²	510	18391 ⁴⁷⁶	517	18908 ⁴⁸⁹	524	19432 ⁵⁰²	532	19964 ⁵¹⁶	63
28	12	17806 ⁴²⁸	515	18321 ⁴⁴⁰	523	18844 ⁴⁵³	530	19374 ⁴⁶⁶	537	19911 ⁴⁷⁹	544	20455 ⁴⁹¹	62
29	12	18213 ⁴⁰⁷	526	18739 ⁴¹⁸	535	19274 ⁴³⁰	541	19815 ⁴⁴¹	550	20365 ⁴⁵⁴	557	20922 ⁴⁶⁷	61
30	12	18597 ³⁸⁴	537	19134 ³⁹⁵	546	19680 ⁴⁰⁶	553	20233 ⁴¹⁸	561	20794 ⁴²⁹	569	21363 ⁴⁴¹	60
31	12	18958 ³⁶¹	548	19506 ³⁷²	556	20062 ³⁸²	564	20626 ³⁹³	571	21197 ⁴⁰³	580	21777 ⁴¹⁴	59
32	13	19296 ³³⁸	558	19854 ³⁴⁸	565	20419 ³⁵⁷	574	20993 ³⁶⁷	582	21575 ³⁷⁸	590	22165 ³⁸⁸	58
33	13	19610 ³¹⁴	567	20177 ³²³	575	20752 ³³³	583	21335 ³⁴²	591	21926 ³⁵¹	600	22526 ³⁶¹	57
34	13	19901 ²⁹¹	575	20476 ²⁹⁹	583	21059 ³⁰⁷	592	21651 ³¹⁶	600	22251 ³²⁵	608	22859 ³³³	56
35	13	20167 ²⁶⁶	582	20749 ²⁷³	591	21340 ²⁸¹	600	21940 ²⁸⁹	608	22548 ²⁹⁷	616	23164 ³⁰⁵	55
36	14	20408 ²⁴¹	590	20998 ²⁴⁹	598	21596 ²⁵⁶	606	22202 ²⁶²	615	22817 ²⁶⁹	624	23441 ²⁷⁷	54
37	14	20624 ²¹⁶	596	21220 ²²²	604	21824 ²²⁸	613	22437 ²³⁵	622	23059 ²⁴²	630	23689 ²⁴⁸	53
38	14	20816 ¹⁹²	601	21417 ¹⁹⁷	610	22027 ²⁰³	618	22645 ²⁰⁸	627	23272 ²¹³	636	23908 ²¹⁹	52
39	14	20982 ¹⁶⁶	605	21587 ¹⁷⁰	615	22202 ¹⁷⁵	623	22825 ¹⁸⁰	632	23457 ¹⁸⁵	641	24098 ¹⁹⁰	51
40	14	21122 ¹⁴⁰	610	21732 ¹⁴⁵	618	22350 ¹⁴⁸	627	22977 ¹⁵²	637	23614 ¹⁵⁷	644	24258 ¹⁶⁰	50
41	14	21236 ¹¹⁴	613	21849 ¹¹⁷	622	22471 ¹²¹	631	23102 ¹²⁵	639	23741 ¹²⁷	648	24389 ¹³¹	49
42	14	21324 ⁸⁸	616	21940 ⁹¹	624	22564 ⁹³	634	23198 ⁹⁶	642	23840 ⁹⁹	650	24499 ¹⁰¹	48
43	14	21387 ⁶³	617	22004 ⁶⁴	626	22630 ⁶⁶	635	23265 ⁶⁷	644	23909 ⁶⁹	653	24562 ⁷²	47
44	15	21423 ³⁶	619	22042 ³⁸	627	22669 ³⁹	636	23305 ⁴⁰	645	23949 ⁴⁰	654	24603 ⁴¹	46
45	15	21434 ¹¹	618	22052 ¹⁰	627	22679 ¹⁰	636	23315 ¹⁰	645	23960 ¹¹	654	24614 ¹¹	45

$$f(a \pm nw) = f(a) \pm n \left[f'(a) \pm n \frac{f''(a)}{2} \right], \quad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$$B = (d - \delta)^\circ.$$

d is always numerically larger than δ .

$$0''.00001 = 0''.036 = \text{unit.}$$

δ	$\frac{v}{2}$	$7^\circ.5$	hor. diff.	$7^\circ.6$	hor. diff.	$7^\circ.7$	hor. diff.	$7^\circ.8$	hor. diff.	$7^\circ.9$	hor. diff.	$8^\circ.0$	d
0		000 ⁸⁶³	00	000 ⁸⁸⁶	00	000 ⁹¹⁰	00	000 ⁹³⁴	00	000 ⁹⁵⁸	00	000 ⁹⁸³	90
1		863 ⁸⁶¹	23	886 ⁸⁸⁵	24	910 ⁹⁰⁸	24	934 ⁹³²	24	958 ⁹⁵⁷	25	983 ⁹⁸¹	89
2		1724 ⁸⁶⁰	47	1771 ⁸⁸³	47	1818 ⁹⁰⁷	48	1866 ⁹³⁰	49	1915 ⁹⁵⁴	49	1964 ⁹⁷⁹	88
3		2584 ⁸⁵⁶	70	2654 ⁸⁷⁹	71	2725 ⁹⁰³	71	2796 ⁹²⁷	73	2869 ⁹⁵¹	74	2943 ⁹⁷⁵	87
4		3440 ⁸⁵²	93	3533 ⁸⁷⁵	95	3628 ⁸⁹⁸	95	3723 ⁹²²	97	3820 ⁹⁴⁶	98	3918 ⁹⁷⁰	86
5	3	4292 ⁸⁴⁷	116	4408 ⁸⁷⁰	118	4526 ⁸⁹³	119	4645 ⁹¹⁷	121	4766 ⁹⁴⁰	122	4888 ⁹⁶⁵	85
6	4	5139 ⁸⁴¹	139	5278 ⁸⁶³	141	5419 ⁸⁸⁶	143	5562 ⁹⁰⁹	144	5706 ⁹³⁴	147	5853 ⁹⁵⁷	84
7	4	5980 ⁸³³	161	6141 ⁸⁵⁶	164	6305 ⁸⁷⁹	166	6471 ⁹⁰²	169	6640 ⁹²⁵	170	6810 ⁹⁴⁹	83
8	5	6813 ⁸²⁴	184	6997 ⁸⁴⁷	187	7184 ⁸⁶⁹	189	7373 ⁸⁹²	192	7565 ⁹¹⁵	194	7759 ⁹³⁹	82
9	5	7637 ⁸¹⁶	207	7844 ⁸³⁷	209	8053 ⁸⁶⁰	212	8265 ⁸⁸³	215	8480 ⁹⁰⁵	218	8698 ⁹²⁸	81
10	5	8453 ⁸⁰⁵	228	8681 ⁸²⁷	232	8913 ⁸⁴⁹	235	9148 ⁸⁷¹	237	9385 ⁸⁹⁴	241	9626 ⁹¹⁷	80
11	6	9258 ⁷⁹³	250	9508 ⁸¹⁵	254	9762 ⁸³⁶	257	10019 ⁸⁵⁸	260	10279 ⁸⁸¹	264	10543 ⁹⁰³	79
12	6	10051 ⁷⁸¹	272	10323 ⁸⁰²	275	10598 ⁸²⁴	279	10877 ⁸⁴⁵	283	11160 ⁸⁶⁷	286	11446 ⁸⁹⁰	78
13	7	10832 ⁷⁶⁸	293	11125 ⁷⁸⁸	297	11422 ⁸⁰⁹	300	11722 ⁸³¹	305	12027 ⁸⁵²	309	12336 ⁸⁷⁴	77
14	8	11600 ⁷⁵³	313	11913 ⁷⁷⁴	318	12231 ⁷⁹⁴	322	12553 ⁸¹⁵	326	12879 ⁸³⁷	331	13210 ⁸⁵⁸	76
15	8	12353 ⁷³⁸	334	12687 ⁷⁵⁸	338	13025 ⁷⁷⁹	343	13368 ⁷⁹⁹	348	13716 ⁸¹⁹	352	14068 ⁸⁴⁰	75
16	9	13091 ⁷²²	354	13445 ⁷⁴²	359	13804 ⁷⁶¹	363	14167 ⁷⁸¹	368	14535 ⁸⁰²	373	14908 ⁸²²	74
17	9	13813 ⁷⁰⁶	374	14187 ⁷²⁴	378	14565 ⁷⁴³	383	14948 ⁷⁶³	389	15337 ⁷⁸³	393	15730 ⁸⁰³	73
18	10	14519 ⁶⁸⁷	392	14911 ⁷⁰⁶	397	15308 ⁷²⁵	403	15711 ⁷⁴⁴	409	16120 ⁷⁶²	413	16533 ⁷⁸²	72
19	10	15206 ⁶⁶⁸	411	15617 ⁶⁸⁶	416	16033 ⁷⁰⁵	422	16455 ⁷²³	427	16882 ⁷⁴²	433	17315 ⁷⁶¹	71
20	11	15874 ⁶⁴⁹	429	16303 ⁶⁶⁶	435	16738 ⁶⁸⁴	440	17178 ⁷⁰²	446	17624 ⁷²⁰	452	18076 ⁷³⁹	70
21	11	16523 ⁶²⁹	446	16969 ⁶⁴⁶	453	17422 ⁶⁶²	458	17880 ⁶⁸⁰	464	18344 ⁶⁹⁸	471	18815 ⁷¹⁵	69
22	11	17152 ⁶⁰⁷	463	17615 ⁶²⁴	469	18084 ⁶⁴¹	476	18560 ⁶⁵⁷	482	19042 ⁶⁷⁴	488	19530 ⁶⁹²	68
23	12	17759 ⁵⁸⁶	480	18239 ⁶⁰¹	486	18725 ⁶¹⁷	492	19217 ⁶³⁴	499	19716 ⁶⁵⁰	506	20222 ⁶⁶⁶	67
24	12	18345 ⁵⁶³	495	18840 ⁵⁷⁸	502	19342 ⁵⁹⁴	509	19851 ⁶⁰⁹	515	20366 ⁶²⁵	522	20888 ⁶⁴¹	66
25	12	18908 ⁵⁴⁰	510	19418 ⁵⁵⁵	518	19936 ⁵⁶⁹	524	20460 ⁵⁸⁴	531	20991 ⁵⁹⁹	538	21529 ⁶¹⁵	65
26	13	19448 ⁵¹⁶	525	19973 ⁵³⁰	532	20505 ⁵⁴⁴	539	21044 ⁵⁵⁸	546	21590 ⁵⁷³	554	22144 ⁵⁸⁷	64
27	13	19964 ⁴⁹¹	539	20503 ⁵⁰⁴	546	21049 ⁵¹⁸	553	21602 ⁵³²	561	22163 ⁵⁴⁵	568	22731 ⁵⁵⁹	63
28	13	20455 ⁴⁶⁷	552	21007 ⁴⁷⁹	560	21567 ⁴⁹²	567	22134 ⁵⁰⁴	574	22708 ⁵¹⁸	582	23290 ⁵³¹	62
29	14	20922 ⁴⁴¹	564	21486 ⁴⁵³	573	22059 ⁴⁶⁴	579	22638 ⁴⁷⁷	588	23226 ⁴⁸⁹	595	23821 ⁵⁰¹	61
30	14	21363 ⁴¹⁴	576	21939 ⁴²⁶	584	22523 ⁴³⁷	592	23115 ⁴⁴⁸	600	23715 ⁴⁶⁰	607	24322 ⁴⁷²	60
31	14	21777 ³⁸⁸	588	22365 ³⁹⁸	595	22960 ⁴⁰⁹	603	23563 ⁴²⁰	612	24175 ⁴³⁰	619	24794 ⁴⁴¹	59
32	15	22165 ³⁶¹	598	22763 ³⁷⁰	606	23369 ³⁸⁰	614	23983 ³⁹⁰	622	24605 ⁴⁰⁰	630	25235 ⁴¹⁰	58
33	15	22526 ³³³	607	23133 ³⁴²	616	23749 ³⁵¹	624	24373 ³⁶⁰	632	25005 ³⁶⁹	640	25645 ³⁷⁹	57
34	15	22859 ³⁰⁵	616	23475 ³¹³	625	24100 ³²¹	633	24733 ³³⁰	641	25374 ³³⁹	650	26024 ³⁴⁷	56
35	15	23164 ²⁷⁷	624	23788 ²⁸⁵	633	24421 ²⁹²	642	25063 ²⁹⁹	650	25713 ³⁰⁶	658	26371 ³¹⁴	55
36	16	23441 ²⁴⁸	632	24073 ²⁵⁴	640	24713 ²⁶¹	649	25362 ²⁶⁸	657	26019 ²⁷⁵	666	26685 ²⁸²	54
37	16	23689 ²¹⁹	638	24327 ²²⁵	647	24974 ²³¹	656	25630 ²³⁷	664	26294 ²⁴³	673	26967 ²⁴⁹	53
38	16	23908 ¹⁹⁰	644	24552 ¹⁹⁵	653	25205 ²⁰⁰	662	25867 ²⁰⁵	670	26537 ²¹¹	679	27216 ²¹⁶	52
39	16	24098 ¹⁶⁰	649	24747 ¹⁶⁵	658	25405 ¹⁶⁹	667	26072 ¹⁷⁴	676	26748 ¹⁷⁷	684	27432 ¹⁸²	51
40	16	24258 ¹³¹	654	24912 ¹³⁴	662	25574 ¹³⁸	672	26246 ¹⁴¹	679	26925 ¹⁴⁵	689	27614 ¹⁴⁹	50
41	16	24389 ¹⁰¹	657	25046 ¹⁰⁴	666	25712 ¹⁰⁷	675	26387 ¹⁰⁹	683	27070 ¹¹²	693	27763 ¹¹⁴	49
42	16	24490 ⁷²	660	25150 ⁷³	669	25819 ⁷⁴	677	26496 ⁷⁷	686	27182 ⁷⁹	695	27877 ⁸¹	48
43	16	24562 ⁴¹	661	25223 ⁴²	670	25893 ⁴⁴	680	26573 ⁴⁴	688	27261 ⁴⁵	697	27958 ⁴⁶	47
44	16	24603 ¹¹	662	25265 ¹²	672	25937 ¹¹	680	26617 ¹²	689	27306 ¹²	698	28004 ¹²	46
45	16	24614	663	25277	671	25948	681	26629	689	27318	698	28016	45
δ	$\frac{v}{2}$	$7^\circ.5$	hor. diff.	$7^\circ.6$	hor. diff.	$7^\circ.7$	hor. diff.	$7^\circ.8$	hor. diff.	$7^\circ.9$	hor. diff.	$8^\circ.0$	d

$$f(a \pm nw) = f(a) \pm n \left[f'(a) \pm n \frac{f''(a)}{2} \right], \quad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$$B = (d - \delta)^\circ.$$

d is always numerically larger than δ .

$$0''.00001 = 0''.036 = \text{unit.}$$

$\frac{a-a_0}{\delta}$	$\frac{v}{2}$	$8^\circ.0$	hor. diff.	$8^\circ.1$	hor. diff.	$8^\circ.2$	hor. diff.	$8^\circ.3$	hor. diff.	$8^\circ.4$	hor. diff.	$8^\circ.5$	$\frac{a-a_0}{d}$
0		000	00	0000	00	0000	00	0000	00	0000	00	0000	90
1		983 ⁹⁸³	25	1008 ¹⁰⁰⁸	25	1033 ¹⁰³³	25	1058 ¹⁰⁵⁸	26	1084 ¹⁰⁸⁴	26	1110 ¹¹¹⁰	89
2		1964 ⁹⁸¹	50	2014 ¹⁰⁰⁶	50	2064 ¹⁰³¹	51	2115 ¹⁰⁵⁷	52	2167 ¹⁰⁸³	52	2219 ¹¹⁰⁹	88
3		2943 ⁹⁷⁹	74	3017 ¹⁰⁰³	76	3093 ¹⁰²⁹	77	3170 ¹⁰⁵⁵	77	3247 ¹⁰⁸⁰	79	3326 ¹¹⁰⁷	87
4	3	3918 ⁹⁷⁵	100	4018 ¹⁰⁰¹	100	4118 ¹⁰²⁵	102	4220 ¹⁰⁵⁰	103	4323 ¹⁰⁷⁶	105	4428 ¹¹⁰²	86
5	3	4888 ⁹⁷⁰	125	5013 ⁹⁹⁵	125	5138 ¹⁰²⁰	127	5265 ¹⁰⁴⁵	129	5394 ¹⁰⁷¹	131	5525 ¹⁰⁹⁷	85
6	4	5853 ⁹⁶⁵	148	6001 ⁹⁸⁸	151	6152 ¹⁰¹⁴	152	6304 ¹⁰³⁹	154	6458 ¹⁰⁶⁴	156	6614 ¹⁰⁸⁹	84
7	5	6810 ⁹⁵⁷	173	6983 ⁹⁸²	175	7158 ¹⁰⁰⁶	177	7335 ¹⁰³¹	179	7514 ¹⁰⁵⁶	182	7696 ¹⁰⁸²	83
8	5	7759 ⁹⁴⁹	197	7956 ⁹⁷³	199	8155 ⁹⁹⁷	202	8357 ¹⁰²²	204	8561 ¹⁰⁴⁷	207	8768 ¹⁰⁷²	82
9	6	8698 ⁹³⁹	221	8919 ⁹⁶³	223	9142 ⁹⁸⁷	226	9368 ¹⁰¹¹	229	9597 ¹⁰³⁶	232	9829 ¹⁰⁶¹	81
10	6	9626 ⁹²⁸	244	9870 ⁹⁵¹	248	10118 ⁹⁷⁶	250	10368 ¹⁰⁰⁰	254	10622 ¹⁰²⁵	256	10878 ¹⁰⁴⁹	80
11	7	10543 ⁹¹⁷	267	10810 ⁹⁴⁰	271	11081 ⁹⁶³	274	11355 ⁹⁸⁷	278	11633 ¹⁰¹¹	281	11914 ¹⁰³⁶	79
12	8	11446 ⁹⁰³	291	11737 ⁹²⁷	294	12031 ⁹⁵⁰	297	12328 ⁹⁷³	302	12630 ⁹⁹⁷	305	12935 ¹⁰²¹	78
13	8	12336 ⁸⁹⁰	313	12649 ⁹¹²	316	12965 ⁹³⁴	321	13286 ⁹⁵⁸	325	13611 ⁹⁸¹	329	13940 ¹⁰⁰⁵	77
14	9	13210 ⁸⁷⁴	335	13545 ⁸⁹⁶	339	13884 ⁹¹⁹	344	14228 ⁹⁴²	347	14575 ⁹⁶⁴	353	14928 ⁹⁸⁸	76
15	9	14068 ⁸⁵⁸	356	14424 ⁸⁷⁹	362	14786 ⁸⁹⁷	365	15151 ⁹²³	371	15522 ⁹⁴⁷	375	15897 ⁹⁶⁹	75
16	10	14908 ⁸⁴⁰	378	15286 ⁸⁶²	383	15669 ⁸⁸³	388	16057 ⁹⁰⁶	392	16449 ⁹²⁷	397	16846 ⁹⁴⁹	74
17	10	15730 ⁸²²	399	16129 ⁸⁴³	404	16533 ⁸⁶⁴	409	16942 ⁸⁸⁵	414	17356 ⁹⁰⁷	419	17775 ⁹²⁹	73
18	11	16533 ⁸⁰³	419	16952 ⁸²³	425	17377 ⁸⁴⁴	429	17806 ⁸⁶⁴	435	18241 ⁸⁸⁵	441	18682 ⁹⁰⁷	72
19	12	17315 ⁷⁸²	439	17754 ⁸⁰²	445	18199 ⁸²²	450	18649 ⁸⁴³	455	19104 ⁸⁶³	462	19566 ⁸⁸⁴	71
20	12	18076 ⁷⁶¹	458	18534 ⁷⁸⁰	464	18998 ⁷⁹⁹	470	19468 ⁸¹⁹	476	19944 ⁸⁴⁰	481	20425 ⁸⁵⁹	70
21	13	18815 ⁷³⁹	477	19292 ⁷⁵⁸	483	19775 ⁷⁷⁷	488	20263 ⁷⁹⁵	495	20758 ⁸¹⁴	502	21260 ⁸³⁵	69
22	13	19530 ⁷¹⁵	495	20025 ⁷³³	501	20526 ⁷⁵¹	508	21034 ⁷⁷¹	514	21548 ⁷⁹⁰	520	22068 ⁸⁰⁸	68
23	13	20222 ⁶⁹²	512	20734 ⁷⁰⁹	519	21253 ⁷²⁷	525	21778 ⁷⁴⁴	532	22310 ⁷⁶²	539	22849 ⁷⁸¹	67
24	14	20888 ⁶⁶⁶	530	21418 ⁶⁸⁴	535	21953 ⁷⁰⁰	543	22496 ⁷¹⁸	549	23045 ⁷³⁵	556	23601 ⁷⁵²	66
25	14	21529 ⁶⁴¹	546	22075 ⁶⁵⁷	552	22627 ⁶⁷⁴	559	23186 ⁶⁹⁰	566	23752 ⁷⁰⁷	573	24325 ⁷²⁴	65
26	15	22144 ⁶¹⁵	560	22704 ⁶²⁹	568	23272 ⁶⁴⁵	575	23847 ⁶⁶¹	583	24430 ⁶⁷⁸	589	25019 ⁶⁹⁴	64
27	15	22731 ⁵⁸⁷	575	23306 ⁶⁰²	583	23889 ⁶¹⁷	590	24479 ⁶³²	598	25077 ⁶⁴⁷	605	25682 ⁶⁶³	63
28	15	23290 ⁵⁵⁹	590	23880 ⁵⁷⁴	597	24477 ⁵⁸⁸	605	25082 ⁶⁰³	612	25694 ⁶¹⁷	619	26313 ⁶³¹	62
29	16	23821 ⁵³¹	603	24424 ⁵⁴⁴	610	25034 ⁵⁵⁷	619	25653 ⁵⁷¹	626	26279 ⁵⁸⁵	633	26912 ⁵⁹⁹	61
30	16	24322 ⁵⁰¹	616	24938 ⁵¹⁴	623	25561 ⁵²⁷	631	26192 ⁵³⁹	639	26831 ⁵⁵²	647	27478 ⁵⁶⁶	60
31	16	24794 ⁴⁷²	627	25421 ⁴⁸³	636	26057 ⁴⁹⁶	643	26700 ⁵⁰⁸	651	27351 ⁵²⁰	659	28010 ⁵³²	59
32	17	25235 ⁴⁴¹	638	25873 ⁴⁵²	647	26520 ⁴⁶³	655	27175 ⁴⁷⁵	662	27837 ⁴⁸⁶	671	28508 ⁴⁹⁸	58
33	17	25645 ⁴¹⁰	649	26294 ⁴²¹	657	26951 ⁴³¹	665	27616 ⁴⁴¹	673	28289 ⁴⁵²	682	28971 ⁴⁶³	57
34	17	26024 ³⁷⁹	658	26682 ³⁸⁸	667	27349 ³⁹⁸	674	28023 ⁴⁰⁷	684	28707 ⁴¹⁸	691	29398 ⁴²⁷	56
35	18	26371 ³⁴⁷	667	27038 ³⁵⁶	675	27713 ³⁶⁴	684	28397 ³⁷⁴	692	29089 ³⁸²	701	29790 ³⁹²	55
36	18	26685 ³¹⁴	675	27360 ³²²	683	28043 ³³⁰	692	28735 ³³⁸	700	29435 ³⁴⁶	709	30144 ³⁵⁴	54
37	18	26967 ²⁸²	682	27649 ²⁸⁹	690	28339 ²⁹⁶	699	29038 ³⁰³	708	29746 ³¹¹	716	30462 ³¹⁸	53
38	18	27216 ²⁴⁹	688	27904 ²⁵⁵	697	28601 ²⁶²	706	29307 ²⁶⁹	713	30020 ²⁷⁴	723	30743 ²⁸¹	52
39	18	27432 ²¹⁶	693	28125 ²²¹	702	28827 ²²⁶	711	29538 ²³¹	719	30257 ²³⁷	728	30985 ²⁴²	51
40	18	27614 ¹⁸²	698	28312 ¹⁸⁷	706	29018 ¹⁹¹	716	29734 ¹⁹⁶	724	30458 ²⁰¹	733	31191 ²⁰⁶	50
41	18	27763 ¹⁴⁹	701	28464 ¹⁵²	710	29174 ¹⁵⁶	719	29893 ¹⁵⁹	728	30621 ¹⁶³	737	31358 ¹⁶⁷	49
42	18	27877 ¹¹⁴	704	28581 ¹¹⁷	713	29294 ¹²⁰	722	30016 ¹²³	731	30747 ¹²⁶	739	31486 ¹²⁸	48
43	18	27958 ⁸¹	706	28664 ⁸³	714	29378 ⁸⁴	724	30102 ⁸⁶	733	30835 ⁸⁸	741	31576 ⁹⁰	47
44	18	28004 ⁴⁶	707	28711 ⁴⁷	716	29427 ⁴⁹	725	30152 ⁵⁰	733	30885 ⁵⁰	743	31628 ⁵²	46
45	18	28016 ¹²	707	28723 ¹²	717	29440 ¹³	724	30164 ¹²	734	30898 ¹³	743	31641 ¹³	45

$$f(a \pm nw) = f(a) \pm n \left[f'(a) \pm n \frac{f''(a)}{2} \right], \quad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$$B = (d - \delta)^\circ.$$

d is always numerically larger than δ .

$$0''.00001 = 0''.036 = \text{unit.}$$

δ	$\frac{v}{2}$	8°.5	hor. diff.	8°.6	hor. diff.	8°.7	hor. diff.	8°.8	hor. diff.	8°.9	hor. diff.	9°.0	d
0		0000	0	0000	0	0000	0	0000	0	0000	0	0000	90
1		1110	27	1137	27	1164	27	1191	27	1218	28	1246	89
2		2219	53	2272	54	2326	54	2380	55	2435	56	2491	88
3		3326	79	3405	81	3486	81	3567	82	3649	84	3733	87
4	3	4428	106	4534	107	4641	108	4749	110	4859	111	4970	86
5	3	5525	131	5656	134	5790	135	5925	137	6062	138	6200	85
6	4	6614	158	6772	160	6932	162	7094	164	7258	165	7423	84
7	5	7696	184	7880	186	8066	188	8254	191	8445	192	8637	83
8	6	8768	210	8978	212	9190	214	9404	217	9621	220	9841	82
9	7	9829	235	10064	238	10302	240	10542	244	10786	246	11032	81
10	7	10878	260	11138	263	11401	266	11667	270	11937	272	12209	80
11	8	11914	285	12199	288	12487	291	12778	295	13073	298	13371	79
12	9	12935	309	13244	313	13557	316	13873	320	14193	324	14517	78
13	9	13940	333	14273	337	14610	341	14951	344	15295	349	15644	77
14	10	14928	356	15284	361	15645	365	16010	369	16379	373	16752	76
15	10	15897	379	16276	384	16660	389	17049	393	17442	398	17840	75
16	11	16846	403	17249	407	17656	411	18067	417	18484	421	18905	74
17	12	17775	425	18200	429	18629	434	19063	440	19503	444	19947	73
18	13	18682	446	19128	451	19579	457	20036	462	20498	467	20965	72
19	13	19566	467	20033	472	20505	478	20983	484	21467	489	21956	71
20	14	20425	488	20913	493	21406	499	21905	505	22410	510	22920	70
21	14	21260	507	21767	513	22280	519	22799	526	23325	531	23856	69
22	15	22068	526	22594	533	23127	539	23666	545	24211	552	24763	68
23	15	22849	545	23394	551	23945	558	24503	565	25068	571	25639	67
24	15	23601	563	24164	570	24734	576	25310	583	25893	590	26483	66
25	16	24325	580	24905	587	25492	594	26086	601	26687	608	27295	65
26	17	25019	596	25615	604	26219	611	26830	618	27448	625	28073	64
27	17	25682	612	26294	619	26913	627	27540	634	28174	642	28816	63
28	18	26313	627	26940	635	27575	642	28217	650	28867	657	29524	62
29	17	26912	641	27553	649	28202	657	28859	664	29523	672	30195	61
30	18	27478	655	28133	662	28795	670	29465	679	30144	686	30830	60
31	19	28010	668	28678	675	29353	683	30036	691	30727	699	31426	59
32	19	28508	679	29187	687	29874	695	30569	704	31273	711	31984	58
33	19	28971	690	29661	698	30359	706	31065	715	31780	722	32502	57
34	19	29398	700	30098	708	30806	717	31523	725	32248	733	32981	56
35	20	29790	709	30499	717	31216	726	31942	734	32676	743	33419	55
36	20	30144	717	30861	726	31587	735	32322	743	33065	751	33816	54
37	20	30462	725	31187	733	31920	742	32662	751	33413	759	34172	53
38	20	30743	731	31474	740	32214	749	32963	757	33720	766	34486	52
39	20	30985	737	31722	746	32468	755	33223	763	33986	772	34758	51
40	21	31191	741	31932	751	32683	759	33442	768	34210	777	34987	50
41	21	31358	745	32103	754	32857	764	33621	772	34393	781	35174	49
42	21	31486	749	32235	757	32992	766	33758	775	34533	784	35317	48
43	21	31576	751	32327	759	33086	768	33854	778	34632	786	35418	47
44	21	31628	752	32380	760	33140	769	33909	779	34688	787	35475	46
45	21	31641	752	32393	761	33154	769	33923	778	34701	788	35489	45

$$f(a \pm nw) = f(a) \pm n \left[f'(a) \pm n \frac{f''(a)}{2} \right], \quad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$$B = (d - \delta)^2.$$

 d is always numerically larger than δ .

 $0^{\circ}.00001 = 0^{\circ}.036 = \text{unit}.$

$\frac{a-a_0}{\delta}$	$\frac{v}{2}$	$9^{\circ}.0$	hor. diff.	$9^{\circ}.1$	hor. diff.	$9^{\circ}.2$	hor. diff.	$9^{\circ}.3$	hor. diff.	$9^{\circ}.4$	hor. diff.	$9^{\circ}.5$	$\frac{a-a_0}{d}$
0		0000	0	0000	0	0000	0	0000	0	0000	0	0000	90
1		1246 ¹²⁴⁶	28	1274 ¹²⁷⁴	29	1303 ¹³⁰³	29	1332 ¹³³²	29	1361 ¹³⁶¹	29	1390 ¹³⁹⁰	89
2		2491 ¹²⁴⁵	56	2547 ¹²⁷³	57	2604 ¹²⁹⁸	58	2662 ¹³²⁶	58	2720 ¹³⁵⁶	59	2779 ¹³⁸⁵	88
3		3733 ¹²⁴²	84	3817 ¹²⁷⁰	85	3902 ¹²⁹³	86	3988 ¹³²²	88	4076 ¹³⁵⁰	88	4164 ¹³⁸⁰	87
4	3	4970 ¹²³⁰	112	5082 ¹²⁵⁸	113	5195 ¹²⁸⁷	115	5310 ¹³¹⁵	116	5426 ¹³⁴⁴	118	5544 ¹³⁷³	86
5	4	6200 ¹²²³	140	6340 ¹²⁵¹	142	6482 ¹²⁷⁹	143	6625 ¹³⁰⁷	145	6770 ¹³³⁶	147	6917 ¹³⁶⁴	85
6	5	7423 ¹²¹⁴	168	7591 ¹²⁴¹	170	7761 ¹²⁶⁹	171	7932 ¹²⁹⁷	174	8106 ¹³²⁵	175	8281 ¹³⁵⁴	84
7	6	8637 ¹²⁰⁴	195	8832 ¹²³¹	198	9030 ¹²⁵⁸	199	9229 ¹²⁸⁶	202	9431 ¹³¹⁴	204	9635 ¹³⁴²	83
8	7	9841 ¹¹⁹¹	222	10063 ¹²¹⁸	225	10288 ¹²⁴⁵	227	10515 ¹²⁷³	230	10745 ¹³⁰⁰	232	10977 ¹³²⁹	82
9	8	11032 ¹¹⁷⁷	249	11281 ¹²⁰³	252	11533 ¹²³⁰	255	11788 ¹²⁵⁷	257	12045 ¹²⁸⁶	261	12306 ¹³¹³	81
10	8	12209 ¹¹⁶²	275	12484 ¹¹⁸⁹	279	12763 ¹²¹⁵	282	13045 ¹²⁴²	286	13331 ¹²⁶⁸	288	13619 ¹²⁹⁶	80
11	9	13371 ¹¹⁴⁶	302	13673 ¹¹⁷¹	305	13978 ¹¹⁹⁸	309	14287 ¹²²⁴	312	14599 ¹²⁵¹	316	14915 ¹²⁷⁸	79
12	10	14517 ¹¹²⁷	327	14844 ¹¹⁵³	332	15176 ¹¹⁷⁹	335	15511 ¹²⁰⁵	339	15850 ¹²³¹	343	16193 ¹²⁵⁸	78
13	11	15644 ¹¹⁰⁸	353	15997 ¹¹³⁴	358	16355 ¹¹⁵⁸	361	16716 ¹¹⁸⁴	365	17081 ¹²¹⁰	370	17451 ¹²³⁶	77
14	11	16752 ¹⁰⁸⁸	379	17131 ¹¹¹²	382	17513 ¹¹³⁷	387	17900 ¹¹⁶²	391	18291 ¹¹⁸⁷	396	18687 ¹²¹²	76
15	12	17840 ¹⁰⁶⁵	403	18243 ¹⁰⁸⁹	407	18650 ¹¹¹⁴	412	19062 ¹¹³⁸	416	19478 ¹¹⁶³	421	19899 ¹¹⁸⁸	75
16	12	18905 ¹⁰⁴²	427	19332 ¹⁰⁶⁵	432	19764 ¹⁰⁸⁹	436	20200 ¹¹¹³	441	20641 ¹¹³⁸	446	21087 ¹¹⁶²	74
17	13	19947 ¹⁰¹⁸	450	20397 ¹⁰⁴¹	456	20853 ¹⁰⁶³	460	21313 ¹⁰⁸⁷	466	21779 ¹¹¹⁰	470	22249 ¹¹³⁵	73
18	14	20965 ⁹⁹¹	473	21438 ¹⁰¹³	478	21916 ¹⁰³⁶	484	22400 ¹⁰⁵⁹	489	22889 ¹⁰⁸²	495	23384 ¹¹⁰⁵	72
19	14	21956 ⁹⁶⁴	495	22451 ⁹⁸⁶	501	22952 ¹⁰⁰⁸	507	23459 ¹⁰³⁰	512	23971 ¹⁰⁵³	518	24489 ¹⁰⁷⁵	71
20	15	22920 ⁹³⁶	517	23437 ⁹⁵⁷	523	23960 ⁹⁷⁸	529	24489 ¹⁰⁰⁰	535	25024 ¹⁰²¹	540	25564 ¹⁰⁴⁴	70
21	16	23856 ⁹⁰⁷	538	24394 ⁹²⁷	544	24938 ⁹⁴⁸	551	25489 ⁹⁶⁸	556	26045 ⁹⁹⁰	563	26608 ¹⁰¹⁰	69
22	16	24763 ⁸⁷⁶	558	25321 ⁸⁹⁶	565	25886 ⁹¹⁵	571	26457 ⁹³⁵	578	27035 ⁹⁵⁵	583	27618 ⁹⁷⁷	68
23	17	25639 ⁸⁴⁴	578	26217 ⁸⁶³	584	26801 ⁸⁸³	591	27392 ⁹⁰²	598	27990 ⁹²²	605	28595 ⁹⁴¹	67
24	18	26483 ⁸¹²	597	27080 ⁸³⁰	604	27684 ⁸⁴⁸	610	28294 ⁸⁶⁷	618	28912 ⁸⁸⁵	624	29536 ⁹⁰⁴	66
25	18	27295 ⁷⁷⁸	615	27910 ⁷⁹⁵	622	28532 ⁸¹³	629	29161 ⁸³¹	636	29797 ⁸⁴⁹	643	30440 ⁸⁶⁷	65
26	18	28073 ⁷⁴³	632	28705 ⁷⁶⁰	640	29345 ⁷⁷⁷	647	29992 ⁷⁹³	654	30646 ⁸¹¹	661	31307 ⁸²⁹	64
27	19	28816 ⁷⁰⁸	649	29465 ⁷²⁴	657	30122 ⁷³⁹	663	30785 ⁷⁵⁶	672	31457 ⁷⁷²	679	32136 ⁷⁸⁸	63
28	19	29524 ⁶⁷¹	665	30189 ⁶⁸⁶	672	30861 ⁷⁰²	680	31541 ⁷¹⁷	688	32229 ⁷³³	695	32924 ⁷⁴⁹	62
29	20	30195 ⁶³⁵	680	30875 ⁶⁴⁹	688	31563 ⁶⁶³	695	32258 ⁶⁷⁸	704	32962 ⁶⁹¹	711	33673 ⁷⁰⁶	61
30	20	30830 ⁵⁹⁶	694	31524 ⁶⁰⁹	702	32226 ⁶²³	710	32936 ⁶³⁶	717	33653 ⁶⁵¹	726	34379 ⁶⁶⁵	60
31	20	31426 ⁵⁵⁸	707	32133 ⁵⁷¹	716	32849 ⁵⁸³	723	33572 ⁵⁹⁶	732	34304 ⁶⁰⁸	740	35044 ⁶²¹	59
32	21	31984 ⁵¹⁸	720	32704 ⁵³⁰	728	33432 ⁵⁴¹	736	34168 ⁵⁵³	744	34912 ⁵⁶⁶	753	35665 ⁵⁷⁷	58
33	21	32502 ⁴⁷⁹	732	33234 ⁴⁸⁹	739	33973 ⁵⁰⁰	748	34721 ⁵¹¹	757	35478 ⁵²¹	764	36242 ⁵³³	57
34	21	32981 ⁴³⁸	742	33723 ⁴⁴⁸	750	34473 ⁴⁵⁸	759	35232 ⁴⁶⁸	767	35999 ⁴⁷⁸	776	36775 ⁴⁸⁸	56
35	22	33419 ³⁹⁷	752	34171 ⁴⁰⁶	760	34931 ⁴¹⁵	769	35700 ⁴²⁴	777	36477 ⁴³³	786	37263 ⁴⁴²	55
36	22	33816 ³⁵⁶	761	34577 ³⁶⁴	769	35346 ³⁷²	778	36124 ³⁷⁹	786	36910 ³⁸⁷	795	37705 ³⁹⁵	54
37	22	34172 ³¹⁴	769	34941 ³²⁰	777	35718 ³²⁷	785	36503 ³³⁵	794	37297 ³⁴²	803	38100 ³⁵⁰	53
38	22	34486 ²⁷²	775	35261 ²⁷⁸	784	36045 ²⁸⁴	793	36838 ²⁹⁰	801	37639 ²⁹⁶	811	38450 ³⁰²	52
39	23	34758 ²²⁹	781	35539 ²³⁴	790	36329 ²³⁹	799	37128 ²⁴⁴	807	37935 ²⁵⁰	817	38752 ³⁰⁸	51
40	23	34987 ¹⁸⁷	786	35773 ¹⁹¹	795	36568 ¹⁹⁵	804	37372 ¹⁹⁹	813	38185 ²⁰²	821	39006 ²⁰⁷	50
41	23	35174 ¹⁴³	790	35964 ¹⁴⁶	799	36763 ¹⁴⁹	808	37571 ¹⁵²	816	38387 ¹⁵⁶	826	39213 ²¹¹	49
42	23	35317 ¹⁰¹	793	36110 ¹⁰³	802	36912 ¹⁰⁵	811	37723 ¹⁰⁷	820	38543 ¹⁰⁹	829	39372 ²¹⁵	48
43	23	35418 ⁵⁷	795	36213 ⁵⁸	804	37017 ⁵⁹	813	37830 ⁶¹	822	38652 ⁶²	831	39483 ²¹⁹	47
44	23	35475 ¹⁴	796	36271 ¹⁴	805	37076 ¹⁵	815	37891 ¹⁴	823	38714 ¹⁴	832	39546 ¹⁴	46
45	23	35489	796	36285	806	37091	814	37905	823	38728	832	39560	45
$\frac{a-a_0}{\delta}$	$\frac{v}{2}$	$9^{\circ}.0$	hor. diff.	$9^{\circ}.1$	hor. diff.	$9^{\circ}.2$	hor. diff.	$9^{\circ}.3$	hor. diff.	$9^{\circ}.4$	hor. diff.	$9^{\circ}.5$	$\frac{a-a_0}{d}$

$$f(a \pm nw) = f(a) \pm n \left[f'(a) \pm n \frac{f''(a)}{2} \right], \quad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$$B = (d - \delta)^\circ.$$

d is always numerically larger than δ .

$$0''.00001 = 0''.036 = \text{unit.}$$

δ	$\frac{v}{2}$	9°.5	hor. diff.	9°.6	hor. diff.	9°.7	hor. diff.	9°.8	hor. diff.	9°.9	hor. diff.	10°.0	d
0		0000	0	0000	0	0000	0	0000	0	0000	0	0000	90
1		1390	30	1420	30	1450	30	1480	31	1511	31	1542	89
2		2779	59	2838	60	2898	61	2959	62	3021	62	3083	88
3	3	4164	89	4253	90	4343	91	4434	92	4526	93	4619	87
4	4	5544	118	5662	120	5782	122	5904	122	6026	124	6150	86
5	5	6917	148	7065	149	7214	152	7366	153	7519	154	7673	85
6	6	8281	177	8458	180	8638	181	8819	183	9002	185	9187	84
7	6	9635	206	9841	209	10050	211	10261	213	10474	215	10689	83
8	7	10977	235	11212	238	11450	240	11690	243	11933	245	12178	82
9	8	12306	263	12569	266	12835	270	13105	272	13377	275	13652	81
10	9	13619	291	13910	295	14205	298	14503	301	14804	304	15108	80
11	10	14915	319	15234	323	15557	326	15883	330	16213	333	16546	79
12	11	16193	347	16540	350	16890	354	17244	358	17602	361	17963	78
13	11	17451	373	17824	378	18202	381	18583	386	18969	389	19358	77
14	12	18687	399	19086	405	19491	408	19899	413	20312	417	20729	76
15	13	19899	426	20325	430	20755	435	21190	440	21630	444	22074	75
16	14	21087	452	21539	456	21995	460	22455	466	22921	471	23392	74
17	15	22249	476	22725	481	23206	487	23693	491	24184	496	24680	73
18	15	23384	500	23884	505	24389	511	24900	517	25417	521	25938	72
19	16	24489	524	25013	529	25542	535	26077	541	26618	546	27164	71
20	17	25564	547	26111	552	26663	559	27222	564	27786	570	28356	70
21	18	26608	569	27177	574	27751	581	28332	588	28920	593	29513	69
22	18	27618	591	28209	596	28805	603	29408	610	30018	615	30633	68
23	19	28595	611	29206	617	29823	625	30448	630	31078	638	31716	67
24	20	29536	631	30167	638	30805	644	31449	652	32101	658	32759	66
25	20	30440	651	31091	657	31748	664	32412	671	33083	679	33762	65
26	21	31307	669	31976	676	32652	683	33335	690	34025	697	34722	64
27	21	32136	686	32822	693	33515	701	34216	709	34925	715	35640	63
28	22	32924	703	33627	711	34338	718	35056	725	35781	733	36514	62
29	22	33673	718	34391	727	35118	734	35852	742	36594	749	37343	61
30	23	34379	734	35113	741	35854	750	36604	757	37361	765	38126	60
31	23	35044	747	35791	756	36547	763	37310	772	38082	780	38862	59
32	23	35665	760	36425	769	37194	777	37971	785	38756	794	39550	58
33	24	36242	773	37015	781	37796	790	38586	797	39383	806	40189	57
34	24	36775	784	37559	792	38351	801	39152	810	39962	817	40779	56
35	25	37263	794	38057	803	38860	811	39671	820	40491	828	41319	55
36	24	37705	803	38508	812	39320	821	40141	829	40970	838	41808	54
37	25	38100	812	38912	821	39733	829	40562	837	41399	847	42246	53
38	25	38450	819	39269	827	40096	837	40933	845	41778	854	42632	52
39	25	38752	825	39577	834	40411	843	41254	851	42105	861	42966	51
40	25	39006	831	39837	839	40676	848	41524	857	42381	866	43247	50
41	26	39213	835	40048	843	40891	853	41744	861	42605	870	43475	49
42	26	39372	838	40210	847	41057	855	41912	865	42777	873	43650	48
43	25	39483	840	40323	849	41172	857	42029	867	42896	876	43772	47
44	26	39546	841	40387	850	41237	859	42096	868	42964	877	43841	46
45	26	39560	842	40402	850	41252	859	42111	868	42979	877	43856	45

$$f(a \pm nw) = f(a) \pm n \left[f'(a) \pm n \frac{f''(a)}{2} \right], \quad \frac{f'''(a)}{2} = \frac{v}{2}.$$

TABLE III.

$$C = k \xi \sec d - (\alpha - \alpha_0)^\circ.$$

$k \xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$.

for $d - \delta_0 \leq 1^\circ$, $0^\circ.000001 = 0^\circ.0036 = \text{unit}$; for $d - \delta_0 > 1^\circ$, $0^\circ.00001 = 0^\circ.036 = \text{unit}$.

$\alpha - \alpha_0$ $d - \delta_0$	$0^\circ.0$	hor. diff.	$0^\circ.1$	hor. diff.	$0^\circ.2$	hor. diff.	$0^\circ.3$	hor. diff.	$0^\circ.4$	hor. diff.	$0^\circ.5$	$\alpha - \alpha_0$ $d - \delta_0$
0.0	0	0	0	1	1	2	3	4	7	6	13	0.0
.1		0	0 ₀	1	1 ₁	2	3 ₂	4	7 ₂	6	13 ₃	.1
.2		1	1 ₀	1	2 ₂	3	5 ₂	4	9 ₃	7	16 ₄	.2
.3		1	1 ₂	3	4 ₂	3	7 ₃	5	12 ₄	8	20 ₅	.3
.4		3	3 ₁	3	6 ₂	4	10 ₄	6	16 ₆	9	25 ₇	.4
0.5		4	4 ₂	4	8 ₄	6	14 ₅	8	22 ₆	10	32 ₈	0.5
.6		6	6 ₂	6	12 ₄	7	19 ₆	9	28 ₈	12	40 ₁₀	.6
.7		8	8 ₂	8	16 ₄	9	25 ₇	11	36 ₉	14	50 ₁₁	.7
.8		10	10 ₂	10	20 ₅	12	32 ₈	13	45 ₁₁	16	61 ₁₃	.8
.9		12	12 ₃	13	25 ₆	15	40 ₈	16	56 ₁₁	18	74 ₁₅	.9
1.0		15	15 ₃	16	31 ₆	17	48 ₈	19	67 ₁₁	22	89 ₁₅	1.0
1.1		2	2 ₀	2	4 ₀	2	6 ₁	2	8 ₁	2	10 ₁	1.1
1.2		2	2 ₁	2	4 ₁	3	7 ₁	2	9 ₂	3	12 ₂	1.2
1.3		3	3 ₀	2	5 ₁	3	8 ₁	3	11 ₂	3	14 ₂	1.3
1.4		3	3 ₀	2	6 ₁	3	9 ₂	4	13 ₁	3	16 ₂	1.4
1.5		3	3 ₁	4	7 ₁	4	11 ₁	3	14 ₂	4	18 ₂	1.5
1.6		4	4 ₀	4	8 ₁	4	12 ₁	4	16 ₂	5	21 ₃	1.6
1.7		4	4 ₁	5	9 ₁	5	14 ₁	4	18 ₂	5	23 ₃	1.7
1.8		5	5 ₁	5	10 ₁	5	15 ₂	5	20 ₂	6	26 ₃	1.8
1.9		6	6 ₁	5	11 ₁	6	17 ₂	6	23 ₃	6	29 ₃	1.9
2.0		6	6 ₀	6	12 ₁	6	18 ₁	7	25 ₂	7	32 ₃	2.0
2.1		7	7 ₁	7	14 ₂	6	20 ₂	8	28 ₃	7	35 ₃	2.1
2.2		7	7 ₀	8	15 ₁	7	22 ₂	8	30 ₂	8	38 ₃	2.2
2.3		8	8 ₁	8	16 ₁	8	24 ₂	9	33 ₃	9	42 ₄	2.3
2.4		9	9 ₁	9	18 ₁	9	27 ₂	9	36 ₃	9	45 ₄	2.4
2.5		10	10 ₀	9	19 ₂	10	29 ₂	10	39 ₃	10	49 ₄	2.5
2.6		10	10 ₁	11	21 ₂	10	31 ₂	11	42 ₃	11	53 ₄	2.6
2.7		11	11 ₁	11	22 ₂	12	34 ₂	11	45 ₃	12	57 ₄	2.7
2.8		12	12 ₁	12	24 ₂	12	36 ₂	12	48 ₃	13	61 ₄	2.8
2.9		13	13 ₁	13	26 ₂	13	39 ₃	13	52 ₄	13	65 ₄	2.9
3.0		14	14 ₁	14	28 ₂	13	41 ₂	15	56 ₄	14	70 ₅	3.0
3.1		15	15 ₁	14	29 ₂	15	44 ₃	15	59 ₄	15	74 ₅	3.1
3.2		16	16 ₁	15	31 ₂	16	47 ₃	16	63 ₄	16	79 ₅	3.2
3.3		17	17 ₁	16	33 ₂	17	50 ₃	17	67 ₄	17	84 ₅	3.3
3.4		18	18 ₁	17	35 ₂	18	53 ₃	18	71 ₄	18	89 ₅	3.4
3.5		19	19 ₁	18	37 ₂	19	56 ₃	19	75 ₄	20	95 ₅	3.5
3.6		20	20 ₁	20	40 ₂	20	60 ₃	20	80 ₄	20	100 ₅	3.6
3.7		21	21 ₁	21	42 ₂	21	63 ₃	21	84 ₄	22	106 ₅	3.7
3.8		22	22 ₁	22	44 ₂	22	66 ₃	23	89 ₄	22	111 ₅	3.8
3.9		23	23 ₁	23	46 ₂	24	70 ₄	23	93 ₄	24	117 ₅	3.9
4.0		24	24 ₁	25	49 ₃	24	73 ₃	25	98 ₅	25	123 ₆	4.0
4.1		26	26 ₁	25	51 ₂	26	77 ₄	26	103 ₅	26	129 ₆	4.1
4.2		27	27 ₁	27	54 ₃	27	81 ₄	27	108 ₅	28	136 ₆	4.2
4.3		28	28 ₁	28	56 ₂	29	85 ₄	28	113 ₅	29	142 ₆	4.3
4.4		30	30 ₁	29	59 ₃	30	89 ₄	30	119 ₅	30	149 ₆	4.4
4.5		31	31 ₁	31	62 ₃	31	93 ₄	31	124 ₅	31	155 ₆	4.5
4.6		32	32 ₁	33	65 ₃	32	97 ₄	33	130 ₆	32	162 ₇	4.6
4.7		34	34 ₁	33	67 ₂	34	101 ₄	34	135 ₅	34	169 ₇	4.7
4.8		35	35 ₁	35	70 ₃	36	106 ₅	35	141 ₆	36	177 ₈	4.8
4.9		37	37 ₂	36	73 ₃	37	110 ₄	37	147 ₆	37	184 ₇	4.9
5.0		38	38 ₁	38	76 ₃	39	115 ₅	38	153 ₆	39	192 ₈	5.0
$\alpha - \alpha_0$ $d - \delta_0$	$0^\circ.0$	hor. diff.	$0^\circ.1$	hor. diff.	$0^\circ.2$	hor. diff.	$0^\circ.3$	hor. diff.	$0^\circ.4$	hor. diff.	$0^\circ.5$	$\alpha - \alpha_0$ $d - \delta_0$

TABLE III.

$$C = k \xi \sec d - (\alpha - \alpha_0)^\circ.$$

$k \xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$.

for $d - \delta_0 \leq 1^\circ$, $0^\circ.000001 = 0''.0036 = \text{unit}$; for $d - \delta_0 > 1^\circ$, $0^\circ.00001 = 0''.036 = \text{unit}$.

$\alpha - \alpha_0$ $d - \delta_0$	$0^\circ.5$	hor. diff.	$0^\circ.6$	hor. diff.	$0^\circ.7$	hor. diff.	$0^\circ.8$	hor. diff.	$0^\circ.9$	hor. diff.	$1^\circ.0$	$\alpha - \alpha_0$ $d - \delta_0$
0.0	13	9	22	13	35	17	52	22	74	28	102	0.0
.1	13 ⁰	10	23 ¹	13	36 ¹	17	53 ¹	22	75 ¹	28	103 ¹	.1
.2	16 ³	10	26 ³	13	39 ³	18	57 ⁴	22	79 ⁴	29	108 ⁵	.2
.3	20 ⁴	10	30 ⁴	14	44 ⁵	19	63 ⁶	23	86 ⁷	29	115 ¹¹	.3
.4	25 ⁷	12	37 ⁸	15	52 ⁹	20	72 ¹⁰	24	96 ¹²	30	126 ¹⁴	.4
0.5	32 ⁸	13	45 ¹⁰	16	61 ¹²	21	82 ¹⁴	26	108 ¹⁵	32	140 ¹⁶	0.5
.6	40 ¹⁰	15	55 ¹²	18	73 ¹⁴	23	96 ¹⁶	27	123 ¹⁸	33	156 ²⁰	.6
.7	50 ¹¹	17	67 ¹³	20	87 ¹⁶	25	112 ¹⁸	29	141 ²¹	35	176 ²³	.7
.8	61 ¹³	19	80 ¹⁶	23	103 ¹⁸	27	130 ²¹	32	162 ²³	37	199 ²⁶	.8
.9	74 ¹⁵	22	96 ¹⁷	25	121 ²⁰	30	151 ²³	34	185 ²⁶	40	225 ²⁹	.9
1.0	89 ¹⁵	24	113 ¹⁷	28	141 ²⁰	33	174 ²³	37	211 ²⁶	43	254 ²⁹	1.0
1.1	10 ¹	3	13 ²	3	16 ²	4	20 ³	4	24 ³	5	29 ⁴	1.1
1.2	12 ²	3	15 ³	4	19 ³	4	23 ³	4	27 ³	5	32 ³	1.2
1.3	14 ²	4	18 ³	4	22 ³	4	26 ³	5	31 ⁴	5	36 ⁴	1.3
1.4	16 ²	4	20 ³	4	24 ³	5	29 ³	5	34 ³	6	40 ⁴	1.4
1.5	18 ²	5	23 ³	5	28 ³	5	33 ³	5	38 ⁴	6	44 ⁴	1.5
1.6	21 ³	5	26 ³	5	31 ³	5	36 ³	6	42 ⁴	7	49 ⁵	1.6
1.7	23 ³	6	29 ³	5	34 ⁴	6	40 ⁴	7	47 ⁵	7	54 ⁵	1.7
1.8	26 ³	6	32 ³	6	38 ⁴	7	45 ⁵	7	52 ⁵	8	60 ⁶	1.8
1.9	29 ³	6	35 ³	7	42 ⁴	7	49 ⁴	8	57 ⁵	8	65 ⁵	1.9
2.0	32 ³	7	39 ⁴	7	46 ⁴	8	54 ⁵	8	62 ⁵	9	71 ⁶	2.0
2.1	35 ³	7	42 ³	8	50 ⁴	9	59 ⁵	9	68 ⁶	9	77 ⁶	2.1
2.2	38 ³	8	46 ⁴	9	55 ⁵	9	64 ⁵	10	74 ⁶	10	84 ⁷	2.2
2.3	42 ⁴	9	51 ⁵	9	60 ⁵	10	70 ⁶	10	80 ⁶	11	91 ⁷	2.3
2.4	45 ⁴	10	55 ⁴	10	65 ⁵	11	76 ⁶	10	86 ⁶	12	98 ⁷	2.4
2.5	49 ⁴	10	59 ⁴	11	70 ⁵	12	82 ⁶	11	93 ⁷	12	105 ⁷	2.5
2.6	53 ⁴	11	64 ⁵	12	76 ⁶	12	88 ⁶	12	100 ⁷	13	113 ⁸	2.6
2.7	57 ⁴	12	69 ⁵	12	81 ⁵	13	94 ⁷	13	107 ⁸	14	121 ⁸	2.7
2.8	61 ⁴	13	74 ⁵	13	87 ⁶	14	101 ⁷	14	115 ⁸	15	130 ⁹	2.8
2.9	65 ⁴	14	79 ⁵	14	93 ⁶	15	108 ⁷	15	123 ⁸	15	138 ⁸	2.9
3.0	70 ⁵	14	84 ⁵	15	99 ⁶	16	115 ⁷	16	131 ⁸	16	147 ⁹	3.0
3.1	74 ⁴	16	90 ⁶	16	106 ⁷	16	122 ⁸	17	139 ⁹	18	157 ¹⁰	3.1
3.2	79 ⁵	17	96 ⁶	17	113 ⁷	17	130 ⁸	18	148 ⁹	18	166 ⁹	3.2
3.3	84 ⁵	18	102 ⁶	18	120 ⁷	18	138 ⁸	19	157 ⁹	19	176 ¹⁰	3.3
3.4	89 ⁵	19	108 ⁶	19	127 ⁷	19	146 ⁸	20	166 ⁹	20	186 ¹⁰	3.4
3.5	95 ⁶	19	114 ⁶	20	134 ⁸	21	155 ⁹	21	176 ¹⁰	21	197 ¹¹	3.5
3.6	100 ⁵	21	121 ⁷	21	142 ⁸	21	163 ⁸	22	185 ⁹	23	208 ¹¹	3.6
3.7	106 ⁶	21	127 ⁶	23	150 ⁸	22	172 ⁹	23	195 ¹⁰	24	219 ¹¹	3.7
3.8	111 ⁵	23	134 ⁷	24	158 ⁸	23	181 ⁹	25	206 ¹⁰	25	231 ¹²	3.8
3.9	117 ⁶	24	141 ⁷	25	166 ⁸	25	191 ¹⁰	25	216 ¹⁰	26	242 ¹¹	3.9
4.0	123 ⁶	26	149 ⁸	25	174 ⁸	26	200 ⁹	27	227 ¹¹	27	254 ¹²	4.0
4.1	129 ⁶	27	156 ⁷	27	183 ⁹	27	210 ¹⁰	28	238 ¹¹	29	267 ¹³	4.1
4.2	136 ⁷	28	164 ⁸	28	192 ⁹	29	221 ¹¹	29	250 ¹²	29	279 ¹²	4.2
4.3	142 ⁷	29	171 ⁷	30	201 ⁹	30	231 ¹⁰	30	261 ¹¹	31	292 ¹³	4.3
4.4	149 ⁶	30	179 ⁸	32	211 ⁹	30	241 ¹¹	32	273 ¹³	33	306 ¹⁴	4.4
4.5	155 ⁷	33	188 ⁸	32	220 ⁹	32	252 ¹²	34	286 ¹³	33	319 ¹³	4.5
4.6	162 ⁷	34	196 ⁸	34	230 ¹⁰	34	264 ¹¹	34	298 ¹²	35	333 ¹⁴	4.6
4.7	169 ⁷	35	204 ⁸	36	240 ¹⁰	35	275 ¹²	36	311 ¹³	36	347 ¹⁵	4.7
4.8	177 ⁸	36	213 ⁹	37	250 ¹⁰	37	287 ¹²	37	324 ¹³	38	362 ¹⁵	4.8
4.9	184 ⁷	38	222 ⁹	38	260 ¹⁰	39	299 ¹²	38	337 ¹³	40	377 ¹⁵	4.9
5.0	192 ⁸	39	231 ⁹	39	270 ¹⁰	41	311 ¹²	40	351 ¹⁴	41	392 ¹⁵	5.0
$\alpha - \alpha_0$ $d - \delta_0$	$0^\circ.5$	hor. diff.	$0^\circ.6$	hor. diff.	$0^\circ.7$	hor. diff.	$0^\circ.8$	hor. diff.	$0^\circ.9$	hor. diff.	$1^\circ.0$	$\alpha - \alpha_0$ $d - \delta_0$

TABLE III.

$$C = k \xi \sec d - (\alpha - \alpha_0)^\circ.$$

$k \xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$.

for $d - \delta_0 \leq 1^\circ$, $0''.000001 = 0''.0036 = \text{unit}$; for $d - \delta_0 > 1^\circ$, $0''.00001 = 0''.036 = \text{unit}$.

$d - \delta_0$	$1^\circ.0$	hor. diff.	$1^\circ.1$	hor. diff.	$1^\circ.2$	hor. diff.	$1^\circ.3$	hor. diff.	$1^\circ.4$	hor. diff.	$1^\circ.5$	$d - \delta_0$
0.0	102	33	135	40	175	48	223	56	279	64	343	0.0
.1	103	34	137	40	177	48	225	56	281	64	345	.1
.2	108	34	142	41	183	48	231	56	287	65	352	.2
.3	115	35	150	42	192	49	241	57	298	65	363	.3
.4	126	36	162	43	205	50	255	58	313	67	380	.4
0.5	140	37	177	44	221	52	273	59	332	68	400	0.5
.6	156	39	195	46	241	54	295	61	356	69	425	.6
.7	176	41	217	48	265	55	320	63	383	72	455	.7
.8	199	43	242	51	293	57	350	65	415	74	489	.8
.9	225	46	271	53	324	60	384	68	452	76	528	.9
1.0	254	49	303	55	358	63	421	71	492	80	572	1.0
1.1	29	5	34	6	40	6	46	8	54	8	62	1.1
1.2	32	6	38	6	44	7	51	8	59	8	67	1.2
1.3	36	6	42	6	48	8	56	8	64	9	73	1.3
1.4	40	6	46	7	53	8	61	9	70	9	79	1.4
1.5	44	7	51	8	59	8	67	9	76	10	86	1.5
1.6	49	7	56	8	64	9	73	9	82	11	93	1.6
1.7	54	8	62	8	70	9	79	11	90	10	100	1.7
1.8	60	8	68	9	77	9	86	11	97	11	108	1.8
1.9	65	9	74	10	84	10	94	11	105	12	117	1.9
2.0	71	10	81	10	91	11	102	11	113	13	126	2.0
2.1	77	10	87	11	98	12	110	12	122	13	135	2.1
2.2	84	11	95	11	106	12	118	13	131	14	145	2.2
2.3	91	11	102	12	114	13	127	14	141	14	155	2.3
2.4	98	12	110	13	123	13	136	15	151	15	166	2.4
2.5	105	14	119	13	132	14	146	15	161	16	177	2.5
2.6	113	14	127	14	141	15	156	15	171	18	189	2.6
2.7	121	15	136	15	151	16	167	16	183	18	201	2.7
2.8	130	15	145	16	161	17	178	17	195	19	214	2.8
2.9	138	17	155	16	171	18	189	18	207	20	227	2.9
3.0	147	18	165	17	182	19	201	19	220	20	240	3.0
3.1	157	18	175	18	193	20	213	20	233	21	254	3.1
3.2	166	19	185	20	205	20	225	22	247	22	269	3.2
3.3	176	20	196	21	217	21	238	22	260	24	284	3.3
3.4	186	21	207	22	229	23	252	23	275	24	299	3.4
3.5	197	22	219	23	242	23	265	25	290	25	315	3.5
3.6	208	23	231	24	255	24	279	26	305	26	331	3.6
3.7	219	24	243	25	268	26	294	26	320	28	348	3.7
3.8	231	25	256	26	282	27	309	27	336	29	365	3.8
3.9	242	27	269	27	296	28	324	29	353	30	383	3.9
4.0	254	28	282	29	311	29	340	30	370	31	401	4.0
4.1	267	29	296	30	326	30	356	31	387	32	419	4.1
4.2	279	31	310	31	341	31	372	33	405	33	438	4.2
4.3	292	33	325	31	356	33	389	34	423	35	458	4.3
4.4	306	33	339	33	372	35	407	35	442	36	478	4.4
4.5	319	35	354	34	388	37	425	36	461	37	498	4.5
4.6	333	36	369	36	405	38	443	37	480	39	519	4.6
4.7	347	38	385	38	423	38	461	39	500	41	541	4.7
4.8	362	39	401	39	440	40	480	41	521	41	562	4.8
4.9	377	41	418	40	458	41	499	43	542	43	585	4.9
5.0	392	42	434	42	476	43	519	44	563	44	607	5.0
$d - \delta_0$	$1^\circ.0$	hor. diff.	$1^\circ.1$	hor. diff.	$1^\circ.2$	hor. diff.	$1^\circ.3$	hor. diff.	$1^\circ.4$	hor. diff.	$1^\circ.5$	$d - \delta_0$

TABLE III.

$$C = k \xi \sec d - (\alpha - \alpha_0)^\circ.$$

$k \xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$.

for $d - \delta_0 \leq 1^\circ$, $0''.000001 = 0''.0036 = \text{unit}$; for $d - \delta_0 > 1^\circ$, $0''.00001 = 0''.036 = \text{unit}$.

$\alpha - \alpha_0$ $d - \delta_0$	1°.5	hor. diff.	1°.6	hor. diff.	1°.7	hor. diff.	1°.8	hor. diff.	1°.9	hor. diff.	2°.0	$\alpha - \alpha_0$ $d - \delta_0$
0.0	343	73	416	83	499	94	593	104	697	116	813	0.0
.1	345 ²	74	419 ³	83	502 ³	94	596 ³	104	700 ³	116	816 ³	.1
.2	352 ¹¹	74	426 ¹²	84	510 ¹³	94	604 ¹³	104	708 ¹⁴	117	825 ¹⁵	.2
.3	363 ¹⁷	75	438 ¹⁷	85	523 ¹⁸	94	617 ²⁰	105	722 ²¹	118	840 ²¹	.3
.4	380 ²⁰	75	455 ²²	86	541 ²³	96	637 ²⁴	106	743 ²⁶	118	861 ²⁸	.4
0.5	400 ²⁵	77	477 ²⁷	87	564 ²⁹	97	661 ³⁰	108	769 ³²	120	889 ³⁴	0.5
.6	425 ³⁰	79	504 ³²	89	593 ³³	98	691 ³⁶	110	801 ³⁷	122	923 ³⁹	.6
.7	455 ³⁴	81	536 ³⁶	90	626 ³⁹	101	727 ⁴¹	111	838 ⁴⁴	124	962 ⁴⁶	.7
.8	489 ³⁹	83	572 ⁴²	93	665 ⁴⁴	103	768 ⁴⁷	114	882 ⁴⁹	126	1008 ⁵²	.8
.9	528 ⁴⁴	86	614 ⁴⁶	95	709 ⁴⁹	106	815 ⁵²	116	931 ⁵⁵	129	1060 ⁵⁷	.9
1.0	572	88	660	98	758	109	867	119	986	131	1117	1.0
1.1	62 ⁵	9	71 ⁵	10	81 ⁵	11	92 ⁵	13	105 ⁶	13	118 ⁶	1.1
1.2	67 ⁶	10	77 ⁶	10	87 ⁶	12	99 ⁷	12	111 ⁶	14	125 ⁷	1.2
1.3	73 ⁶	10	83 ⁶	11	94 ⁷	12	106 ⁷	13	119 ⁷	14	133 ⁸	1.3
1.4	79 ⁷	10	89 ⁷	12	101 ⁷	12	113 ⁸	13	126 ⁹	15	141 ⁹	1.4
1.5	86 ⁷	10	96 ⁸	12	108 ⁸	13	121 ⁸	14	135 ⁹	15	150 ⁹	1.5
1.6	93 ⁷	11	104 ⁸	12	116 ⁹	13	129 ⁹	15	144 ⁹	15	159 ¹⁰	1.6
1.7	100 ⁸	12	112 ⁹	13	125 ⁹	13	138 ¹⁰	15	153 ¹¹	16	169 ¹¹	1.7
1.8	108 ⁸	13	121 ⁹	13	134 ¹⁰	14	148 ¹⁰	16	164 ¹⁰	16	180 ¹¹	1.8
1.9	117 ⁹	13	130 ⁹	14	144 ¹⁰	14	158 ¹⁰	16	174 ¹⁰	17	191 ¹¹	1.9
2.0	126 ⁹	13	139 ¹⁰	15	154 ¹⁰	15	169 ¹¹	17	186 ¹¹	17	203 ¹²	2.0
2.1	135 ¹⁰	14	149 ¹¹	15	164 ¹¹	16	180 ¹¹	17	197 ¹¹	19	216 ¹³	2.1
2.2	145 ¹⁰	15	160 ¹¹	15	175 ¹²	16	191 ¹¹	19	210 ¹³	19	229 ¹³	2.2
2.3	155 ¹¹	16	171 ¹¹	16	187 ¹²	17	204 ¹³	19	223 ¹³	20	243 ¹⁴	2.3
2.4	166 ¹¹	16	182 ¹²	17	199 ¹³	18	217 ¹⁴	20	237 ¹⁴	20	257 ¹⁵	2.4
2.5	177 ¹²	17	194 ¹³	18	212 ¹³	19	231 ¹⁴	21	252 ¹⁵	20	272 ¹⁵	2.5
2.6	189 ¹²	18	207 ¹³	18	225 ¹⁴	20	245 ¹⁴	21	266 ¹⁵	21	287 ¹⁶	2.6
2.7	201 ¹³	19	220 ¹³	19	239 ¹⁴	20	259 ¹⁵	22	281 ¹⁶	22	303 ¹⁷	2.7
2.8	214 ¹³	19	233 ¹⁴	20	253 ¹⁵	21	274 ¹⁵	23	297 ¹⁶	23	320 ¹⁷	2.8
2.9	227 ¹³	20	247 ¹⁴	21	268 ¹⁵	22	290 ¹⁶	23	313 ¹⁶	25	338 ¹⁸	2.9
3.0	240 ¹⁴	21	261 ¹⁴	22	283 ¹⁵	23	306 ¹⁶	24	330 ¹⁷	26	356 ¹⁸	3.0
3.1	254 ¹⁴	22	276 ¹⁵	23	299 ¹⁶	24	323 ¹⁷	25	348 ¹⁸	26	374 ¹⁸	3.1
3.2	269 ¹⁵	22	291 ¹⁵	24	315 ¹⁶	25	340 ¹⁷	26	366 ¹⁸	28	394 ²⁰	3.2
3.3	284 ¹⁵	23	307 ¹⁶	25	332 ¹⁷	26	358 ¹⁸	27	385 ¹⁹	29	414 ²⁰	3.3
3.4	299 ¹⁵	25	324 ¹⁷	26	350 ¹⁸	27	377 ¹⁹	28	405 ²⁰	29	434 ²⁰	3.4
3.5	315 ¹⁶	26	341 ¹⁷	27	368 ¹⁸	28	396 ¹⁹	29	425 ²⁰	30	455 ²¹	3.5
3.6	331 ¹⁷	27	358 ¹⁸	28	386 ¹⁹	29	415 ¹⁹	30	445 ²²	32	477 ²²	3.6
3.7	348 ¹⁷	28	376 ¹⁸	29	405 ²⁰	30	435 ²¹	32	467 ²²	32	499 ²³	3.7
3.8	365 ¹⁸	29	394 ¹⁹	31	425 ²⁰	31	456 ²¹	33	489 ²²	33	522 ²³	3.8
3.9	383 ¹⁸	30	413 ¹⁹	32	445 ²⁰	32	477 ²¹	34	511 ²²	35	546 ²⁴	3.9
4.0	401 ¹⁸	31	432 ¹⁹	33	465 ²⁰	34	499 ²²	35	534 ²³	36	570 ²⁴	4.0
4.1	419 ¹⁸	33	452 ²⁰	34	486 ²¹	35	521 ²²	36	557 ²³	38	595 ²⁵	4.1
4.2	438 ¹⁹	35	473 ²⁰	35	508 ²²	36	544 ²³	38	582 ²⁴	38	620 ²⁵	4.2
4.3	458 ²⁰	35	493 ²²	37	530 ²³	38	568 ²⁴	38	606 ²⁶	40	646 ²⁶	4.3
4.4	478 ²⁰	37	515 ²²	38	553 ²³	39	592 ²⁴	40	632 ²⁵	41	673 ²⁷	4.4
4.5	498 ²¹	39	537 ²²	39	576 ²⁴	40	616 ²⁴	41	657 ²⁵	43	700 ²⁷	4.5
4.6	519 ²¹	40	559 ²²	41	600 ²⁴	41	641 ²⁵	43	684 ²⁷	44	728 ²⁸	4.6
4.7	541 ²²	41	582 ²³	42	624 ²⁴	43	667 ²⁶	44	711 ²⁷	45	756 ²⁸	4.7
4.8	562 ²³	43	605 ²³	43	648 ²⁴	45	693 ²⁶	46	739 ²⁸	46	785 ²⁹	4.8
4.9	585 ²³	44	629 ²⁴	45	674 ²⁵	46	720 ²⁷	47	767 ²⁸	48	815 ³⁰	4.9
5.0	607 ²²	46	653 ²⁴	46	699 ²⁵	48	747 ²⁷	49	796 ²⁹	50	846 ³¹	5.0
$\alpha - \alpha_0$ $d - \delta_0$	1°.5	hor. diff.	1°.6	hor. diff.	1°.7	hor. diff.	1°.8	hor. diff.	1°.9	hor. diff.	2°.0	$\alpha - \alpha_0$ $d - \delta_0$

TABLE III.

$$C = k \xi \sec d - (\alpha - \alpha_0)^\circ.$$

 $k \xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$.

0°.00001 = 0".036 = unit.

$\alpha - \alpha_0$ $d - \delta_0$	2°.0	hor. diff.	2°.1	hor. diff.	2°.2	hor. diff.	2°.3	hor. diff.	2°.4	hor. diff.	2°.5	$\alpha - \alpha_0$ $d - \delta_0$
0.0	00081	13	00094	14	00108	16	00124	16	00140	19	00159	0.0
.1	82 ¹	12	94 ⁰	15	109 ¹	15	124 ⁰	17	141 ¹	18	159 ⁰	.1
.2	82 ²	13	95 ²	15	110 ²	15	125 ²	17	142 ²	18	160 ²	.2
.3	84 ²	13	97 ²	15	112 ²	15	127 ²	17	144 ²	18	162 ³	.3
.4	86 ³	13	99 ³	15	114 ³	15	129 ³	17	146 ³	19	165 ³	.4
0.5	89 ³	13	102 ³	15	117 ³	15	132 ⁴	17	149 ⁵	19	168 ⁵	0.5
.6	92 ⁴	13	105 ⁵	15	120 ⁵	16	136 ⁵	18	154 ⁵	19	173 ⁵	.6
.7	96 ⁵	14	110 ⁵	15	125 ⁵	16	141 ⁵	18	159 ⁵	19	178 ⁵	.7
.8	101 ⁵	14	115 ⁵	15	130 ⁵	16	146 ⁶	18	164 ⁶	19	183 ⁷	.8
.9	106 ⁶	14	120 ⁶	15	135 ⁷	17	152 ⁷	18	170 ⁷	20	190 ⁷	.9
1.0	112 ⁶	14	126 ⁷	16	142 ⁷	17	159 ⁷	18	177 ⁸	20	197 ⁸	1.0
1.1	118 ⁷	15	133 ⁷	16	149 ⁸	17	166 ⁸	19	185 ⁸	20	205 ⁹	1.1
1.2	125 ⁸	15	140 ⁸	17	157 ⁸	17	174 ⁹	19	193 ⁹	21	214 ⁹	1.2
1.3	133 ⁸	15	148 ⁹	17	165 ⁹	18	183 ⁹	19	202 ¹⁰	21	223 ¹⁰	1.3
1.4	141 ⁹	16	157 ⁹	17	174 ¹⁰	18	192 ¹¹	20	212 ¹¹	21	233 ¹²	1.4
1.5	150 ⁹	16	166 ¹⁰	18	184 ¹⁰	19	203 ¹⁰	20	223 ¹¹	22	245 ¹¹	1.5
1.6	159 ¹⁰	17	176 ¹¹	18	194 ¹¹	19	213 ¹²	21	234 ¹²	22	256 ¹³	1.6
1.7	169 ¹¹	18	187 ¹¹	18	205 ¹²	20	225 ¹²	21	246 ¹³	23	269 ¹³	1.7
1.8	180 ¹¹	18	198 ¹²	19	217 ¹²	20	237 ¹³	22	259 ¹³	23	282 ¹⁴	1.8
1.9	191 ¹²	19	210 ¹²	19	229 ¹²	21	250 ¹³	22	272 ¹⁴	24	296 ¹⁴	1.9
2.0	203 ¹²	19	222 ¹²	20	242 ¹³	22	264 ¹⁴	23	287 ¹⁵	24	311 ¹⁵	2.0
2.1	216 ¹³	19	235 ¹³	21	256 ¹³	22	278 ¹⁴	24	302 ¹⁵	25	327 ¹⁶	2.1
2.2	229 ¹³	20	249 ¹⁴	22	271 ¹⁵	22	293 ¹⁶	24	317 ¹⁷	26	343 ¹⁷	2.2
2.3	243 ¹⁴	20	263 ¹⁴	23	286 ¹⁵	23	309 ¹⁶	25	334 ¹⁷	26	360 ¹⁸	2.3
2.4	257 ¹⁵	22	279 ¹⁵	23	302 ¹⁶	24	326 ¹⁷	25	351 ¹⁸	27	378 ¹⁹	2.4
2.5	272 ¹⁵	22	294 ¹⁷	24	318 ¹⁷	25	343 ¹⁸	26	369 ¹⁹	28	397 ²⁰	2.5
2.6	287 ¹⁶	24	311 ¹⁷	24	335 ¹⁸	26	361 ¹⁸	27	388 ¹⁹	29	417 ²⁰	2.6
2.7	303 ¹⁷	25	328 ¹⁷	25	353 ¹⁸	26	379 ²⁰	28	407 ²⁰	30	437 ²¹	2.7
2.8	320 ¹⁸	25	345 ¹⁹	26	371 ¹⁹	28	399 ²⁰	28	427 ²¹	31	458 ²²	2.8
2.9	338 ¹⁸	26	364 ¹⁹	26	390 ²⁰	29	419 ²⁰	29	448 ²¹	32	480 ²²	2.9
3.0	356 ¹⁸	26	382 ¹⁸	28	410 ²⁰	29	439 ²⁰	31	470 ²²	32	502 ²²	3.0
3.1	374 ¹⁸	28	402 ²⁰	29	431 ²¹	30	461 ²²	31	492 ²³	33	525 ²³	3.1
3.2	394 ²⁰	28	422 ²¹	30	452 ²²	31	483 ²³	32	515 ²⁴	35	550 ²⁴	3.2
3.3	414 ²⁰	29	443 ²²	31	474 ²²	32	506 ²³	33	539 ²⁴	35	574 ²⁶	3.3
3.4	434 ²¹	31	465 ²²	31	496 ²³	33	529 ²⁵	34	563 ²⁶	37	600 ²⁶	3.4
3.5	455 ²²	32	487 ²³	32	519 ²⁴	35	554 ²⁵	35	589 ²⁶	37	626 ²⁷	3.5
3.6	477 ²²	33	510 ²³	33	543 ²⁵	36	579 ²⁵	36	615 ²⁷	38	653 ²⁷	3.6
3.7	499 ²³	34	533 ²⁴	35	568 ²⁵	36	604 ²⁷	38	642 ²⁸	39	681 ²⁸	3.7
3.8	522 ²⁴	35	557 ²⁵	36	593 ²⁶	38	631 ²⁷	39	670 ²⁸	40	710 ²⁹	3.8
3.9	546 ²⁴	36	582 ²⁵	37	619 ²⁷	39	658 ²⁸	39	697 ²⁹	43	740 ³⁰	3.9
4.0	570 ²⁵	37	607 ²⁵	39	646 ²⁷	40	686 ²⁸	40	726 ²⁹	44	770 ³⁰	4.0
4.1	595 ²⁵	37	632 ²⁸	41	673 ²⁷	41	714 ²⁸	42	756 ³⁰	45	801 ³¹	4.1
4.2	620 ²⁵	40	660 ²⁸	41	701 ²⁸	42	743 ²⁹	44	787 ³¹	46	833 ³²	4.2
4.3	646 ²⁶	41	687 ²⁸	43	730 ²⁹	43	773 ³⁰	45	818 ³¹	47	865 ³²	4.3
4.4	673 ²⁷	42	715 ²⁹	44	759 ³⁰	45	804 ³¹	46	850 ³²	48	898 ³³	4.4
4.5	700 ²⁸	44	744 ²⁹	45	789 ³¹	46	835 ³²	47	882 ³⁴	50	932 ³⁴	4.5
4.6	728 ²⁸	45	773 ³⁰	47	820 ³¹	47	867 ³³	49	916 ³⁴	51	967 ³⁵	4.6
4.7	756 ²⁹	47	803 ³¹	48	851 ³²	49	900 ³⁴	50	950 ³⁶	53	1003 ³⁶	4.7
4.8	785 ³⁰	49	834 ³¹	49	883 ³²	51	934 ³⁴	52	986 ³⁶	53	1039 ³⁷	4.8
4.9	815 ³⁰	50	865 ³¹	51	916 ³³	52	968 ³⁴	53	1021 ³⁵	55	1076 ³⁷	4.9
5.0	846 ³¹	51	897 ³²	52	949 ³³	54	1003 ³⁵	54	1057 ³⁶	57	1114 ³⁸	5.0
$\alpha - \alpha_0$ $d - \delta_0$	2°.0	hor. diff.	2°.1	hor. diff.	2°.2	hor. diff.	2°.3	hor. diff.	2°.4	hor. diff.	2°.5	$\alpha - \alpha_0$ $d - \delta_0$

TABLE III.

$$C = k \xi \sec d - (\alpha - \alpha_0)^\circ.$$

 $k \xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$.

0°.00001 = 0°.036 = unit.

$\alpha - \alpha_0$ $d - d_0$	2°.5	hor. diff.	2°.6	hor. diff.	2°.7	hor. diff.	2°.8	hor. diff.	2°.9	hor. diff.	3°.0	$\alpha - \alpha_0$ $d - d_0$
0.0	00159	20	00179	21	00200	23	00223	25	00248	26	00274	0.0
.1	159	20	179	21	200	24	224	24	248	27	275	.1
.2	160	20	180	22	202	23	225	25	250	26	276	.2
.3	162	20	182	22	204	23	227	25	252	26	278	.3
.4	165	20	185	22	207	23	230	25	255	27	282	.4
.5	168	21	189	21	210	24	234	25	259	27	286	.5
.6	173	20	193	22	215	24	239	25	264	27	291	.6
.7	178	20	198	22	220	24	244	26	270	27	297	.7
.8	183	21	204	22	226	24	250	26	276	28	304	.8
.9	190	21	211	22	233	25	258	26	284	27	311	.9
1.0	197	21	218	23	241	25	266	26	292	28	320	1.0
1.1	205	22	227	23	250	25	275	26	301	29	330	1.1
1.2	214	22	236	23	259	26	285	27	312	28	340	1.2
1.3	223	23	246	24	270	25	295	28	323	29	352	1.3
1.4	233	23	256	25	281	26	307	28	335	29	364	1.4
1.5	245	23	268	25	293	26	319	28	347	30	377	1.5
1.6	256	24	280	25	305	28	333	28	361	31	392	1.6
1.7	269	24	293	26	319	28	347	29	376	31	407	1.7
1.8	282	25	307	26	333	29	362	29	391	32	423	1.8
1.9	296	26	322	27	349	28	377	31	408	32	440	1.9
2.0	311	26	337	28	365	29	394	31	425	32	457	2.0
2.1	327	27	354	28	382	30	412	31	443	33	476	2.1
2.2	343	28	371	28	399	31	430	32	462	34	496	2.2
2.3	360	28	388	30	418	31	449	33	482	35	517	2.3
2.4	378	29	407	30	437	32	469	34	503	35	538	2.4
2.5	397	30	427	30	457	33	490	34	524	36	560	2.5
2.6	417	30	447	32	479	33	512	35	547	37	584	2.6
2.7	437	31	468	32	500	35	535	36	571	37	608	2.7
2.8	458	32	490	33	523	35	558	37	595	38	633	2.8
2.9	480	32	512	35	547	36	583	37	620	39	659	2.9
3.0	502	34	536	35	571	37	608	38	646	40	686	3.0
3.1	525	35	560	36	596	38	634	39	673	41	714	3.1
3.2	550	35	585	37	622	39	661	40	701	42	743	3.2
3.3	574	37	611	38	649	39	688	42	730	42	772	3.3
3.4	600	37	637	40	677	40	717	43	760	44	804	3.4
3.5	626	39	665	40	705	42	747	43	790	46	836	3.5
3.6	653	40	693	41	734	43	777	45	822	46	868	3.6
3.7	681	41	722	43	765	44	809	45	854	48	902	3.7
3.8	710	42	752	43	795	46	841	46	887	49	936	3.8
3.9	740	43	783	44	827	47	874	48	922	49	971	3.9
4.0	770	44	814	46	860	48	908	49	957	51	1008	4.0
4.1	801	45	846	47	893	49	942	51	993	52	1045	4.1
4.2	833	46	879	49	928	50	978	52	1030	53	1083	4.2
4.3	865	48	913	50	963	51	1014	53	1067	55	1122	4.3
4.4	898	50	948	51	999	53	1052	54	1106	56	1162	4.4
4.5	932	51	983	53	1036	54	1090	55	1145	58	1203	4.5
4.6	967	52	1019	54	1073	56	1129	57	1186	59	1245	4.6
4.7	1003	53	1056	56	1112	57	1169	58	1227	60	1287	4.7
4.8	1039	55	1094	57	1151	58	1209	60	1269	62	1331	4.8
4.9	1076	57	1133	58	1191	60	1251	61	1312	64	1376	4.9
5.0	1114	58	1172	60	1232	62	1294	63	1357	64	1421	5.0
$\alpha - \alpha_0$ $d - d_0$	2°.5	hor. diff.	2°.6	hor. diff.	2°.7	hor. diff.	2°.8	hor. diff.	2°.9	hor. diff.	3°.0	$\alpha - \alpha_0$ $d - d_0$

TABLE III.

$$C = k \xi \sec d - (\alpha - \alpha_0)^\circ.$$

$k \xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$.

$0^\circ.00001 = 0''.036 = \text{unit.}$

$\alpha - \alpha_0$ $d - d_0$	3°.0	hor. diff.	3°.1	hor. diff.	3°.2	hor. diff.	3°.3	hor. diff.	3°.4	hor. diff.	3°.5	$\alpha - \alpha_0$ $d - d_0$
0.0	00274	29	00303	30	00333	32	00365	34	00399	37	00436	0.0
.1	275	28	303	30	333	33	366	34	400	36	436	.1
.2	276	29	305	30	335	32	367	35	402	36	438	.2
.3	278	29	307	30	337	33	370	34	404	37	441	.3
.4	282	28	310	31	341	32	373	35	408	36	444	.4
.5	286	28	314	31	345	33	378	34	412	37	449	.5
.6	291	29	320	31	351	32	383	35	418	37	455	.6
.7	297	29	326	31	357	33	390	35	425	37	462	.7
.8	304	29	333	31	364	33	397	36	433	37	470	.8
.9	311	30	341	32	373	33	406	36	442	37	479	.9
1.0	320	30	350	32	382	34	416	35	451	38	489	1.0
1.1	330	30	360	32	392	34	426	36	462	38	500	1.1
1.2	340	31	371	32	403	35	438	36	474	39	513	1.2
1.3	352	31	383	33	416	34	450	37	487	39	526	1.3
1.4	364	31	395	34	429	35	464	37	501	39	540	1.4
1.5	377	32	409	34	443	36	479	37	516	40	556	1.5
1.6	392	32	424	34	458	36	494	38	532	41	573	1.6
1.7	407	32	439	35	474	37	511	38	549	41	590	1.7
1.8	423	33	456	35	491	37	528	40	568	41	609	1.8
1.9	440	33	473	36	509	38	547	40	587	42	629	1.9
2.0	457	35	492	36	528	39	567	40	607	42	649	2.0
2.1	476	35	511	37	548	39	587	41	628	43	671	2.1
2.2	496	36	532	37	569	40	609	42	651	43	694	2.2
2.3	517	36	553	38	591	41	632	42	674	44	718	2.3
2.4	538	37	575	39	614	41	655	43	698	46	744	2.4
2.5	560	38	598	40	638	42	680	44	724	46	770	2.5
2.6	584	39	623	40	663	43	706	45	751	46	797	2.6
2.7	608	40	648	41	689	43	732	46	778	47	825	2.7
2.8	633	41	674	42	716	44	760	46	806	48	854	2.8
2.9	659	42	701	43	744	45	789	47	836	49	885	2.9
3.0	686	43	729	44	773	46	819	48	867	50	917	3.0
3.1	714	44	758	44	802	47	849	49	898	51	949	3.1
3.2	743	44	787	46	833	48	881	50	931	52	983	3.2
3.3	772	46	818	47	865	49	914	51	965	53	1018	3.3
3.4	804	46	850	48	898	50	948	52	1000	54	1054	3.4
3.5	836	46	882	50	932	51	983	52	1035	56	1091	3.5
3.6	868	48	916	50	966	52	1018	54	1072	57	1129	3.6
3.7	902	49	951	51	1002	53	1055	55	1110	58	1168	3.7
3.8	936	51	987	52	1039	54	1093	57	1150	58	1208	3.8
3.9	971	52	1023	54	1077	55	1132	58	1190	59	1249	3.9
4.0	1008	52	1060	55	1115	57	1172	59	1231	61	1292	4.0
4.1	1045	54	1099	56	1155	58	1213	60	1273	62	1335	4.1
4.2	1083	55	1138	58	1196	59	1255	61	1316	64	1380	4.2
4.3	1122	57	1179	58	1237	61	1298	62	1360	65	1425	4.3
4.4	1162	58	1220	60	1280	62	1342	64	1406	66	1472	4.4
4.5	1203	59	1262	61	1323	64	1387	65	1452	67	1519	4.5
4.6	1245	61	1306	62	1368	65	1433	67	1500	68	1568	4.6
4.7	1287	63	1350	64	1414	66	1480	68	1548	70	1618	4.7
4.8	1331	64	1395	65	1460	68	1528	69	1597	72	1669	4.8
4.9	1376	65	1441	67	1508	69	1577	71	1648	73	1721	4.9
5.0	1421	67	1488	69	1557	70	1627	73	1700	74	1774	5.0
$\alpha - \alpha_0$ $d - d_0$	3°.0	hor. diff.	3°.1	hor. diff.	3°.2	hor. diff.	3°.3	hor. diff.	3°.4	hor. diff.	3°.5	$\alpha - \alpha_0$ $d - d_0$

TABLE III.

$$C = k \xi \sec d - (\alpha - \alpha_0)^\circ.$$

 $k \xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$.

 $0^\circ.00001 = 0^\circ.036 = \text{unit}.$

$\alpha - \alpha_0$ $d - \delta_0$	3°.5	hor. diff.	3°.6	hor. diff.	3°.7	hor. diff.	3°.8	hor. diff.	3°.9	hor. diff.	4°.0	$\alpha - \alpha_0$ $d - \delta_0$
0.0	00436	38	00474	41	00515	42	00557	46	00603	48	00651	0.0
.1	436 ⁰	39	475 ¹	41	516 ¹	42	558 ¹	46	604 ¹	48	652 ¹	.1
.2	438 ²	39	477 ²	40	517 ²	43	560 ²	46	606 ²	48	654 ²	.2
.3	441 ³	38	479 ³	41	520 ³	43	563 ³	46	609 ³	48	657 ³	.3
.4	444 ⁴	39	483 ⁴	41	524 ⁴	43	567 ⁴	46	613 ⁴	48	661 ⁴	.4
.5	449 ⁵	39	488 ⁵	41	529 ⁵	43	572 ⁵	46	618 ⁵	48	666 ⁵	.5
.6	455 ⁶	39	494 ⁶	41	535 ⁶	44	579 ⁶	46	625 ⁶	48	673 ⁶	.6
.7	462 ⁷	39	501 ⁷	42	543 ⁷	44	587 ⁷	46	633 ⁷	48	681 ⁷	.7
.8	470 ⁸	40	510 ⁸	41	551 ⁸	44	595 ⁸	47	642 ⁸	48	690 ⁸	.8
.9	479 ⁹	40	519 ⁹	42	561 ⁹	44	605 ⁹	47	652 ⁹	49	701 ⁹	.9
1.0	489 ¹⁰	40	529 ¹⁰	42	571 ¹⁰	45	616 ¹⁰	47	663 ¹⁰	49	712 ¹⁰	1.0
1.1	500 ¹¹	41	541 ¹¹	42	583 ¹¹	45	628 ¹¹	47	675 ¹¹	50	725 ¹¹	1.1
1.2	513 ¹²	41	554 ¹²	43	597 ¹²	45	642 ¹²	47	689 ¹²	50	739 ¹²	1.2
1.3	526 ¹³	41	567 ¹³	44	611 ¹³	45	656 ¹³	48	704 ¹³	50	754 ¹³	1.3
1.4	540 ¹⁴	42	582 ¹⁴	44	626 ¹⁴	46	672 ¹⁴	48	720 ¹⁴	51	771 ¹⁴	1.4
1.5	556 ¹⁵	42	598 ¹⁵	44	642 ¹⁵	47	689 ¹⁵	48	737 ¹⁵	51	788 ¹⁵	1.5
1.6	573 ¹⁶	42	615 ¹⁶	45	660 ¹⁶	47	707 ¹⁶	49	756 ¹⁶	51	807 ¹⁶	1.6
1.7	590 ¹⁷	43	633 ¹⁷	45	678 ¹⁷	48	726 ¹⁷	49	775 ¹⁷	52	827 ¹⁷	1.7
1.8	609 ¹⁸	43	652 ¹⁸	46	698 ¹⁸	48	746 ¹⁸	50	796 ¹⁸	53	849 ¹⁸	1.8
1.9	629 ¹⁹	44	673 ¹⁹	46	719 ¹⁹	48	767 ¹⁹	51	818 ¹⁹	53	871 ¹⁹	1.9
2.0	649 ²⁰	45	694 ²⁰	47	741 ²⁰	49	790 ²⁰	51	841 ²⁰	54	895 ²⁰	2.0
2.1	671 ²¹	46	717 ²¹	47	764 ²¹	50	814 ²¹	52	866 ²¹	54	920 ²¹	2.1
2.2	694 ²²	46	740 ²²	48	788 ²²	51	839 ²²	53	892 ²²	55	947 ²²	2.2
2.3	718 ²³	47	765 ²³	49	814 ²³	51	865 ²³	53	918 ²³	56	974 ²³	2.3
2.4	744 ²⁴	47	791 ²⁴	49	840 ²⁴	52	892 ²⁴	54	946 ²⁴	57	1003 ²⁴	2.4
2.5	770 ²⁵	48	818 ²⁵	50	868 ²⁵	53	921 ²⁵	55	976 ²⁵	57	1033 ²⁵	2.5
2.6	797 ²⁶	49	846 ²⁶	51	897 ²⁶	53	950 ²⁶	56	1006 ²⁶	58	1064 ²⁶	2.6
2.7	825 ²⁷	50	875 ²⁷	52	927 ²⁷	54	981 ²⁷	56	1037 ²⁷	59	1096 ²⁷	2.7
2.8	854 ²⁸	51	905 ²⁸	53	958 ²⁸	55	1013 ²⁸	57	1070 ²⁸	60	1130 ²⁸	2.8
2.9	885 ²⁹	52	937 ²⁹	53	990 ²⁹	56	1046 ²⁹	58	1104 ²⁹	61	1165 ²⁹	2.9
3.0	917 ³⁰	52	969 ³⁰	55	1024 ³⁰	56	1080 ³⁰	59	1139 ³⁰	62	1201 ³⁰	3.0
3.1	949 ³¹	54	1003 ³¹	55	1058 ³¹	58	1116 ³¹	60	1176 ³¹	62	1238 ³¹	3.1
3.2	983 ³²	54	1037 ³²	57	1094 ³²	58	1152 ³²	61	1213 ³²	64	1277 ³²	3.2
3.3	1018 ³³	55	1073 ³³	57	1130 ³³	60	1190 ³³	62	1252 ³³	64	1316 ³³	3.3
3.4	1054 ³⁴	56	1110 ³⁴	58	1168 ³⁴	61	1229 ³⁴	63	1292 ³⁴	65	1357 ³⁴	3.4
3.5	1091 ³⁵	57	1148 ³⁵	59	1207 ³⁵	62	1269 ³⁵	64	1333 ³⁵	67	1400 ³⁵	3.5
3.6	1129 ³⁶	58	1187 ³⁶	61	1248 ³⁶	62	1310 ³⁶	65	1375 ³⁶	68	1443 ³⁶	3.6
3.7	1168 ³⁷	59	1227 ³⁷	62	1289 ³⁷	64	1353 ³⁷	66	1419 ³⁷	69	1488 ³⁷	3.7
3.8	1208 ³⁸	61	1269 ³⁸	63	1332 ³⁸	65	1397 ³⁸	67	1464 ³⁸	70	1534 ³⁸	3.8
3.9	1249 ³⁹	63	1312 ³⁹	63	1375 ³⁹	66	1441 ³⁹	69	1510 ³⁹	71	1581 ³⁹	3.9
4.0	1292 ⁴⁰	63	1355 ⁴⁰	65	1420 ⁴⁰	67	1487 ⁴⁰	70	1557 ⁴⁰	72	1629 ⁴⁰	4.0
4.1	1335 ⁴¹	64	1399 ⁴¹	67	1466 ⁴¹	69	1535 ⁴¹	71	1606 ⁴¹	73	1679 ⁴¹	4.1
4.2	1380 ⁴²	65	1445 ⁴²	68	1513 ⁴²	70	1583 ⁴²	72	1655 ⁴²	75	1730 ⁴²	4.2
4.3	1425 ⁴³	67	1492 ⁴³	69	1561 ⁴³	71	1632 ⁴³	74	1706 ⁴³	76	1782 ⁴³	4.3
4.4	1472 ⁴⁴	68	1540 ⁴⁴	70	1610 ⁴⁴	73	1683 ⁴⁴	75	1758 ⁴⁴	77	1835 ⁴⁴	4.4
4.5	1519 ⁴⁵	70	1589 ⁴⁵	72	1661 ⁴⁵	74	1735 ⁴⁵	76	1811 ⁴⁵	79	1890 ⁴⁵	4.5
4.6	1568 ⁴⁶	71	1639 ⁴⁶	74	1713 ⁴⁶	75	1788 ⁴⁶	78	1866 ⁴⁶	80	1946 ⁴⁶	4.6
4.7	1618 ⁴⁷	73	1691 ⁴⁷	74	1765 ⁴⁷	77	1842 ⁴⁷	79	1921 ⁴⁷	82	2003 ⁴⁷	4.7
4.8	1669 ⁴⁸	74	1743 ⁴⁸	75	1818 ⁴⁸	79	1897 ⁴⁸	81	1978 ⁴⁸	84	2062 ⁴⁸	4.8
4.9	1721 ⁴⁹	76	1797 ⁴⁹	77	1874 ⁴⁹	80	1954 ⁴⁹	82	2036 ⁴⁹	85	2121 ⁴⁹	4.9
5.0	1774 ⁵⁰	78	1852 ⁵⁰	78	1930 ⁵⁰	82	2012 ⁵⁰	83	2095 ⁵⁰	86	2181 ⁵⁰	5.0
$\alpha - \alpha_0$ $d - \delta_0$	3°.5	hor. diff.	3°.6	hor. diff.	3°.7	hor. diff.	3°.8	hor. diff.	3°.9	hor. diff.	4°.0	$\alpha - \alpha_0$ $d - \delta_0$

TABLE III.

$$C = k \xi \sec d - (\alpha - \alpha_0)^\circ.$$

 $k \xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$.

0°.00001 = 0°.036 = unit.

$\alpha - \alpha_0$ $d - \delta_0$	4°.0	hor. diff.	4°.1	hor. diff.	4°.2	hor. diff.	4°.3	hor. diff.	4°.4	hor. diff.	4°.5	$\alpha - \alpha_0$ $d - \delta_0$
0.0	00651 ¹	50	00701 ¹	53	00754 ¹	55	00809 ¹	58	00867 ¹	61	00928 ¹	0.0
.1	652 ²	50	702 ²	53	755 ²	55	810 ²	58	868 ²	61	929 ²	.1
.2	654 ³	50	704 ³	53	757 ³	55	812 ³	58	870 ³	61	931 ³	.2
.3	657 ⁴	50	707 ⁴	53	760 ⁴	55	815 ⁴	58	873 ⁴	61	934 ⁴	.3
.4	661 ⁵	50	711 ⁵	53	764 ⁵	56	820 ⁵	58	878 ⁵	61	939 ⁵	.4
0.5	666 ⁶	51	717 ⁶	53	770 ⁶	56	826 ⁶	58	884 ⁶	61	945 ⁶	0.5
.6	673 ⁷	51	724 ⁷	53	777 ⁷	56	833 ⁷	58	891 ⁷	62	953 ⁷	.6
.7	681 ⁸	51	732 ⁸	53	785 ⁸	56	841 ⁸	59	900 ⁸	62	962 ⁸	.7
.8	690 ⁹	51	741 ⁹	54	795 ⁹	56	851 ⁹	59	910 ⁹	62	972 ⁹	.8
.9	701 ¹⁰	51	752 ¹⁰	54	806 ¹⁰	57	863 ¹⁰	59	922 ¹⁰	62	984 ¹⁰	.9
1.0	712 ¹¹	52	764 ¹¹	54	818 ¹¹	57	875 ¹¹	59	934 ¹¹	62	996 ¹¹	1.0
1.1	725 ¹²	52	777 ¹²	55	832 ¹²	57	889 ¹²	59	948 ¹²	63	1011 ¹²	1.1
1.2	739 ¹³	52	791 ¹³	55	846 ¹³	58	904 ¹³	60	964 ¹³	63	1027 ¹³	1.2
1.3	754 ¹⁴	53	807 ¹⁴	55	862 ¹⁴	58	920 ¹⁴	61	981 ¹⁴	63	1044 ¹⁴	1.3
1.4	771 ¹⁵	53	824 ¹⁵	56	880 ¹⁵	58	938 ¹⁵	61	999 ¹⁵	63	1062 ¹⁵	1.4
1.5	788 ¹⁶	54	842 ¹⁶	56	898 ¹⁶	59	957 ¹⁶	61	1018 ¹⁶	64	1082 ¹⁶	1.5
1.6	807 ¹⁷	55	862 ¹⁷	56	918 ¹⁷	59	977 ¹⁷	62	1039 ¹⁷	65	1104 ¹⁷	1.6
1.7	827 ¹⁸	55	882 ¹⁸	57	939 ¹⁸	60	999 ¹⁸	62	1061 ¹⁸	65	1126 ¹⁸	1.7
1.8	849 ¹⁹	55	904 ¹⁹	58	962 ¹⁹	60	1022 ¹⁹	63	1085 ¹⁹	65	1150 ¹⁹	1.8
1.9	871 ²⁰	56	927 ²⁰	59	986 ²⁰	60	1046 ²⁰	64	1110 ²⁰	66	1176 ²⁰	1.9
2.0	895 ²¹	57	952 ²¹	59	1011 ²¹	61	1072 ²¹	64	1136 ²¹	67	1203 ²¹	2.0
2.1	920 ²²	58	978 ²²	59	1037 ²²	62	1099 ²²	64	1163 ²²	68	1231 ²²	2.1
2.2	947 ²³	58	1005 ²³	59	1064 ²³	63	1127 ²³	66	1193 ²³	67	1260 ²³	2.2
2.3	974 ²⁴	59	1033 ²⁴	60	1093 ²⁴	64	1157 ²⁴	66	1223 ²⁴	68	1291 ²⁴	2.3
2.4	1003 ²⁵	60	1063 ²⁵	60	1123 ²⁵	65	1188 ²⁵	66	1254 ²⁵	70	1324 ²⁵	2.4
2.5	1033 ²⁶	60	1093 ²⁶	62	1155 ²⁶	65	1220 ²⁶	67	1287 ²⁶	70	1357 ²⁶	2.5
2.6	1064 ²⁷	61	1125 ²⁷	63	1188 ²⁷	65	1253 ²⁷	68	1321 ²⁷	71	1392 ²⁷	2.6
2.7	1096 ²⁸	62	1158 ²⁸	64	1222 ²⁸	66	1288 ²⁸	69	1357 ²⁸	72	1429 ²⁸	2.7
2.8	1130 ²⁹	62	1192 ²⁹	65	1257 ²⁹	67	1324 ²⁹	70	1394 ²⁹	73	1467 ²⁹	2.8
2.9	1165 ³⁰	63	1228 ³⁰	66	1294 ³⁰	68	1362 ³⁰	70	1432 ³⁰	74	1506 ³⁰	2.9
3.0	1201 ³¹	64	1265 ³¹	66	1331 ³¹	69	1400 ³¹	72	1472 ³¹	75	1547 ³¹	3.0
3.1	1238 ³²	64	1302 ³²	69	1371 ³²	70	1441 ³²	72	1513 ³²	76	1589 ³²	3.1
3.2	1277 ³³	65	1342 ³³	69	1411 ³³	71	1482 ³³	74	1556 ³³	76	1632 ³³	3.2
3.3	1316 ³⁴	66	1382 ³⁴	71	1453 ³⁴	72	1525 ³⁴	74	1599 ³⁴	78	1677 ³⁴	3.3
3.4	1357 ³⁵	68	1425 ³⁵	71	1496 ³⁵	73	1569 ³⁵	75	1644 ³⁵	79	1723 ³⁵	3.4
3.5	1400 ³⁶	69	1469 ³⁶	71	1540 ³⁶	74	1614 ³⁶	77	1691 ³⁶	79	1770 ³⁶	3.5
3.6	1443 ³⁷	70	1513 ³⁷	73	1586 ³⁷	75	1661 ³⁷	78	1739 ³⁷	80	1819 ³⁷	3.6
3.7	1488 ³⁸	71	1559 ³⁸	74	1633 ³⁸	76	1709 ³⁸	79	1788 ³⁸	81	1869 ³⁸	3.7
3.8	1534 ³⁹	72	1606 ³⁹	75	1681 ³⁹	77	1758 ³⁹	81	1839 ³⁹	82	1921 ³⁹	3.8
3.9	1581 ⁴⁰	74	1655 ⁴⁰	76	1731 ⁴⁰	78	1809 ⁴⁰	82	1891 ⁴⁰	83	1974 ⁴⁰	3.9
4.0	1629 ⁴¹	75	1704 ⁴¹	78	1782 ⁴¹	79	1861 ⁴¹	83	1944 ⁴¹	85	2029 ⁴¹	4.0
4.1	1679 ⁴²	76	1755 ⁴²	79	1834 ⁴²	80	1914 ⁴²	84	1998 ⁴²	87	2085 ⁴²	4.1
4.2	1730 ⁴³	77	1807 ⁴³	80	1887 ⁴³	82	1969 ⁴³	86	2055 ⁴³	87	2142 ⁴³	4.2
4.3	1782 ⁴⁴	79	1861 ⁴⁴	81	1942 ⁴⁴	83	2025 ⁴⁴	87	2112 ⁴⁴	88	2200 ⁴⁴	4.3
4.4	1835 ⁴⁵	80	1915 ⁴⁵	83	1998 ⁴⁵	84	2082 ⁴⁵	88	2170 ⁴⁵	91	2261 ⁴⁵	4.4
4.5	1890 ⁴⁶	81	1971 ⁴⁶	84	2055 ⁴⁶	86	2141 ⁴⁶	90	2231 ⁴⁶	91	2322 ⁴⁶	4.5
4.6	1946 ⁴⁷	82	2028 ⁴⁷	86	2114 ⁴⁷	87	2201 ⁴⁷	91	2292 ⁴⁷	93	2385 ⁴⁷	4.6
4.7	2003 ⁴⁸	84	2087 ⁴⁸	87	2174 ⁴⁸	88	2262 ⁴⁸	93	2355 ⁴⁸	95	2450 ⁴⁸	4.7
4.8	2062 ⁴⁹	85	2147 ⁴⁹	88	2235 ⁴⁹	91	2326 ⁴⁹	93	2419 ⁴⁹	96	2515 ⁴⁹	4.8
4.9	2121 ⁵⁰	87	2208 ⁵⁰	89	2297 ⁵⁰	93	2390 ⁵⁰	94	2484 ⁵⁰	98	2582 ⁵⁰	4.9
5.0	2181 ⁵¹	89	2270 ⁵¹	91	2361 ⁵¹	94	2455 ⁵¹	96	2551 ⁵¹	99	2650 ⁵¹	5.0
$d - \delta_0$ $\alpha - \alpha_0$	4°.0	hor. diff.	4°.1	hor. diff.	4°.2	hor. diff.	4°.3	hor. diff.	4°.4	hor. diff.	4°.5	$d - \delta_0$ $\alpha - \alpha_0$

TABLE III.

$$C = k \xi \sec d - (\alpha - \alpha_0)^\circ.$$

 $k \xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$.

 $0.00001 = 0.036 = \text{unit.}$

$\alpha - \alpha_0$ $d - \delta_0$	4°.5	hor. diff.	4°.6	hor. diff.	4°.7	hor. diff.	4°.8	hor. diff.	4°.9	hor. diff.	5°.0	$\alpha - \alpha_0$ $d - \delta_0$
0.0	00928	63	00991	66	01057	69	01126	71	01197	75	01272	0.0
.1	929	63	992	66	1058	69	1127	71	1198	75	1273	.1
.2	931	63	994	66	1060	69	1129	72	1201	75	1276	.2
.3	934	63	997	67	1064	69	1133	72	1205	75	1280	.3
.4	939	63	1002	67	1069	69	1138	72	1210	75	1285	.4
0.5	945	64	1009	66	1075	70	1145	72	1217	75	1292	0.5
.6	953	64	1017	66	1083	70	1153	72	1225	76	1301	.6
.7	962	64	1026	66	1092	70	1162	73	1235	76	1311	.7
.8	972	64	1036	67	1103	70	1173	73	1246	76	1322	.8
.9	984	64	1048	67	1115	71	1186	73	1259	76	1335	.9
1.0	996	65	1061	68	1129	70	1199	74	1273	76	1349	1.0
1.1	1011	65	1076	68	1144	71	1215	74	1289	76	1365	1.1
1.2	1027	65	1092	68	1160	72	1232	74	1306	77	1383	1.2
1.3	1044	66	1110	68	1178	72	1250	75	1325	77	1402	1.3
1.4	1062	67	1129	69	1198	72	1270	75	1345	78	1423	1.4
1.5	1082	67	1149	70	1219	72	1291	76	1367	78	1445	1.5
1.6	1104	67	1171	70	1241	73	1314	76	1390	79	1469	1.6
1.7	1126	68	1194	71	1265	73	1338	77	1415	79	1494	1.7
1.8	1150	69	1219	71	1290	74	1364	77	1441	80	1521	1.8
1.9	1176	69	1245	71	1316	75	1391	77	1468	81	1549	1.9
2.0	1203	69	1272	72	1344	75	1419	78	1497	82	1579	2.0
2.1	1231	70	1301	73	1374	75	1449	79	1528	82	1610	2.1
2.2	1260	71	1331	74	1405	76	1481	80	1561	82	1643	2.2
2.3	1291	72	1363	74	1437	77	1514	80	1594	83	1677	2.3
2.4	1324	72	1396	75	1471	78	1549	80	1629	84	1713	2.4
2.5	1357	73	1430	76	1506	78	1584	82	1666	85	1751	2.5
2.6	1392	74	1466	77	1543	79	1622	82	1704	86	1790	2.6
2.7	1429	74	1503	78	1581	80	1661	83	1744	86	1830	2.7
2.8	1467	75	1542	78	1620	81	1701	84	1785	87	1872	2.8
2.9	1506	76	1582	79	1661	82	1743	85	1828	88	1916	2.9
3.0	1547	77	1624	80	1704	82	1786	86	1872	89	1961	3.0
3.1	1589	77	1666	82	1748	83	1831	87	1918	90	2008	3.1
3.2	1632	79	1711	82	1793	84	1877	88	1965	91	2056	3.2
3.3	1677	80	1757	82	1839	86	1925	89	2014	92	2106	3.3
3.4	1723	81	1804	84	1888	86	1974	90	2064	93	2157	3.4
3.5	1770	82	1852	85	1937	88	2025	91	2116	94	2210	3.5
3.6	1819	84	1903	85	1988	89	2077	92	2169	95	2264	3.6
3.7	1869	85	1954	87	2041	90	2131	93	2224	96	2320	3.7
3.8	1921	86	2007	88	2095	91	2186	94	2280	98	2378	3.8
3.9	1974	87	2061	90	2151	92	2243	95	2338	99	2437	3.9
4.0	2029	88	2117	91	2208	93	2301	97	2398	99	2497	4.0
4.1	2085	89	2174	92	2266	95	2361	98	2459	100	2559	4.1
4.2	2142	90	2232	94	2326	96	2422	99	2521	102	2623	4.2
4.3	2200	92	2292	95	2387	97	2484	101	2585	103	2688	4.3
4.4	2261	92	2353	97	2450	98	2548	102	2650	105	2755	4.4
4.5	2322	94	2416	98	2514	100	2614	103	2717	106	2823	4.5
4.6	2385	96	2481	98	2579	102	2681	105	2786	107	2893	4.6
4.7	2450	96	2546	100	2646	103	2749	107	2856	108	2964	4.7
4.8	2515	98	2613	102	2715	104	2819	108	2927	110	3037	4.8
4.9	2582	100	2682	103	2785	106	2891	109	3000	112	3112	4.9
5.0	2650	102	2752	105	2857	107	2964	111	3075	113	3188	5.0
$d - \delta_0$ $\alpha - \alpha_0$	4°.5	hor. diff.	4°.6	hor. diff.	4°.7	hor. diff.	4°.8	hor. diff.	4°.9	hor. diff.	5°.0	$d - \delta_0$ $\alpha - \alpha_0$

TABLE III.

$$C = k \xi \sec d - (\alpha - \alpha_0)^\circ.$$

$k \xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$.

$0''.00001 = 0''.036 = \text{unit.}$

$\alpha - \alpha_0$ $d - d_0$	5°.0	hor. diff.	5°.1	hor. diff.	5°.2	hor. diff.	5°.3	hor. diff.	5°.4	hor. diff.	5°.5	$\alpha - \alpha_0$ $d - d_0$
0.0	01272	79	01351	81	01432	85	01517	88	01605	91	01696	0.0
.1	1273	79	1352	81	1433	85	1518	87	1605	91	1696	.1
.2	1276	78	1354	82	1436	84	1520	88	1608	91	1699	.2
.3	1280	78	1358	82	1440	84	1524	88	1612	92	1704	.3
.4	1285	79	1364	81	1445	85	1530	88	1618	91	1709	.4
0.5	1292	79	1371	81	1452	85	1537	88	1625	92	1717	0.5
.6	1301	78	1379	82	1461	85	1546	88	1634	92	1726	.6
.7	1311	78	1389	82	1471	86	1557	88	1645	92	1737	.7
.8	1322	79	1401	82	1483	86	1569	88	1657	92	1749	.8
.9	1335	79	1414	83	1497	86	1583	88	1671	92	1763	.9
1.0	1349	80	1429	83	1512	86	1598	89	1687	92	1779	1.0
1.1	1365	81	1446	83	1529	86	1615	89	1704	93	1797	1.1
1.2	1383	81	1464	83	1547	87	1634	89	1723	94	1817	1.2
1.3	1402	81	1483	84	1567	87	1654	90	1744	94	1838	1.3
1.4	1423	81	1504	84	1588	88	1676	90	1766	94	1860	1.4
1.5	1445	82	1527	84	1611	88	1699	91	1790	95	1885	1.5
1.6	1469	82	1551	85	1636	88	1724	92	1816	95	1911	1.6
1.7	1494	83	1577	85	1662	89	1751	92	1843	96	1939	1.7
1.8	1521	83	1604	86	1690	89	1779	93	1872	96	1968	1.8
1.9	1549	84	1633	86	1719	90	1809	93	1902	97	1999	1.9
2.0	1579	84	1663	87	1750	91	1841	94	1935	97	2032	2.0
2.1	1610	85	1695	88	1783	91	1874	95	1969	97	2066	2.1
2.2	1643	86	1729	88	1817	92	1909	95	2004	98	2102	2.2
2.3	1677	87	1764	89	1853	93	1946	95	2041	99	2140	2.3
2.4	1713	87	1800	90	1890	94	1984	96	2080	100	2180	2.4
2.5	1751	87	1838	91	1929	94	2023	98	2121	100	2221	2.5
2.6	1790	88	1878	92	1970	95	2065	98	2163	101	2264	2.6
2.7	1830	90	1920	92	2012	96	2108	98	2206	103	2309	2.7
2.8	1872	91	1963	93	2056	96	2152	100	2252	103	2355	2.8
2.9	1916	91	2007	94	2101	97	2198	101	2299	104	2403	2.9
3.0	1961	92	2053	95	2148	98	2246	101	2347	106	2453	3.0
3.1	2008	92	2100	97	2197	99	2296	102	2398	106	2504	3.1
3.2	2056	93	2149	98	2247	100	2347	103	2450	107	2557	3.2
3.3	2106	94	2200	99	2299	101	2400	104	2504	108	2612	3.3
3.4	2157	96	2253	99	2352	102	2454	105	2559	109	2668	3.4
3.5	2210	97	2307	100	2407	103	2510	106	2616	110	2726	3.5
3.6	2264	98	2362	102	2464	104	2568	107	2675	111	2786	3.6
3.7	2320	99	2419	103	2522	105	2627	109	2736	112	2848	3.7
3.8	2378	100	2478	104	2582	106	2688	110	2798	113	2911	3.8
3.9	2437	101	2538	105	2643	108	2751	111	2862	114	2976	3.9
4.0	2497	103	2600	106	2706	109	2815	112	2927	116	3043	4.0
4.1	2559	104	2663	107	2770	111	2881	113	2994	117	3111	4.1
4.2	2623	105	2728	108	2836	113	2949	114	3063	118	3181	4.2
4.3	2688	107	2795	109	2904	114	3018	115	3133	120	3253	4.3
4.4	2755	108	2863	111	2974	114	3088	117	3205	121	3326	4.4
4.5	2823	109	2932	113	3045	115	3160	119	3279	122	3401	4.5
4.6	2893	111	3004	114	3118	116	3234	121	3355	123	3478	4.6
4.7	2964	113	3077	115	3192	118	3310	122	3432	125	3557	4.7
4.8	3037	115	3152	116	3268	120	3388	123	3511	126	3637	4.8
4.9	3112	116	3228	118	3346	121	3467	124	3591	128	3719	4.9
5.0	3188	117	3305	119	3424	123	3547	126	3673	130	3803	5.0
$\alpha - \alpha_0$ $d - d_0$	5°.0	hor. diff.	5°.1	hor. diff.	5°.2	hor. diff.	5°.3	hor. diff.	5°.4	hor. diff.	5°.5	$\alpha - \alpha_0$ $d - d_0$

TABLE III.

$$C = k \xi \sec d - (\alpha - \alpha_0)^\circ.$$

 $k \xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$.

 $0''.00001 = 0''.036 = \text{unit.}$

$\alpha - \alpha_0$ $d - \delta_0$	5°.5	hor. diff.	5°.6	hor. diff.	5°.7	hor. diff.	5°.8	hor. diff.	5°.9	hor. diff.	6°.0	$\alpha - \alpha_0$ $d - \delta_0$
0.0	01696 ⁰	94	01790 ¹	98	01888 ¹	101	01989 ¹	105	02094 ¹	109	02203 ¹	0.0
.1	1696 ³	95	1791 ³	98	1889 ³	101	1990 ³	105	2095 ³	109	2204 ³	.1
.2	1699 ⁵	95	1794 ⁴	98	1892 ⁴	101	1993 ⁵	105	2098 ⁵	109	2207 ⁵	.2
.3	1704 ⁵	94	1798 ⁶	98	1896 ⁶	102	1998 ⁶	105	2103 ⁶	109	2212 ⁶	.3
.4	1709 ⁸	95	1804 ⁸	98	1902 ⁸	102	2004 ⁸	105	2109 ⁸	109	2218 ⁸	.4
0.5	1717 ⁹	95	1812 ⁹	98	1910 ¹⁰	102	2012 ¹⁰	105	2117 ¹⁰	109	2226 ¹⁰	0.5
.6	1726 ¹¹	95	1821 ¹¹	99	1920 ¹¹	102	2022 ¹¹	105	2127 ¹²	109	2236 ¹²	.6
.7	1737 ¹²	95	1832 ¹³	99	1931 ¹³	102	2033 ¹³	106	2139 ¹³	109	2248 ¹⁴	.7
.8	1749 ¹⁴	96	1845 ¹⁵	99	1944 ¹⁵	102	2046 ¹⁵	106	2152 ¹⁵	110	2262 ¹⁵	.8
.9	1763 ¹⁶	97	1860 ¹⁶	99	1959 ¹⁶	102	2061 ¹⁷	106	2167 ¹⁸	110	2277 ¹⁸	.9
1.0	1779 ¹⁸	97	1876 ¹⁸	99	1975 ¹⁸	103	2078 ¹⁹	107	2185 ¹⁹	110	2295 ¹⁹	1.0
1.1	1797 ²⁰	97	1894 ²⁰	99	1993 ²⁰	104	2097 ²⁰	107	2204 ²¹	110	2314 ²¹	1.1
1.2	1817 ²¹	97	1914 ²¹	99	2013 ²²	104	2117 ²²	108	2225 ²²	110	2335 ²³	1.2
1.3	1838 ²²	97	1935 ²³	100	2035 ²⁴	104	2139 ²⁴	108	2247 ²⁴	111	2358 ²⁵	1.3
1.4	1860 ²⁵	98	1958 ²⁵	101	2059 ²⁵	104	2163 ²⁶	108	2271 ²⁶	112	2383 ²⁷	1.4
1.5	1885 ²⁶	98	1983 ²⁶	101	2084 ²⁷	105	2189 ²⁸	108	2297 ²⁸	113	2410 ²⁸	1.5
1.6	1911 ²⁸	98	2009 ²⁸	102	2111 ²⁹	106	2217 ²⁹	108	2325 ³⁰	113	2438 ³⁰	1.6
1.7	1939 ²⁹	98	2037 ³⁰	103	2140 ³⁰	106	2246 ³¹	109	2355 ³²	113	2468 ³²	1.7
1.8	1968 ³¹	99	2067 ³²	103	2170 ³²	107	2277 ³²	110	2387 ³³	113	2500 ³⁴	1.8
1.9	1999 ³³	100	2099 ³³	103	2202 ³³	107	2309 ³⁴	111	2420 ³⁵	114	2534 ³⁶	1.9
2.0	2032 ³⁵	100	2132 ³⁵	104	2236 ³⁶	108	2344 ³⁷	111	2455 ³⁷	115	2570 ³⁸	2.0
2.1	2066 ³⁶	101	2167 ³⁷	105	2272 ³⁸	109	2381 ³⁸	111	2492 ³⁹	116	2608 ³⁹	2.1
2.2	2102 ³⁸	102	2204 ³⁹	106	2310 ³⁹	109	2419 ⁴⁰	112	2531 ⁴¹	116	2647 ⁴¹	2.2
2.3	2140 ⁴⁰	103	2243 ⁴⁰	106	2349 ⁴¹	110	2459 ⁴²	113	2572 ⁴²	116	2688 ⁴³	2.3
2.4	2180 ⁴¹	103	2283 ⁴²	107	2390 ⁴³	111	2501 ⁴³	113	2614 ⁴⁴	117	2741 ⁴⁶	2.4
2.5	2221 ⁴³	104	2325 ⁴⁴	108	2433 ⁴⁵	111	2544 ⁴⁵	114	2658 ⁴⁶	119	2777 ⁴⁷	2.5
2.6	2264 ⁴⁵	105	2369 ⁴⁶	109	2478 ⁴⁶	111	2589 ⁴⁷	115	2704 ⁴⁸	120	2824 ⁴⁹	2.6
2.7	2309 ⁴⁶	106	2415 ⁴⁷	109	2524 ⁴⁸	112	2636 ⁴⁹	116	2752 ⁵⁰	121	2873 ⁵⁰	2.7
2.8	2355 ⁴⁸	107	2462 ⁴⁹	110	2572 ⁵⁰	113	2685 ⁵¹	117	2802 ⁵¹	121	2923 ⁵²	2.8
2.9	2403 ⁵⁰	108	2511 ⁵⁰	111	2622 ⁵¹	114	2736 ⁵²	117	2853 ⁵⁴	122	2975 ⁵⁴	2.9
3.0	2453 ⁵¹	108	2561 ⁵²	112	2673 ⁵³	115	2788 ⁵⁴	119	2907 ⁵⁵	122	3029 ⁵⁶	3.0
3.1	2504 ⁵³	109	2613 ⁵⁴	113	2726 ⁵⁵	116	2842 ⁵⁶	120	2962 ⁵⁷	123	3085 ⁵⁸	3.1
3.2	2557 ⁵⁵	110	2667 ⁵⁶	114	2781 ⁵⁷	117	2898 ⁵⁸	121	3019 ⁵⁸	124	3143 ⁶⁰	3.2
3.3	2612 ⁵⁶	111	2723 ⁵⁷	115	2838 ⁵⁸	118	2956 ⁶⁰	121	3077 ⁶¹	126	3203 ⁶²	3.3
3.4	2668 ⁵⁸	112	2780 ⁶⁰	116	2896 ⁶⁰	120	3016 ⁶¹	122	3138 ⁶³	127	3265 ⁶³	3.4
3.5	2726 ⁶⁰	114	2840 ⁶¹	116	2956 ⁶²	121	3077 ⁶³	124	3201 ⁶⁴	127	3328 ⁶⁵	3.5
3.6	2786 ⁶²	115	2901 ⁶²	117	3018 ⁶⁴	122	3140 ⁶⁵	125	3265 ⁶⁶	128	3393 ⁶⁷	3.6
3.7	2848 ⁶³	115	2963 ⁶⁵	119	3082 ⁶⁶	123	3205 ⁶⁷	126	3331 ⁶⁸	129	3460 ⁶⁹	3.7
3.8	2911 ⁶⁵	117	3028 ⁶⁶	120	3148 ⁶⁷	124	3272 ⁶⁸	127	3399 ⁷⁰	130	3529 ⁷¹	3.8
3.9	2976 ⁶⁷	118	3094 ⁶⁸	121	3215 ⁶⁹	125	3340 ⁷⁰	129	3469 ⁷¹	131	3600 ⁷³	3.9
4.0	3043 ⁶⁸	119	3162 ⁶⁹	122	3284 ⁷¹	126	3410 ⁷²	130	3540 ⁷³	133	3673 ⁷⁵	4.0
4.1	3111 ⁷⁰	120	3231 ⁷²	124	3355 ⁷³	127	3482 ⁷⁴	131	3613 ⁷⁶	135	3748 ⁷⁶	4.1
4.2	3181 ⁷²	122	3303 ⁷³	125	3428 ⁷⁵	128	3556 ⁷⁶	133	3689 ⁷⁷	135	3824 ⁷⁸	4.2
4.3	3253 ⁷³	123	3376 ⁷⁵	126	3502 ⁷⁶	130	3632 ⁷⁸	134	3766 ⁷⁹	136	3902 ⁸¹	4.3
4.4	3326 ⁷⁵	125	3451 ⁷⁷	127	3578 ⁷⁸	132	3710 ⁸⁰	135	3845 ⁸⁰	138	3983 ⁸²	4.4
4.5	3401 ⁷⁷	127	3528 ⁷⁸	128	3656 ⁸⁰	134	3790 ⁸¹	135	3925 ⁸²	140	4065 ⁸⁴	4.5
4.6	3478 ⁷⁹	128	3606 ⁸⁰	130	3736 ⁸¹	135	3871 ⁸³	136	4007 ⁸⁵	142	4149 ⁸⁶	4.6
4.7	3557 ⁸⁰	129	3686 ⁸¹	131	3817 ⁸³	137	3954 ⁸⁴	138	4092 ⁸⁶	143	4235 ⁸⁸	4.7
4.8	3637 ⁸²	130	3767 ⁸⁴	133	3900 ⁸⁵	138	4038 ⁸⁶	140	4178 ⁸⁸	145	4323 ⁸⁹	4.8
4.9	3779 ⁸⁴	132	3851 ⁸⁵	134	3985 ⁸⁷	139	4124 ⁸⁸	142	4266 ⁹⁰	146	4412 ⁹¹	4.9
5.0	3803 ⁸⁴	133	3936 ⁸⁵	136	4072 ⁸⁷	140	4212 ⁸⁸	144	4356 ⁹⁰	147	4503 ⁹¹	5.0
$\alpha - \alpha_0$ $d - \delta_0$	5°.5	hor. diff.	5°.6	hor. diff.	5°.7	hor. diff.	5°.8	hor. diff.	5°.9	hor. diff.	6°.0	$\alpha - \alpha_0$ $d - \delta_0$

TABLE III.

$$C = k \xi \sec d - (\alpha - \alpha_0)^\circ.$$

$k \xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$.

$0''.00001 = 0''.036 = \text{unit.}$

$\alpha - \alpha_0$ $d - d_0$	6°.0	hor. diff.	6°.1	hor. diff.	6°.2	hor. diff.	6°.3	hor. diff.	6°.4	hor. diff.	6°.5	$\alpha - \alpha_0$ $d - d_0$
0.0	02203	112	02315	116	02431	120	02551	124	02675	128	02803	0.0
.1	2204	112	2316	116	2432	120	2552	124	2676	128	2804	.1
.2	2207	112	2319	116	2435	120	2555	124	2679	128	2807	.2
.3	2212	112	2324	116	2440	120	2560	124	2684	128	2812	.3
.4	2218	113	2331	116	2447	120	2567	124	2691	128	2819	.4
0.5	2226	113	2339	117	2456	120	2576	124	2700	128	2828	0.5
.6	2236	113	2349	117	2466	121	2587	124	2711	128	2839	.6
.7	2248	113	2361	117	2478	121	2599	124	2723	129	2852	.7
.8	2262	113	2375	117	2492	121	2613	125	2738	129	2867	.8
.9	2277	114	2391	117	2508	121	2629	126	2755	129	2884	.9
1.0	2295	114	2409	117	2526	122	2648	125	2773	129	2902	1.0
1.1	2314	115	2429	117	2546	122	2668	126	2794	129	2923	1.1
1.2	2335	115	2450	118	2568	122	2690	126	2816	130	2946	1.2
1.3	2358	115	2473	119	2592	122	2714	127	2841	130	2971	1.3
1.4	2383	115	2498	119	2617	123	2740	127	2867	131	2998	1.4
1.5	2410	115	2525	119	2644	124	2768	128	2896	131	3027	1.5
1.6	2438	116	2554	120	2674	124	2798	128	2926	132	3058	1.6
1.7	2468	117	2585	121	2706	124	2830	128	2958	133	3091	1.7
1.8	2500	118	2618	121	2739	125	2864	129	2993	133	3126	1.8
1.9	2534	119	2653	121	2774	125	2899	130	3029	133	3162	1.9
2.0	2570	119	2689	122	2811	126	2937	130	3067	134	3201	2.0
2.1	2608	119	2727	123	2850	127	2977	130	3107	135	3242	2.1
2.2	2647	120	2767	124	2891	127	3018	131	3149	136	3285	2.2
2.3	2688	121	2809	124	2933	129	3062	131	3193	136	3329	2.3
2.4	2741	122	2853	125	2978	129	3107	132	3239	137	3376	2.4
2.5	2777	122	2899	125	3024	130	3154	133	3287	138	3425	2.5
2.6	2824	123	2947	125	3072	131	3203	134	3337	139	3476	2.6
2.7	2873	123	2996	127	3123	131	3254	135	3389	140	3529	2.7
2.8	2923	124	3047	128	3175	132	3307	136	3443	140	3583	2.8
2.9	2975	126	3101	128	3229	133	3362	137	3499	141	3640	2.9
3.0	3029	127	3156	129	3285	134	3419	138	3557	142	3699	3.0
3.1	3085	128	3213	130	3343	135	3478	139	3617	143	3760	3.1
3.2	3143	129	3272	131	3403	136	3539	140	3679	144	3823	3.2
3.3	3203	130	3333	132	3465	137	3602	140	3742	145	3887	3.3
3.4	3265	130	3395	133	3528	138	3666	142	3808	146	3954	3.4
3.5	3328	131	3459	135	3594	139	3733	143	3876	147	4023	3.5
3.6	3393	132	3525	137	3662	140	3802	144	3946	148	4094	3.6
3.7	3460	134	3594	137	3731	141	3872	146	4018	149	4167	3.7
3.8	3529	135	3664	138	3802	143	3945	146	4091	150	4241	3.8
3.9	3600	136	3736	140	3876	144	4020	147	4167	151	4318	3.9
4.0	3673	137	3810	141	3951	145	4096	149	4245	152	4397	4.0
4.1	3748	138	3886	142	4028	146	4174	151	4325	153	4478	4.1
4.2	3824	140	3964	143	4107	148	4255	151	4406	155	4561	4.2
4.3	3902	142	4044	144	4188	149	4337	153	4490	155	4645	4.3
4.4	3983	143	4126	145	4271	150	4421	155	4576	156	4732	4.4
4.5	4065	144	4209	147	4356	151	4507	156	4663	158	4821	4.5
4.6	4149	145	4294	149	4443	152	4595	157	4752	160	4912	4.6
4.7	4235	147	4382	149	4531	154	4685	159	4844	161	5005	4.7
4.8	4323	148	4471	151	4622	156	4778	159	4937	163	5100	4.8
4.9	4412	150	4562	153	4715	157	4872	161	5033	164	5197	4.9
5.0	4503	152	4655	154	4809	159	4968	162	5130	166	5296	5.0
$\alpha - \alpha_0$ $d - d_0$	6°.0	hor. diff.	6°.1	hor. diff.	6°.2	hor. diff.	6°.3	hor. diff.	6°.4	hor. diff.	6°.5	$\alpha - \alpha_0$ $d - d_0$

TABLE III.

$$C = k \xi \sec d - (\alpha - \alpha_0)^\circ.$$

 $k \xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$.

 $0''.00001 = 0''.036 = \text{unit.}$

$\alpha - \alpha_0$ $d - \delta_0$	6°.5	hor. diff.	6°.6	hor. diff.	6°.7	hor. diff.	6°.8	hor. diff.	6°.9	hor. diff.	7°.0	$\alpha - \alpha_0$ $d - \delta_0$
0.0	02803	132	02935	136	03071	140	03211	144	03355	149	03504	0.0
.1	2804	132	2936	136	3072	141	3213	144	3357	149	3506	.1
.2	2807	132	2939	136	3075	141	3216	144	3360	149	3509	.2
.3	2812	132	2944	136	3080	141	3221	144	3365	149	3514	.3
.4	2819	132	2951	136	3087	141	3228	145	3373	148	3521	.4
.5	2828	132	2960	136	3096	141	3237	145	3382	148	3530	.5
.6	2839	132	2971	136	3107	141	3248	146	3394	148	3542	.6
.7	2852	132	2984	136	3120	142	3262	145	3407	149	3556	.7
.8	2867	133	3000	136	3136	142	3278	145	3423	149	3572	.8
.9	2884	133	3017	136	3153	142	3295	146	3441	149	3590	.9
1.0	2902	134	3036	137	3173	142	3315	146	3461	150	3611	1.0
1.1	2923	134	3057	138	3195	142	3337	146	3483	151	3634	1.1
1.2	2946	134	3080	138	3218	143	3361	147	3508	150	3658	1.2
1.3	2971	135	3106	138	3244	143	3387	147	3534	151	3685	1.3
1.4	2998	135	3133	139	3272	143	3415	147	3562	152	3714	1.4
1.5	3027	135	3162	140	3302	143	3445	148	3593	152	3745	1.5
1.6	3058	135	3193	141	3334	143	3477	149	3626	152	3778	1.6
1.7	3091	136	3227	141	3368	144	3512	149	3661	153	3814	1.7
1.8	3126	136	3262	141	3403	145	3548	150	3698	153	3851	1.8
1.9	3162	138	3300	141	3441	145	3586	151	3737	153	3890	1.9
2.0	3201	138	3339	142	3481	146	3627	151	3778	154	3932	2.0
2.1	3242	138	3380	143	3523	147	3670	151	3821	155	3976	2.1
2.2	3285	139	3424	143	3567	147	3714	153	3867	155	4022	2.2
2.3	3329	140	3469	144	3613	148	3761	153	3914	157	4071	2.3
2.4	3376	141	3517	145	3662	148	3810	154	3964	157	4121	2.4
2.5	3425	141	3566	146	3712	149	3861	155	4016	158	4174	2.5
2.6	3476	142	3618	146	3764	150	3914	156	4070	159	4229	2.6
2.7	3529	142	3671	148	3819	151	3970	156	4126	160	4286	2.7
2.8	3583	144	3727	148	3875	152	4027	157	4184	161	4345	2.8
2.9	3640	144	3784	150	3934	152	4086	158	4244	162	4406	2.9
3.0	3699	145	3844	150	3994	154	4148	158	4306	163	4469	3.0
3.1	3760	146	3906	151	4057	155	4212	158	4370	164	4534	3.1
3.2	3823	147	3970	151	4121	156	4277	160	4437	165	4602	3.2
3.3	3887	148	4035	153	4188	157	4345	161	4506	166	4672	3.3
3.4	3954	149	4103	154	4257	158	4415	162	4577	167	4744	3.4
3.5	4023	150	4173	155	4328	159	4487	163	4650	168	4818	3.5
3.6	4094	151	4245	156	4401	160	4561	164	4725	169	4894	3.6
3.7	4167	152	4319	157	4476	162	4638	165	4803	170	4973	3.7
3.8	4241	154	4395	158	4553	163	4716	166	4882	171	5053	3.8
3.9	4318	156	4474	159	4633	163	4796	168	4964	172	5136	3.9
4.0	4397	157	4554	160	4714	165	4879	169	5048	173	5221	4.0
4.1	4478	158	4636	161	4797	166	4963	171	5134	174	5308	4.1
4.2	4561	159	4720	163	4883	167	5050	172	5222	176	5398	4.2
4.3	4645	161	4806	164	4970	169	5139	173	5312	177	5489	4.3
4.4	4732	163	4895	165	5060	170	5230	175	5405	178	5583	4.4
4.5	4821	164	4985	167	5152	171	5323	176	5499	180	5679	4.5
4.6	4912	165	5077	169	5246	172	5418	178	5596	181	5777	4.6
4.7	5005	167	5172	170	5342	174	5516	178	5694	183	5877	4.7
4.8	5100	168	5268	172	5440	175	5615	180	5795	185	5980	4.8
4.9	5197	169	5366	174	5540	176	5716	182	5898	186	6084	4.9
5.0	5296	171	5467	175	5642	178	5820	183	6003	188	6191	5.0

TABLE III.

$$C = k \xi \sec d - (\alpha - \alpha_0)^\circ.$$

 $k \xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$.

 $0''.00001 = 0''.036 = \text{unit.}$

$\alpha - \alpha_0$ $d - \delta_0$	$7^\circ.0$	hor. diff.	$7^\circ.1$	hor. diff.	$7^\circ.2$	hor. diff.	$7^\circ.3$	hor. diff.	$7^\circ.4$	hor. diff.	$7^\circ.5$	$\alpha - \alpha_0$ $d - \delta_0$
0.0	03504	153	03657	157	03814	162	03976	166	04142	171	04313	0.0
.1	3506	152	3658	158	3816	162	3978	166	4144	171	4315	.1
.2	3509	153	3662	157	3819	162	3981	166	4147	171	4318	.2
.3	3514	154	3668	156	3824	163	3987	166	4153	171	4324	.3
.4	3521	154	3675	157	3832	162	3994	167	4161	171	4332	.4
.5	3530	154	3684	158	3842	162	4004	167	4171	171	4342	.5
.6	3542	154	3696	158	3854	163	4017	167	4184	171	4355	.6
.7	3556	154	3710	158	3868	163	4031	167	4198	172	4370	.7
.8	3572	154	3726	159	3885	163	4048	167	4215	172	4387	.8
.9	3590	154	3744	159	3903	164	4067	167	4234	173	4407	.9
1.0	3611	154	3765	159	3924	164	4088	168	4256	172	4428	1.0
1.1	3634	154	3788	159	3947	165	4112	168	4280	172	4452	1.1
1.2	3658	155	3813	159	3972	165	4137	169	4306	173	4479	1.2
1.3	3685	155	3840	160	4000	165	4165	169	4334	173	4507	1.3
1.4	3714	155	3869	161	4030	166	4196	169	4365	173	4538	1.4
1.5	3745	156	3901	161	4062	166	4228	170	4398	174	4572	1.5
1.6	3778	157	3935	161	4096	167	4263	170	4433	175	4608	1.6
1.7	3814	157	3971	162	4133	166	4299	171	4470	176	4646	1.7
1.8	3851	158	4009	162	4171	167	4338	172	4510	176	4686	1.8
1.9	3890	159	4049	163	4212	168	4380	172	4552	176	4728	1.9
2.0	3932	160	4092	163	4255	168	4423	173	4596	177	4773	2.0
2.1	3976	161	4137	163	4300	169	4469	173	4642	178	4820	2.1
2.2	4022	162	4184	164	4348	169	4517	174	4691	179	4870	2.2
2.3	4071	161	4232	166	4398	169	4567	175	4742	180	4922	2.3
2.4	4121	163	4284	166	4450	170	4620	176	4796	180	4976	2.4
2.5	4174	163	4337	167	4504	171	4675	176	4851	181	5032	2.5
2.6	4229	163	4392	168	4560	172	4732	177	4909	182	5091	2.6
2.7	4286	164	4450	169	4619	173	4792	177	4969	183	5152	2.7
2.8	4345	165	4510	169	4679	174	4853	179	5032	183	5215	2.8
2.9	4406	166	4572	170	4742	175	4917	179	5096	184	5280	2.9
3.0	4469	166	4636	171	4807	176	4983	180	5163	185	5348	3.0
3.1	4534	168	4702	172	4874	177	5051	181	5232	186	5418	3.1
3.2	4602	169	4771	173	4944	178	5122	182	5304	187	5491	3.2
3.3	4672	170	4842	174	5016	179	5195	183	5378	188	5566	3.3
3.4	4744	171	4915	175	5090	180	5270	184	5454	189	5643	3.4
3.5	4818	172	4990	176	5166	181	5347	185	5532	190	5722	3.5
3.6	4894	173	5067	177	5244	183	5427	186	5613	191	5804	3.6
3.7	4973	174	5147	178	5325	184	5509	187	5696	192	5888	3.7
3.8	5053	176	5229	179	5408	185	5593	188	5781	193	5974	3.8
3.9	5136	177	5313	181	5494	185	5679	190	5869	194	6063	3.9
4.0	5221	178	5399	182	5581	187	5768	191	5959	195	6154	4.0
4.1	5308	179	5487	184	5671	188	5859	192	6051	196	6247	4.1
4.2	5398	180	5578	185	5763	189	5952	194	6146	197	6343	4.2
4.3	5489	182	5671	186	5857	191	6048	195	6243	199	6442	4.3
4.4	5583	183	5766	187	5953	192	6145	197	6342	201	6543	4.4
4.5	5679	184	5863	189	6052	193	6245	198	6443	202	6645	4.5
4.6	5777	185	5962	191	6153	194	6347	200	6547	203	6750	4.6
4.7	5877	187	6064	192	6256	196	6452	201	6653	205	6858	4.7
4.8	5980	188	6168	194	6362	197	6559	202	6761	207	6968	4.8
4.9	6084	190	6274	195	6469	199	6668	204	6872	208	7080	4.9
5.0	6191	191	6382	197	6579	200	6779	206	6985	209	7194	5.0
$\alpha - \alpha_0$ $d - \delta_0$	$7^\circ.0$	hor. diff.	$7^\circ.1$	hor. diff.	$7^\circ.2$	hor. diff.	$7^\circ.3$	hor. diff.	$7^\circ.4$	hor. diff.	$7^\circ.5$	$\alpha - \alpha_0$ $d - \delta_0$

TABLE III.

$$C = k \xi \sec d - (\alpha - \alpha_0)^\circ.$$

 $k \xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$.

 $0^\circ.00001 = 0^\circ.036 = \text{unit}.$

$\alpha - \alpha_0$ $d - \delta_0$	7°.5	hor. diff.	7°.6	hor. diff.	7°.7	hor. diff.	7°.8	hor. diff.	7°.9	hor. diff.	8°.0	$\alpha - \alpha_0$ $d - \delta_0$
0.0	04313	176	04489	180	04669	185	04854	190	05044	195	05239	0.0
.1	4315	2	4491	2	4671	3	4856	3	5046	4	5241	.1
.2	4318	3	4494	3	4674	6	4859	6	5050	6	5245	.2
.3	4324	6	4500	6	4680	9	4865	9	5056	9	5251	.3
.4	4332	10	4508	10	4689	10	4874	10	5065	10	5260	.4
.5	4342	13	4518	13	4699	13	4884	13	5075	13	5270	.5
.6	4355	15	4531	15	4712	15	4897	15	5088	15	5284	.6
.7	4370	17	4546	17	4727	18	4913	18	5104	18	5300	.7
.8	4387	20	4563	20	4745	20	4931	20	5122	20	5318	.8
.9	4407	21	4583	22	4765	22	4951	23	5143	23	5339	.9
1.0	4428	24	4605	24	4787	25	4974	25	5166	25	5362	1.0
1.1	4452	27	4629	27	4812	27	4999	28	5191	28	5388	1.1
1.2	4479	28	4656	29	4839	30	5027	30	5219	30	5416	1.2
1.3	4507	31	4685	32	4869	32	5057	32	5249	33	5446	1.3
1.4	4538	34	4717	34	4901	34	5089	35	5282	35	5479	1.4
1.5	4572	36	4751	36	4935	37	5124	37	5317	37	5515	1.5
1.6	4608	38	4787	39	4972	39	5161	39	5354	40	5553	1.6
1.7	4646	40	4826	41	5011	41	5200	42	5394	43	5593	1.7
1.8	4686	42	4867	43	5052	43	5242	44	5437	45	5636	1.8
1.9	4728	45	4910	45	5095	46	5286	47	5482	48	5682	1.9
2.0	4773	47	4955	48	5141	48	5333	49	5529	50	5730	2.0
2.1	4820	50	5003	50	5189	51	5382	51	5579	52	5780	2.1
2.2	4870	52	5053	52	5240	53	5433	54	5631	54	5833	2.2
2.3	4922	54	5105	55	5293	56	5487	56	5685	57	5888	2.3
2.4	4976	56	5160	57	5349	58	5543	59	5742	60	5946	2.4
2.5	5032	59	5217	59	5407	60	5602	61	5802	62	6006	2.5
2.6	5091	61	5276	62	5467	63	5663	64	5864	65	6069	2.6
2.7	5152	63	5338	65	5530	65	5727	66	5928	67	6134	2.7
2.8	5215	65	5403	66	5595	67	5793	68	5995	69	6201	2.8
2.9	5280	68	5469	69	5662	70	5861	71	6064	72	6271	2.9
3.0	5348	70	5538	71	5732	72	5932	73	6136	74	6344	3.0
3.1	5418	73	5609	74	5804	75	6005	75	6210	76	6419	3.1
3.2	5491	75	5683	76	5879	77	6080	78	6286	79	6497	3.2
3.3	5566	77	5759	78	5956	79	6158	80	6365	81	6577	3.3
3.4	5643	79	5837	80	6035	82	6238	83	6446	84	6659	3.4
3.5	5722	82	5917	83	6117	84	6321	85	6530	86	6744	3.5
3.6	5804	84	6000	85	6201	86	6406	88	6616	89	6831	3.6
3.7	5888	86	6085	88	6287	89	6494	90	6705	91	6921	3.7
3.8	5974	89	6173	90	6376	91	6584	92	6796	94	7013	3.8
3.9	6063	91	6263	93	6467	94	6676	95	6890	96	7108	3.9
4.0	6154	93	6356	95	6561	96	6771	97	6986	98	7205	4.0
4.1	6247	96	6451	97	6657	98	6868	99	7084	100	7305	4.1
4.2	6343	99	6548	101	6755	101	6968	102	7186	103	7407	4.2
4.3	6442	101	6647	102	6856	103	7070	104	7289	106	7512	4.3
4.4	6543	102	6749	104	6959	106	7174	107	7395	108	7619	4.4
4.5	6645	105	6853	107	7065	108	7281	109	7503	111	7729	4.5
4.6	6750	108	6960	109	7173	110	7390	112	7614	113	7842	4.6
4.7	6858	110	7069	111	7283	113	7502	114	7727	115	7957	4.7
4.8	6968	112	7180	113	7396	115	7616	117	7842	118	8074	4.8
4.9	7080	114	7293	116	7511	117	7733	119	7960	121	8194	4.9
5.0	7194	115	7409	117	7628	119	7852	122	8081	123	8316	5.0
$\alpha - \alpha_0$ $d - \delta_0$	7°.5	hor. diff.	7°.6	hor. diff.	7°.7	hor. diff.	7°.8	hor. diff.	7°.9	hor. diff.	8°.0	$\alpha - \alpha_0$ $d - \delta_0$

TABLE III.

$$C = k \xi \sec d - (\alpha - \alpha_0)^\circ.$$

 $k \xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$.

 $0''.00001 = 0''.036 = \text{unit.}$

$\alpha - \alpha_0$ $d - \delta_0$	$8^\circ.0$	hor. diff.	$8^\circ.1$	hor. diff.	$8^\circ.2$	hor. diff.	$8^\circ.3$	hor. diff.	$8^\circ.4$	hor. diff.	$8^\circ.5$	$\alpha - \alpha_0$ $d - \delta_0$
0.0	05239	200	05439	205	05644	211	05855	215	06070	221	06291	0.0
.1	5241	200	5441	205	5646	211	5857	215	6072	221	6293	.1
.2	5245	200	5445	205	5650	211	5861	215	6076	221	6297	.2
.3	5251	200	5451	205	5656	211	5867	215	6082	221	6303	.3
.4	5260	200	5460	205	5665	211	5876	215	6091	221	6312	.4
.5	5270	201	5471	205	5676	211	5887	216	6103	221	6324	.5
.6	5284	201	5485	205	5690	211	5901	216	6117	221	6338	.6
.7	5300	201	5501	205	5706	211	5917	217	6134	221	6355	.7
.8	5318	202	5520	205	5725	211	5936	217	6153	222	6375	.8
.9	5339	202	5541	205	5746	212	5958	217	6175	222	6397	.9
1.0	5362	202	5564	206	5770	212	5982	217	6199	223	6422	1.0
1.1	5388	202	5590	206	5796	213	6009	217	6226	223	6449	1.1
1.2	5416	203	5619	206	5825	213	6038	218	6256	223	6479	1.2
1.3	5446	204	5650	207	5857	213	6070	218	6288	224	6512	1.3
1.4	5479	204	5683	208	5891	213	6104	219	6323	224	6547	1.4
1.5	5515	204	5719	209	5928	213	6141	219	6360	225	6585	1.5
1.6	5553	204	5757	210	5967	213	6180	220	6400	225	6625	1.6
1.7	5593	205	5798	210	6008	214	6222	221	6443	225	6668	1.7
1.8	5636	206	5842	210	6052	215	6267	221	6488	226	6714	1.8
1.9	5682	206	5888	211	6099	215	6314	222	6536	226	6762	1.9
2.0	5730	206	5936	212	6148	216	6364	222	6586	227	6813	2.0
2.1	5780	207	5987	212	6200	216	6416	223	6639	227	6866	2.1
2.2	5833	208	6041	213	6254	217	6471	223	6694	229	6923	2.2
2.3	5888	209	6097	213	6310	218	6528	224	6752	230	6982	2.3
2.4	5946	209	6155	214	6369	219	6588	225	6813	230	7043	2.4
2.5	6006	210	6216	215	6431	220	6651	225	6876	231	7107	2.5
2.6	6069	210	6279	216	6495	221	6716	226	6942	232	7174	2.6
2.7	6134	211	6345	217	6562	222	6784	226	7010	234	7244	2.7
2.8	6201	213	6414	217	6631	223	6854	227	7081	235	7316	2.8
2.9	6271	214	6485	218	6703	224	6927	228	7155	235	7390	2.9
3.0	6344	214	6558	219	6777	225	7002	229	7231	236	7467	3.0
3.1	6419	215	6634	220	6854	226	7080	230	7310	237	7547	3.1
3.2	6497	216	6713	220	6933	227	7160	232	7392	237	7629	3.2
3.3	6577	217	6794	221	7015	228	7243	233	7476	238	7714	3.3
3.4	6659	218	6877	223	7100	229	7329	234	7563	239	7802	3.4
3.5	6744	219	6963	224	7187	231	7418	234	7652	240	7892	3.5
3.6	6831	221	7052	225	7277	232	7509	235	7744	241	7985	3.6
3.7	6921	222	7143	226	7369	233	7602	236	7838	242	8080	3.7
3.8	7013	223	7236	228	7464	234	7698	237	7935	243	8178	3.8
3.9	7108	224	7332	229	7561	235	7796	239	8035	244	8279	3.9
4.0	7205	226	7431	230	7661	236	7897	240	8137	246	8383	4.0
4.1	7305	227	7532	232	7764	237	8001	241	8242	247	8489	4.1
4.2	7407	229	7636	233	7869	238	8107	242	8349	249	8598	4.2
4.3	7512	230	7742	235	7977	238	8215	244	8459	250	8709	4.3
4.4	7619	231	7850	237	8087	239	8326	246	8572	251	8823	4.4
4.5	7729	232	7961	238	8199	241	8440	247	8687	253	8940	4.5
4.6	7842	233	8075	239	8314	242	8556	249	8805	254	9059	4.6
4.7	7957	234	8191	240	8431	244	8675	250	8925	256	9181	4.7
4.8	8074	235	8309	242	8551	246	8797	251	9048	258	9306	4.8
4.9	8194	236	8430	244	8674	247	8921	253	9174	259	9433	4.9
5.0	8316	238	8554	245	8799	249	9048	254	9302	260	9562	5.0
$\alpha - \alpha_0$ $d - \delta_0$	$8^\circ.0$	hor. diff.	$8^\circ.1$	hor. diff.	$8^\circ.2$	hor. diff.	$8^\circ.3$	hor. diff.	$8^\circ.4$	hor. diff.	$8^\circ.5$	$\alpha - \alpha_0$ $d - \delta_0$

TABLE III.

$$C = k \xi \sec d - (\alpha - \alpha_0)^\circ.$$

 $k \xi d$ sec is always numerically larger than $(\alpha - \alpha_0)^\circ$.

 $0''.00001 = 0''.036 = \text{unit.}$

$\alpha - \alpha_0$ $d - d_0$	8°.5	hor. diff.	8°.6	hor. diff.	8°.7	hor. diff.	8°.8	hor. diff.	8°.9	hor. diff.	9°.0	$\alpha - \alpha_0$ $d - d_0$
0.0	06291	226	06517	231	06748	237	06985	242	07227	248	07475	0.0
.1	6293	226	6519	231	6750	237	6987	242	7229	248	7477	.1
.2	6297	226	6523	231	6754	237	6991	242	7233	248	7481	.2
.3	6303	226	6529	232	6761	237	6998	242	7240	248	7488	.3
.4	6312	226	6538	232	6770	237	7007	243	7250	248	7498	.4
0.5	6324	226	6550	232	6782	237	7019	243	7262	248	7510	0.5
.6	6338	226	6564	233	6797	237	7034	243	7277	248	7525	.6
.7	6355	227	6582	232	6814	237	7051	244	7295	248	7543	.7
.8	6375	226	6601	233	6834	237	7071	245	7316	248	7564	.8
.9	6397	227	6624	233	6857	237	7094	245	7339	249	7588	.9
1.0	6422	227	6649	233	6882	238	7120	245	7365	249	7614	1.0
1.1	6449	228	6677	233	6910	239	7149	245	7394	249	7643	1.1
1.2	6479	228	6707	234	6941	240	7181	245	7426	249	7675	1.2
1.3	6512	228	6740	234	6974	241	7215	245	7460	250	7710	1.3
.4	6547	229	6776	234	7010	241	7251	245	7496	251	7747	1.4
1.5	6585	229	6814	235	7049	241	7290	246	7536	251	7787	1.5
1.6	6625	230	6855	236	7091	241	7332	246	7578	252	7830	1.6
.7	6668	230	6898	238	7136	241	7377	246	7623	252	7875	1.7
1.8	6714	231	6945	238	7183	241	7424	247	7671	253	7924	1.8
1.9	6762	231	6993	239	7232	242	7474	248	7722	253	7975	1.9
2.0	6813	232	7045	239	7284	243	7527	248	7775	254	8029	2.0
2.1	6866	234	7100	239	7339	243	7582	249	7831	255	8086	2.1
2.2	6923	234	7157	239	7396	244	7640	250	7890	255	8145	2.2
2.3	6982	234	7216	240	7456	245	7701	251	7952	255	8207	2.3
.4	7043	235	7278	241	7519	246	7765	251	8016	256	8272	2.4
2.5	7107	236	7343	242	7585	246	7831	252	8083	257	8340	2.5
2.6	7174	236	7410	243	7653	247	7900	253	8153	258	8411	2.6
.7	7244	236	7480	244	7724	248	7972	253	8225	259	8484	2.7
2.8	7316	237	7553	244	7797	249	8046	255	8301	259	8560	2.8
2.9	7390	239	7629	244	7873	250	8123	256	8379	260	8639	2.9
3.0	7467	240	7707	245	7952	251	8203	257	8460	261	8721	3.0
3.1	7547	241	7788	246	8034	252	8286	258	8544	262	8806	3.1
3.2	7629	242	7871	247	8118	253	8371	259	8630	263	8893	3.2
3.3	7714	243	7957	248	8205	254	8459	260	8719	264	8983	3.3
.4	7802	244	8046	249	8295	255	8550	261	8811	265	9076	3.4
3.5	7892	245	8137	250	8387	256	8643	262	8905	267	9172	3.5
3.6	7985	246	8231	251	8482	257	8739	263	9002	269	9271	3.6
.7	8080	248	8328	252	8580	258	8838	264	9102	270	9372	3.7
3.8	8178	249	8427	254	8681	259	8940	265	9205	271	9476	3.8
3.9	8279	250	8529	255	8784	261	9045	266	9311	272	9583	3.9
4.0	8383	251	8634	256	8890	262	9152	267	9419	273	9692	4.0
4.1	8489	252	8741	258	8999	263	9262	268	9530	274	9804	4.1
4.2	8598	253	8851	259	9110	265	9375	269	9644	276	9920	4.2
4.3	8709	255	8964	260	9324	266	9490	271	9761	277	10038	4.3
.4	8823	256	9079	262	9441	267	9608	273	9881	277	10158	4.4
4.5	8940	257	9197	263	9460	269	9729	274	10003	279	10282	4.5
4.6	9059	259	9318	264	9582	270	9852	276	10128	280	10408	4.6
.7	9181	260	9441	266	9707	272	9979	277	10256	282	10538	4.7
4.8	9306	261	9567	268	9835	273	10108	278	10386	284	10670	4.8
4.9	9433	263	9696	269	9965	275	10240	279	10519	285	10804	4.9
5.0	9562	265	9827	271	10098	276	10374	281	10655	287	10942	5.0
$\alpha - \alpha_0$ $d - d_0$	8°.5	hor. diff.	8°.6	hor. diff.	8°.7	hor. diff.	8°.8	hor. diff.	8°.9	hor. diff.	9°.0	$\alpha - \alpha_0$ $d - d_0$

TABLE III.

$$C = k \xi \sec d - (\alpha - \alpha_0)^\circ.$$

 $k \xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$.

0°.00001 = 0°.036 = unit.

$\alpha - \alpha_0$ $d - \delta_0$	9°.0	hor. diff.	9°.1	hor. diff.	9°.2	hor. diff.	9°.3	hor. diff.	9°.4	hor. diff.	9°.5	$\alpha - \alpha_0$ $d - \delta_0$
0.0	07475	255	07730	259	07989	265	08254	272	08526	277	08803	0.0
.1	7477	255	7732	259	7991	265	8256	272	8528	277	8805	.1
.2	7481	255	7736	259	7995	265	8260	272	8532	277	8809	.2
.3	7488	255	7743	259	8002	265	8367	272	8539	277	8816	.3
.4	7498	255	7753	259	8012	265	8377	272	8549	277	8826	.4
0.5	7510	255	7765	260	8025	265	8290	272	8562	277	8839	0.5
.6	7525	256	7781	260	8041	265	8306	272	8578	277	8855	.6
.7	7543	256	7799	260	8059	265	8324	273	8597	277	8874	.7
.8	7564	256	7820	260	8080	266	8346	273	8619	277	8896	.8
.9	7588	256	7844	260	8104	266	8370	273	8643	278	8921	.9
1.0	7614	256	7870	261	8131	266	8397	273	8670	279	8949	1.0
1.1	7643	257	7900	261	8161	266	8427	273	8700	280	8980	1.1
1.2	7675	257	7932	261	8193	267	8460	274	8734	280	9014	1.2
1.3	7710	257	7967	261	8228	268	8496	274	8770	280	9050	1.3
1.4	7747	257	8004	263	8267	268	8535	274	8809	280	9089	1.4
1.5	7787	258	8045	263	8308	268	8576	275	8851	281	9132	1.5
1.6	7830	258	8088	264	8352	268	8620	276	8896	281	9177	1.6
1.7	7875	259	8134	264	8398	269	8667	276	8943	282	9225	1.7
1.8	7924	259	8183	265	8448	269	8717	277	8994	283	9277	1.8
1.9	7975	260	8235	265	8500	270	8770	278	9048	283	9331	1.9
2.0	8029	260	8289	266	8555	271	8826	278	9104	283	9387	2.0
2.1	8086	260	8346	267	8613	272	8885	278	9163	284	9447	2.1
2.2	8145	261	8406	268	8674	273	8946	280	9226	284	9510	2.2
2.3	8207	262	8469	268	8737	273	9010	281	9291	284	9575	2.3
2.4	8272	263	8535	269	8804	274	9078	281	9359	285	9644	2.4
2.5	8340	264	8604	269	8873	275	9148	281	9429	287	9716	2.5
2.6	8411	265	8676	269	8945	276	9221	282	9503	288	9791	2.6
2.7	8484	266	8750	270	9020	277	9297	282	9579	289	9868	2.7
2.8	8560	267	8827	271	9098	278	9376	283	9659	290	9949	2.8
2.9	8639	267	8906	273	9179	279	9458	284	9742	290	10032	2.9
3.0	8721	268	8989	274	9263	279	9542	285	9827	291	10118	3.0
3.1	8806	269	9075	274	9349	280	9629	286	9915	292	10207	3.1
3.2	8893	270	9163	276	9439	281	9720	287	10007	293	10300	3.2
3.3	8983	271	9254	277	9531	282	9813	288	10101	294	10395	3.3
3.4	9076	272	9348	277	9625	283	9908	290	10198	295	10493	3.4
3.5	9172	273	9445	278	9723	284	10007	291	10298	296	10594	3.5
3.6	9271	274	9545	279	9824	285	10109	292	10401	297	10698	3.6
3.7	9372	275	9647	280	9927	286	10213	294	10507	298	10805	3.7
3.8	9476	276	9752	282	10034	287	10321	295	10616	299	10915	3.8
3.9	9583	277	9860	283	10143	289	10432	296	10728	300	11028	3.9
4.0	9692	279	9971	284	10255	290	10545	297	10842	302	11144	4.0
4.1	9804	281	10085	285	10370	291	10661	298	10959	304	11263	4.1
4.2	9920	281	10201	287	10488	293	10781	299	11080	305	11385	4.2
4.3	10038	282	10320	289	10609	294	10903	300	11203	306	11509	4.3
4.4	10158	284	10442	290	10732	296	11028	302	11330	307	11637	4.4
4.5	10282	286	10568	291	10859	296	11155	304	11459	309	11768	4.5
4.6	10408	288	10696	292	10988	298	11286	305	11591	311	11902	4.6
4.7	10538	289	10827	293	11120	299	11419	307	11726	312	12038	4.7
4.8	10670	290	10960	295	11255	301	11556	308	11864	314	12178	4.8
4.9	10804	292	11096	297	11393	302	11695	309	12004	316	12320	4.9
5.0	10942	293	11235	299	11534	304	11838	310	12148	317	12465	5.0
$\alpha - \alpha_0$ $d - \delta_0$	9°.0	hor. diff.	9°.1	hor. diff.	9°.2	hor. diff.	9°.3	hor. diff.	9°.4	hor. diff.	9°.5	$\alpha - \alpha_0$ $d - \delta_0$

TABLE III.

$$C = k \varepsilon \sec d - (\alpha - \alpha_0)^\circ.$$

 $k \varepsilon \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$.

 $0^\circ.00001 = 0^\circ.036 = \text{unit.}$

$\alpha - \alpha_0$ $d - \delta_0$	9°.5	hor. diff.	9°.6	hor. diff.	9°.7	hor. diff.	9°.8	hor. diff.	9°.9	hor. diff.	10°.0	$\alpha - \alpha_0$ $d - \delta_0$
0.0	08803	283	09086	289	09375	295	09670	301	09971	309	10280	0.0
.1	8805	283	9088	289	9377	295	9672	301	9973	309	10282	.1
.2	8809	283	9092	289	9381	295	9676	301	9977	309	10286	.2
.3	8816	283	9099	289	9388	296	9684	301	9985	309	10294	.3
.4	8826	284	9110	289	9399	296	9695	301	9996	309	10305	.4
0.5	8839	284	9123	289	9412	296	9708	301	10009	309	10318	0.5
.6	8855	284	9139	289	9428	297	9725	301	10026	309	10335	.6
.7	8874	284	9158	289	9447	297	9744	301	10045	310	10355	.7
.8	8896	284	9180	290	9470	297	9767	301	10068	310	10378	.8
.9	8921	284	9205	291	9496	297	9793	301	10094	310	10404	.9
1.0	8949	284	9233	291	9524	297	9821	302	10123	310	10433	1.0
1.1	8980	284	9264	291	9555	298	9853	303	10156	310	10466	1.1
1.2	9014	284	9298	291	9589	298	9887	304	10191	310	10501	1.2
1.3	9050	285	9335	292	9627	298	9925	304	10229	310	10539	1.3
1.4	9089	286	9375	293	9667	298	9965	305	10270	311	10581	1.4
1.5	9132	286	9418	293	9711	298	10009	305	10314	312	10626	1.5
1.6	9177	287	9464	293	9757	299	10056	305	10361	313	10674	1.6
1.7	9225	287	9512	294	9806	299	10105	307	10412	313	10725	1.7
1.8	9277	287	9564	295	9859	299	10158	307	10465	313	10778	1.8
1.9	9331	288	9619	295	9914	300	10214	307	10521	314	10835	1.9
2.0	9387	289	9676	296	9972	301	10273	308	10581	314	10895	2.0
2.1	9447	290	9737	296	10033	303	10336	308	10644	314	10958	2.1
2.2	9510	291	9801	296	10097	304	10401	308	10709	316	11025	2.2
2.3	9575	292	9867	298	10165	304	10469	309	10778	316	11094	2.3
2.4	9644	292	9936	299	10235	305	10540	310	10850	316	11166	2.4
2.5	9716	293	10009	299	10308	306	10614	310	10924	318	11242	2.5
2.6	9791	294	10085	299	10384	307	10691	311	11002	319	11321	2.6
2.7	9868	295	10163	300	10463	308	10771	312	11083	319	11402	2.7
2.8	9949	295	10244	302	10546	308	10854	313	11167	320	11487	2.8
2.9	10032	296	10328	303	10631	309	10940	314	11254	321	11575	2.9
3.0	10118	297	10415	304	10719	310	11029	315	11344	322	11666	3.0
3.1	10207	298	10505	305	10810	311	11121	316	11437	323	11760	3.1
3.2	10300	299	10599	306	10905	311	11216	317	11533	324	11857	3.2
3.3	10395	300	10695	308	11003	311	11314	318	11632	325	11957	3.3
3.4	10493	301	10794	309	11103	313	11416	319	11735	326	12061	3.4
3.5	10594	302	10896	310	11206	314	11520	321	11841	326	12167	3.5
3.6	10698	303	11001	311	11312	315	11627	323	11950	327	12277	3.6
3.7	10805	304	11109	312	11421	317	11738	323	12061	328	12389	3.7
3.8	10915	306	11221	313	11534	317	11851	325	12176	329	12505	3.8
3.9	11028	307	11335	314	11649	319	11968	325	12293	331	12624	3.9
4.0	11144	308	11452	315	11767	320	12087	327	12414	332	12746	4.0
4.1	11263	310	11573	316	11889	321	12210	328	12538	333	12871	4.1
4.2	11385	311	11696	317	12013	322	12335	330	12665	335	13000	4.2
4.3	11509	313	11822	318	12140	324	12464	331	12795	336	13131	4.3
4.4	11637	314	11951	320	12271	325	12596	332	12928	337	13265	4.4
4.5	11768	315	12083	321	12404	327	12731	333	13064	339	13403	4.5
4.6	11902	316	12218	322	12540	329	12869	334	13203	341	13544	4.6
4.7	12038	318	12356	324	12680	330	13010	335	13345	343	13688	4.7
4.8	12178	319	12497	325	12822	332	13154	337	13491	344	13835	4.8
4.9	12320	321	12641	327	12968	333	13301	338	13639	346	13985	4.9
5.0	12465	323	12788	328	13116	335	13451	340	13791	347	14138	5.0
$\alpha - \alpha_0$ $d - \delta_0$	9°.5	hor. diff.	9°.6	hor. diff.	9°.7	hor. diff.	9°.8	hor. diff.	9°.9	hor. diff.	10°.0	$\alpha - \alpha_0$ $d - \delta_0$

TABLES IV-XI FOR THE COMPUTATION OF CORRECTIONS FOR REFRACTION.

Table IV	gives	N .	Argument $(s - \alpha)$ and δ } for { $-45^\circ < (s - \alpha) < +45^\circ$ }
" V	"	Y .	" $(s - \alpha)$ " δ } for { $-40^\circ < \delta < +90^\circ$ }
" VI	"	Δ .	" $(s - \alpha)$ } for { $-180^\circ < (s - \alpha) < +180^\circ$ }
" VII	"	{ $\log N \cos (\Delta - \delta)$ $N \cos (\Delta - \delta)$ }	" $(s - \alpha)$ }

The above tables are computed for the latitude of the Students' Observatory: $37^\circ 52'.4$.

Table VIII for conversion of degrees into radians.

" IX	giving $x^2 + y^2$.	Arguments x and y .
" X	" D .	Argument $xN + yY$.
" XI	" Z .	" N and Y .

TABLE IV.

X

X has the same sign as $(s - \alpha)$.

$s - \alpha$ δ	0°	$d.$	5°	$d.$	10°	$d.$	15°	$d.$	20°	$d.$	25°	$d.$	30°	$d.$	35°	$d.$	40°	$d.$	45°	$s - \alpha$ δ
-40°	0.000	331	0.331	351	0.682	397	1.079	476	1.555	619	2.174	962	3.058							-40°
39		306	0.306	323	0.629	362	0.991	429	1.420	547	1.967	756	2.723							39
38		285	0.285	299	.584	333	0.917	390	1.307	489	1.796	660	2.456							38
37		266	.266	279	.545	308	0.853	357	1.210	442	1.652	584	2.236	845	3.081					37
36		250	.250	261	.511	287	0.798	331	1.129	402	1.531	522	2.053	737	2.790					36
35		236	.236	246	.482	268	0.750	307	1.057	369	1.426	473	1.899	651	2.550					35
34		223	.223	232	.455	252	0.707	287	.994	341	1.335	431	1.766	578	2.349	877	3.226			34
33		212	.212	220	.432	238	0.670	269	.939	317	1.256	395	1.651	527	2.178	759	2.937			33
32		201	.201	210	.411	225	0.636	253	.889	297	1.186	365	1.551	479	2.030	676	2.706			32
31		192	.192	199	.391	214	0.605	240	.845	278	1.123	340	1.463	439	1.902	608	2.510			31
-30		184	.184	190	.374	204	0.578	227	.805	262	1.067	318	1.385	404	1.789	552	2.341	823	3.164	-30
29		177	.177	182	.359	194	0.553	216	.769	248	1.017	297	1.314	376	1.690	504	2.194	736	2.930	29
28		170	.170	174	.344	187	0.531	205	.736	235	.971	280	1.251	351	1.602	463	2.065	663	2.728	28
27		163	.163	168	.331	179	0.510	196	.706	224	.930	266	1.196	326	1.522	429	1.951	602	2.553	27
26		157	.157	162	.319	172	0.491	188	.679	213	.892	252	1.144	307	1.451	398	1.849	551	2.400	26
25		152	.152	156	.308	165	0.473	181	.654	204	.858	238	1.096	291	1.387	371	1.758	507	2.265	25
24		147	.147	151	.298	159	0.457	174	.631	195	.826	227	1.053	275	1.328	348	1.676	469	2.145	24
23		142	.142	146	.288	154	0.442	168	.610	187	.797	217	1.014	260	1.274	328	1.602	436	2.038	23
22		138	.138	141	.279	149	0.428	162	.590	180	.770	207	0.977	248	1.225	309	1.534	407	1.941	22
21		134	.134	137	.271	144	0.415	156	.571	174	.745	199	0.944	236	1.180	293	1.473	382	1.855	21
-20		130	.130	133	.263	140	0.403	151	.554	168	.722	191	0.913	226	1.139	278	1.417	358	1.775	-20
19		127	.127	129	.256	136	0.392	146	.538	162	.700	184	0.884	216	1.100	265	1.365	338	1.703	19
18		123	.123	126	.249	133	0.382	141	.523	157	.680	177	0.857	208	1.065	252	1.317	319	1.636	18
17		120	.120	123	.243	129	0.372	138	.510	151	.661	171	0.832	200	1.032	240	1.272	304	1.576	17
16		117	.117	120	.237	125	0.362	135	.497	146	.643	166	0.809	192	1.001	230	1.231	289	1.520	16
15		115	.115	117	.232	122	0.354	130	.484	143	.627	160	0.787	185	0.972	221	1.193	275	1.468	15
14		112	.112	114	.226	119	0.345	128	.473	138	.611	156	0.767	178	.945	213	1.158	262	1.420	14
13		110	.110	111	.221	117	0.338	124	.462	135	.597	152	0.749	171	.920	205	1.125	251	1.376	13
12		107	.107	110	.217	114	0.331	121	.452	131	.583	148	0.731	166	.897	197	1.094	241	1.335	12
11		105	.105	107	.212	112	0.324	118	.442	128	.570	144	0.714	161	.875	190	1.065	231	1.296	11
-10		103	.103	105	.208	109	0.317	116	.433	125	.558	139	0.697	157	.854	184	1.038	222	1.260	-10
9		101	.101	103	.204	107	0.311	113	.424	122	.546	136	.682	152	.834	178	1.012	214	1.226	9
8		99	.099	101	.200	105	0.305	111	.416	119	.535	132	.667	149	.816	172	0.988	206	1.194	8
7		98	.098	99	.197	103	0.300	108	.408	117	.525	129	.654	145	.799	167	.966	199	1.165	7
6		96	.096	97	.193	101	0.294	107	.401	114	.515	126	.641	141	.782	163	.945	192	1.137	6
5		94	.094	96	.190	99	0.289	105	.394	112	.506	123	.629	138	.767	157	.924	187	1.111	5
4		93	.093	94	.187	98	0.285	102	.387	110	.497	121	.618	134	.752	153	.905	181	1.086	4
3		91	.091	93	.184	96	0.280	101	.381	108	.489	118	.607	131	.738	149	.887	176	1.063	3
2		90	.090	91	.181	95	0.276	99	.375	106	.481	116	.597	128	.725	145	.870	171	1.041	2
1		89	.089	90	.179	93	0.272	97	.369	104	.473	114	.587	125	.712	142	.854	166	1.020	1
0		87	.087	89	.176	92	0.268	96	.364	102	.466	111	.577	123	.700	139	.839	161	1.000	0
δ $s - \alpha$	0°	$d.$	5°	$d.$	10°	$d.$	15°	$d.$	20°	$d.$	25°	$d.$	30°	$d.$	35°	$d.$	40°	$d.$	45°	$s - \alpha$ δ

$$X = \tan (s - \alpha) \cos \Delta \sec (\Delta - \delta)$$

TABLE IV.

 X X has the same sign as $(s - \alpha_0)$.

$s - \alpha$ δ	0°	$d.$	5°	$d.$	10°	$d.$	15°	$d.$	20°	$d.$	25°	$d.$	30°	$d.$	35°	$d.$	40°	$d.$	45°	$s - \alpha$ δ
0°	0.000	87	0.087	89	0.176	92	0.268	96	0.364	102	0.466	111	0.577	123	0.700	139	0.839	161	1.000	0°
1		86	.086	88	.174	90	.264	95	.359	100	.459	109	.568	121	.689	136	.825	156	.981	1
2		85	.085	87	.172	89	.261	93	.354	99	.453	107	.560	118	.678	133	.811	153	.964	2
3		84	.084	86	.170	87	.257	92	.349	98	.447	105	.552	116	.668	130	.798	149	.947	3
4		83	.083	85	.168	86	.254	91	.345	96	.441	104	.545	113	.658	128	.786	145	.931	4
5		82	.082	84	.166	85	.251	90	.341	94	.435	102	.537	112	.649	125	.774	142	.916	5
6		81	.081	83	.164	84	.248	89	.337	93	.430	100	.530	110	.640	122	.762	139	.901	6
7		80	.080	82	.162	83	.245	88	.333	92	.425	99	.524	108	.632	120	.752	136	.888	7
8		80	.080	80	.160	83	.243	86	.329	91	.420	98	.518	106	.624	118	.742	133	.875	8
9		79	.079	80	.159	82	.241	85	.326	90	.416	96	.512	104	.616	116	.732	130	.862	9
+10		78	.078	79	.157	81	.238	85	.323	88	.411	95	.506	103	.609	114	.723	127	.850	+10
11		77	.077	79	.156	80	.236	83	.319	88	.407	94	.501	101	.602	112	.714	125	.839	11
12		77	.077	78	.155	79	.234	82	.316	87	.403	93	.496	99	.595	111	.706	123	.829	12
13		76	.076	77	.153	79	.232	81	.313	86	.399	92	.491	98	.589	109	.698	121	.819	13
14		75	.075	77	.152	78	.230	81	.311	85	.396	90	.486	98	.584	106	.690	119	.809	14
15		75	.075	76	.151	77	.228	80	.308	84	.392	90	.482	96	.578	105	.683	117	.800	15
16		74	.074	76	.150	76	.226	80	.306	83	.389	89	.478	95	.573	103	.676	115	.791	16
17		74	.074	75	.149	76	.225	79	.304	82	.386	88	.474	94	.568	102	.670	113	.783	17
18		73	.073	75	.148	75	.223	79	.302	81	.383	87	.470	93	.563	100	.663	112	.775	18
19		73	.073	74	.147	75	.222	78	.300	80	.380	86	.466	92	.558	99	.657	110	.767	19
+20		73	.073	73	.146	75	.221	77	.298	80	.378	85	.463	91	.554	98	.652	108	.760	+20
21		72	.072	73	.145	74	.219	77	.296	80	.376	84	.460	90	.550	97	.647	106	.753	21
22		72	.072	72	.144	74	.218	76	.294	80	.374	83	.457	89	.546	96	.642	105	.747	22
23		71	.071	73	.144	73	.217	76	.293	78	.371	83	.454	88	.542	95	.637	104	.741	23
24		71	.071	72	.143	73	.216	75	.291	78	.369	82	.451	88	.539	93	.632	103	.735	24
25		71	.071	72	.143	72	.215	75	.290	77	.367	82	.449	87	.536	92	.628	101	.729	25
26		71	.071	71	.142	72	.214	74	.288	78	.366	80	.446	87	.533	91	.624	100	.724	26
27		70	.070	71	.141	72	.213	74	.287	77	.364	79	.443	87	.530	91	.621	98	.719	27
28		70	.070	71	.141	71	.212	74	.286	77	.363	78	.441	86	.527	90	.617	98	.715	28
29		70	.070	70	.140	72	.212	73	.285	76	.361	78	.439	86	.525	89	.614	96	.710	29
+30		70	.070	70	.140	71	.211	73	.284	76	.360	77	.437	85	.522	89	.611	95	.706	+30
31		69	.069	71	.140	71	.211	73	.284	75	.359	77	.436	84	.520	88	.608	94	.702	31
32		69	.069	70	.139	71	.210	73	.283	75	.358	77	.435	83	.518	87	.605	94	.699	32
33		69	.069	70	.139	71	.210	72	.282	75	.357	77	.434	82	.516	87	.603	93	.696	33
34		69	.069	70	.139	70	.209	73	.282	74	.356	77	.433	82	.515	86	.601	92	.693	34
35		69	.069	70	.139	70	.209	72	.281	75	.356	76	.432	81	.513	86	.599	91	.690	35
36		69	.069	69	.138	71	.209	72	.281	74	.355	76	.431	81	.512	85	.597	90	.687	36
37		69	.069	69	.138	71	.209	72	.281	74	.355	76	.431	80	.511	84	.595	90	.685	37
38		69	.069	69	.138	71	.209	71	.280	74	.354	77	.431	79	.510	84	.594	89	.683	38
39		69	.069	69	.138	71	.209	71	.280	74	.354	77	.431	78	.509	84	.593	88	.681	39
+40		69	.069	69	.138	71	.209	71	.280	74	.354	76	.430	79	.509	82	.591	88	.679	+40
δ $s - \alpha$	0°	$d.$	5°	$d.$	10°	$d.$	15°	$d.$	20°	$d.$	25°	$d.$	30°	$d.$	35°	$d.$	40°	$d.$	45°	δ $s - \alpha$

TABLE IV.

 X X has the same sign as $(s - \alpha_0)$.

δ	$s - \alpha$	0°	$d.$	5°	$d.$	10°	$d.$	15°	$d.$	20°	$d.$	25°	$d.$	30°	$d.$	35°	$d.$	40°	$d.$	45°	$s - \alpha$	δ
$+40^\circ$	0.000	69	0.069	69	0.138	71	0.209	71	0.280	74	0.354	76	0.430	79	0.509	82	0.591	88	0.679	$+40^\circ$		
41		69	.069	70	.139	70	.209	71	.280	74	.354	76	.430	78	.508	83	.591	86	.677	41		
42		69	.069	70	.139	70	.209	72	.281	73	.354	76	.430	78	.508	82	.590	86	.676	42		
43		69	.069	70	.139	70	.209	72	.281	73	.354	76	.430	78	.508	82	.590	85	.675	43		
44		69	.069	70	.139	70	.209	72	.281	73	.354	76	.430	78	.508	81	.589	85	.674	44		
45		69	.069	70	.139	71	.210	72	.282	73	.355	75	.430	78	.508	81	.589	84	.673	45		
46		70	.070	70	.140	70	.210	72	.282	73	.355	76	.431	77	.508	81	.589	84	.673	46		
47		70	.070	70	.140	71	.211	72	.283	73	.356	75	.431	78	.509	80	.589	84	.673	47		
48		70	.070	70	.140	71	.211	72	.283	74	.357	75	.432	77	.509	80	.589	84	.673	48		
49		70	.070	71	.141	71	.212	72	.284	74	.358	75	.433	77	.510	80	.590	83	.673	49		
$+50$		71	.071	70	.141	72	.213	72	.285	74	.359	75	.434	77	.511	80	.591	82	.673	$+50$		
51		71	.071	71	.142	72	.214	72	.286	74	.360	75	.435	77	.512	80	.592	82	.674	51		
52		71	.071	72	.143	71	.214	73	.287	74	.361	75	.436	77	.513	80	.593	82	.675	52		
53		71	.071	72	.143	72	.215	73	.288	74	.362	75	.437	78	.515	79	.594	82	.676	53		
54		72	.072	72	.144	72	.216	74	.290	74	.364	75	.439	77	.516	79	.595	82	.677	54		
55		72	.072	72	.144	73	.217	74	.291	74	.365	76	.441	77	.518	79	.597	81	.678	55		
56		72	.072	73	.145	73	.218	74	.292	75	.367	76	.443	77	.520	79	.599	81	.680	56		
57		73	.073	73	.146	74	.220	74	.294	75	.369	76	.445	77	.522	79	.601	81	.682	57		
58		73	.073	74	.147	74	.221	74	.295	76	.371	76	.447	77	.524	79	.603	81	.684	58		
59		73	.073	75	.148	74	.222	75	.297	76	.373	76	.449	78	.527	79	.606	80	.686	59		
$+60$		74	.074	75	.149	75	.224	75	.299	76	.375	77	.452	78	.530	78	.608	81	.689	$+60$		
61		75	.075	75	.150	75	.225	76	.301	76	.377	77	.454	78	.532	79	.611	80	.691	61		
62		75	.075	76	.151	76	.227	76	.303	77	.380	77	.457	78	.535	79	.614	80	.694	62		
63		76	.076	76	.152	77	.229	76	.305	78	.383	77	.460	79	.539	79	.618	79	.697	63		
64		77	.077	77	.154	77	.231	77	.308	77	.385	79	.464	78	.542	79	.621	80	.701	64		
65		77	.077	78	.155	78	.233	77	.310	78	.388	79	.467	79	.546	79	.625	79	.704	65		
66		78	.078	78	.156	79	.235	78	.313	78	.391	79	.470	80	.550	79	.629	79	.708	66		
67		79	.079	79	.158	79	.237	79	.316	79	.395	79	.474	80	.554	79	.633	79	.712	67		
68		80	.080	79	.159	80	.239	80	.319	79	.398	80	.478	80	.558	79	.637	80	.717	68		
69		80	.080	81	.161	80	.241	81	.322	80	.402	80	.482	81	.563	79	.642	80	.722	69		
$+70$		81	.081	82	.163	81	.244	81	.325	81	.406	81	.487	80	.567	80	.647	80	.727	$+70$		
71		82	.082	82	.164	83	.247	81	.328	82	.408	82	.492	80	.572	81	.653	79	.732	71		
72		83	.083	83	.166	83	.249	83	.332	82	.414	82	.496	81	.577	81	.658	80	.738	72		
73		84	.084	84	.168	84	.252	84	.336	83	.419	82	.501	82	.583	81	.664	80	.744	73		
74		85	.085	86	.171	84	.255	85	.340	84	.424	83	.507	82	.589	82	.671	79	.750	74		
75		86	.086	87	.173	86	.259	85	.344	85	.429	84	.513	82	.595	82	.677	80	.757	75		
76		88	.088	87	.175	87	.262	86	.348	86	.434	85	.519	83	.602	82	.684	80	.764	76		
77		89	.089	89	.178	87	.265	88	.353	86	.439	86	.525	84	.609	82	.691	80	.771	77		
78		90	.090	90	.180	89	.269	89	.358	87	.445	87	.532	84	.616	83	.699	80	.779	78		
79		91	.091	92	.183	90	.273	90	.363	88	.451	87	.538	86	.624	83	.707	80	.787	79		
$+80$		93	.093	92	.185	92	.277	91	.368	90	.458	88	.546	86	.632	83	.715	81	.796	$+80$		
81		94	.094	94	.188	94	.282	92	.374	90	.464	89	.553	87	.640	84	.724	81	.805	81		
82		96	.096	95	.191	95	.286	94	.380	92	.472	89	.561	88	.649	84	.733	81	.814	82		
83		98	.098	97	.195	96	.291	95	.386	93	.479	91	.570	88	.658	85	.743	81	.824	83		
84		99	.099	99	.198	98	.296	96	.392	95	.487	92	.579	89	.668	85	.753	81	.834	84		
85		101	.101	101	.202	99	.301	98	.399	96	.495	93	.588	90	.678	86	.764	81	.845	85		
86		103	.103	103	.206	101	.307	99	.406	98	.504	94	.598	91	.689	86	.775	82	.857	86		
87		105	.105	105	.210	103	.313	101	.414	99	.513	95	.608	92	.700	87	.787	82	.869	87		
88		107	.107	107	.214	105	.319	103	.422	101	.523	96	.619	93	.712	88	.800	82	.882	88		
89		110	.110	108	.218	108	.326	105	.431	102	.533	98	.631	93	.724	89	.813	82	.895	89		
$+90$		112	.112	111	.223	110	.333	107	.440	103	.543	100	.643	94	.737	90	.827	82	.909	$+90$		
δ	$s - \alpha$	0°	$d.$	5°	$d.$	10°	$d.$	15°	$d.$	20°	$d.$	25°	$d.$	30°	$d.$	35°	$d.$	40°	$d.$	45°	$s - \alpha$	δ

TABLE V.

 Y

δ	$s-\alpha$	0°	$d.$	5°	$d.$	10°	$d.$	15°	$d.$	20°	$d.$	25°	$d.$	30°	$d.$	35°	$d.$	40°	$d.$	45°	$s-\alpha$	δ
-40°																					-40°	
39																					39	
38																					38	
37																					37	
36																					36	
35																					35	
-34	+3.055	19	+3.074	59	+3.133																-34	
33	2.883	18	2.901	53	2.954	95	+3.049														33	
32	2.729	15	2.744	49	2.793	85	2.878	130	+3.008												32	
31	2.588	14	2.602	44	2.646	78	2.724	118	2.842	170	+3.012										31	
-30	+2.459	14	+2.473	40	+2.513	70	+2.583	108	+2.691	154	+2.845	219	+3.064								-30	
29	2.341	13	2.354	37	2.391	64	2.455	99	2.554	140	2.694	198	2.892	282	+3.174						29	
28	2.233	11	2.244	34	2.278	60	2.338	90	2.428	128	2.556	181	2.737	254	2.991						28	
27	2.132	10	2.142	32	2.174	55	2.229	83	2.312	118	2.430	165	2.595	231	2.826	333	+3.159				27	
26	2.038	10	2.048	30	2.078	51	2.129	77	2.206	109	2.315	151	2.466	211	2.677	300	2.977				26	
25	1.952	9	1.961	27	1.988	48	2.036	71	2.107	101	2.208	139	2.347	194	2.541	273	2.814				25	
24	1.871	8	1.879	26	1.905	44	1.949	67	2.016	93	2.109	129	2.238	178	2.416	249	2.665	363	+3.028		24	
23	1.795	8	1.803	23	1.826	42	1.868	62	1.930	87	2.017	120	2.137	165	2.302	228	2.530	329	2.859		23	
22	1.723	8	1.731	22	1.753	39	1.792	58	1.850	82	1.932	111	2.043	153	2.196	210	2.406	301	2.707		22	
21	1.656	7	1.663	21	1.684	37	1.721	55	1.776	76	1.852	104	1.956	142	2.098	194	2.292	276	2.568		21	
-20	+1.592	7	+1.599	20	+1.619	35	+1.654	52	+1.706	71	+1.777	98	+1.875	132	+2.007	180	+2.187	254	+2.441		-20	
19	1.532	7	1.539	19	1.558	32	1.590	49	1.639	68	1.707	91	1.798	124	1.922	168	2.090	234	2.324		19	
18	1.476	6	1.481	18	1.499	31	1.530	46	1.576	64	1.640	87	1.727	116	1.843	157	2.000	216	2.216		18	
17	1.421	6	1.427	17	1.444	29	1.473	44	1.517	60	1.577	83	1.660	109	1.769	147	1.916	201	2.117		17	
16	1.370	5	1.375	17	1.392	27	1.419	42	1.461	57	1.518	78	1.596	103	1.699	138	1.837	188	2.025		16	
15	1.321	5	1.326	16	1.342	26	1.368	40	1.408	54	1.462	73	1.535	98	1.633	130	1.763	176	1.939		15	
14	1.274	5	1.279	15	1.294	25	1.319	38	1.357	52	1.409	69	1.478	93	1.571	122	1.693	166	1.859		14	
13	1.229	5	1.234	14	1.248	24	1.272	37	1.309	49	1.358	66	1.424	88	1.512	116	1.628	156	1.784		13	
12	1.186	5	1.191	13	1.204	24	1.228	34	1.262	48	1.310	63	1.373	83	1.456	110	1.566	147	1.713		12	
11	1.145	5	1.150	13	1.163	22	1.185	33	1.218	45	1.263	60	1.323	80	1.403	104	1.507	139	1.646		11	
-10	+1.106	4	+1.110	12	+1.122	22	+1.144	31	+1.175	43	+1.218	58	+1.276	76	+1.352	99	+1.451	132	+1.583		-10	
9	1.068	4	1.072	12	1.084	20	1.104	31	1.135	41	1.176	56	1.232	72	1.304	94	1.398	126	1.524		9	
8	1.031	4	1.035	11	1.046	20	1.066	30	1.096	40	1.136	53	1.189	69	1.258	90	1.348	119	1.467		8	
7	0.996	3	0.999	11	1.010	20	1.030	28	1.058	38	1.096	51	1.147	67	1.214	86	1.300	114	1.414		7	
6	0.961	4	0.965	11	0.976	18	0.994	28	1.022	36	1.058	50	1.108	64	1.172	82	1.254	109	1.363		6	
5	0.928	4	0.932	10	0.942	18	0.960	27	0.987	35	1.022	48	1.070	61	1.131	79	1.210	104	1.314		5	
4	0.896	4	0.900	10	0.910	17	0.927	26	0.953	34	0.987	46	1.033	59	1.092	76	1.168	99	1.267		4	
3	0.865	4	0.869	9	0.878	17	0.895	25	0.920	33	0.953	44	0.997	57	1.054	74	1.128	95	1.223		3	
2	0.835	4	0.839	9	0.848	16	0.864	24	0.888	33	0.921	42	0.963	55	1.018	71	1.089	91	1.180		2	
1	0.806	3	0.809	9	0.818	16	0.834	24	0.858	31	0.889	41	0.930	53	0.983	68	1.051	88	1.139		1	
0	+0.778	3	+0.781	9	+0.790	15	+0.805	23	+0.828	30	+0.858	40	+0.898	51	+0.949	66	+1.015	85	+1.100		0	
δ	$s-\alpha$	0°	$d.$	5°	$d.$	10°	$d.$	15°	$d.$	20°	$d.$	25°	$d.$	30°	$d.$	35°	$d.$	40°	$d.$	45°	$s-\alpha$	δ

$$Y = \tan (\Delta - \delta).$$

TABLE V.

Y

δ	$s-\alpha$	0°	$d.$	5°	$d.$	10°	$d.$	15°	$d.$	20°	$d.$	25°	$d.$	30°	$d.$	35°	$d.$	40°	$d.$	45°	$s-\alpha$	δ
0°		+0.778	3	+0.781	9	+0.790	15	+0.805	23	+0.828	30	+0.858	40	+0.898	51	+0.949	66	+1.015	85	+1.100		0°
1		0.750	3	0.753	9	0.762	15	0.777	22	0.799	29	0.828	39	0.867	50	0.917	63	0.980	82	1.062		1
2		0.723	3	0.726	9	0.735	14	0.749	22	0.771	28	0.799	38	0.837	48	0.885	62	0.947	78	1.025		2
3		0.697	3	0.700	8	0.708	14	0.722	21	0.743	28	0.771	37	0.808	47	0.855	59	0.914	76	0.990		3
4		0.671	3	0.674	8	0.682	14	0.696	20	0.716	28	0.744	35	0.779	46	0.825	58	0.883	73	0.956		4
5		0.646	3	0.649	8	0.657	13	0.670	20	0.690	27	0.717	35	0.752	44	0.796	56	0.852	72	0.924		5
6		0.622	2	0.624	8	0.632	13	0.645	20	0.665	26	0.691	34	0.725	43	0.768	54	0.822	70	0.892		6
7		0.598	2	0.600	8	0.608	13	0.621	19	0.640	25	0.665	33	0.698	42	0.740	54	0.794	67	0.861		7
8		0.574	3	0.577	7	0.584	13	0.597	18	0.615	25	0.640	33	0.673	41	0.714	52	0.766	65	0.831		8
9		0.551	3	0.554	7	0.561	13	0.574	18	0.592	24	0.616	32	0.648	40	0.688	50	0.738	64	0.802		9
+ 10		+0.529	2	+0.531	7	+0.538	13	+0.551	17	+0.568	24	+0.592	31	+0.623	39	+0.662	50	+0.712	62	+0.774		+ 10
11		0.507	2	0.509	7	0.516	12	0.528	17	0.545	24	0.569	30	0.599	38	0.637	49	0.686	60	0.746		11
12		0.485	2	0.487	7	0.494	12	0.506	17	0.523	23	0.546	30	0.576	37	0.613	47	0.660	59	0.719		12
13		0.464	2	0.466	7	0.473	11	0.484	17	0.501	23	0.524	29	0.553	36	0.589	46	0.635	58	0.693		13
14		0.442	3	0.445	7	0.452	11	0.463	16	0.479	23	0.502	28	0.530	36	0.566	45	0.611	56	0.667		14
15		0.422	2	0.424	7	0.431	11	0.442	16	0.458	22	0.480	28	0.508	35	0.543	45	0.588	55	0.643		15
16		0.401	3	0.404	6	0.410	11	0.421	16	0.437	21	0.458	28	0.486	35	0.521	43	0.564	54	0.618		16
17		0.381	2	0.383	7	0.390	11	0.401	16	0.417	21	0.438	27	0.465	34	0.499	43	0.542	52	0.594		17
18		0.361	3	0.364	6	0.370	11	0.381	15	0.396	21	0.417	27	0.444	33	0.477	42	0.519	52	0.571		18
19		0.342	2	0.344	6	0.350	11	0.361	15	0.376	21	0.397	26	0.423	33	0.456	41	0.497	51	0.548		19
+ 20		+0.322	3	+0.325	6	+0.331	10	+0.341	15	+0.356	20	+0.376	26	+0.402	33	+0.435	41	+0.476	50	+0.526		+ 20
21		0.303	2	0.305	6	0.311	11	0.322	15	0.337	20	0.357	25	0.382	33	0.415	39	0.454	49	0.503		21
22		0.284	2	0.286	6	0.292	11	0.303	14	0.317	20	0.337	25	0.362	32	0.394	39	0.433	49	0.482		22
23		0.266	2	0.268	6	0.274	10	0.284	14	0.298	20	0.318	25	0.343	31	0.374	39	0.413	47	0.460		23
24		0.247	2	0.249	6	0.255	10	0.265	14	0.279	20	0.299	24	0.323	31	0.354	39	0.393	46	0.439		24
25		0.229	2	0.231	5	0.236	10	0.246	15	0.261	19	0.280	24	0.304	31	0.335	38	0.373	46	0.419		25
26		0.210	2	0.212	6	0.218	10	0.228	14	0.242	19	0.261	24	0.285	31	0.316	37	0.353	45	0.398		26
27		0.192	2	0.194	6	0.200	10	0.210	14	0.224	18	0.242	25	0.267	29	0.296	37	0.333	45	0.378		27
28		0.174	2	0.176	6	0.182	9	0.191	15	0.206	18	0.224	24	0.248	30	0.278	36	0.314	44	0.358		28
29		0.156	2	0.158	6	0.164	9	0.173	14	0.187	20	0.207	22	0.229	30	0.259	36	0.295	44	0.339		29
+ 30		+0.138	2	+0.140	6	+0.146	9	+0.155	14	+0.169	20	+0.189	22	+0.211	29	+0.240	36	+0.276	44	+0.320		+ 30
31		0.120	2	0.122	6	0.128	10	0.138	13	0.151	20	0.171	22	0.193	29	0.222	35	0.257	43	0.300		31
32		0.103	2	0.105	5	0.110	10	0.120	14	0.134	19	0.153	22	0.175	29	0.204	35	0.239	42	0.281		32
33		0.085	2	0.087	6	0.093	9	0.102	14	0.116	18	0.134	23	0.157	29	0.186	34	0.220	43	0.263		33
34		0.068	2	0.070	5	0.075	10	0.085	13	0.098	18	0.116	23	0.139	29	0.168	34	0.202	42	0.244		34
35		0.050	2	0.052	5	0.058	9	0.067	14	0.081	18	0.099	22	0.121	29	0.150	34	0.184	42	0.226		35
36		0.033	2	0.035	5	0.040	10	0.050	13	0.063	18	0.081	23	0.104	28	0.132	34	0.166	41	0.207		36
37		0.015	2	0.017	6	0.023	9	0.032	14	0.046	18	0.064	22	0.086	28	0.114	34	0.148	41	0.189		37
38		-0.002	2	+0.000	5	+0.005	10	+0.015	13	0.028	18	0.046	23	0.069	28	0.097	33	0.130	41	0.171		38
39		-0.020	2	-0.018	6	-0.012	9	-0.003	14	+0.011	17	0.028	23	0.051	28	0.079	34	0.113	40	0.153		39
+ 40		-0.037	2	-0.035	5	-0.030	10	-0.020	13	-0.007	18	+0.011	23	+0.034	27	+0.061	34	+0.095	41	+0.136		+ 40
δ	$s-\alpha$	0°	$d.$	5°	$d.$	10°	$d.$	15°	$d.$	20°	$d.$	25°	$d.$	30°	$d.$	35°	$d.$	40°	$d.$	45°	$s-\alpha$	δ

TABLE V.

Y

δ	$s-\alpha$	0°	$d.$	5°	$d.$	10°	$d.$	15°	$d.$	20°	$d.$	25°	$d.$	30°	$d.$	35°	$d.$	40°	$d.$	45°	$s-\alpha$	δ
+40°	-0.037	2	-0.035	5	-0.030	10	-0.020	13	-0.007	18	+0.011	23	+0.034	27	+0.061	34	+0.095	41	+0.136	+40°		
41	0.055	2	0.053	6	0.047	9	-0.038	14	0.024	18	-0.006	22	+0.016	28	0.044	34	0.078	40	0.118	41		
42	0.072	2	0.070	5	0.065	10	0.055	13	0.042	18	0.024	23	-0.001	27	0.026	34	0.060	40	0.100	42		
43	0.090	2	0.088	6	0.082	9	0.073	14	0.059	18	0.041	22	0.019	28	+0.009	33	0.042	41	0.083	43		
44	0.107	2	0.105	5	0.100	10	0.090	13	0.077	18	0.059	23	0.036	28	-0.008	33	0.025	40	0.065	44		
45	0.125	2	0.123	5	0.118	10	0.108	14	0.094	18	0.076	22	0.054	28	0.026	34	+0.008	40	0.048	45		
46	0.143	2	0.141	6	0.135	9	0.126	14	0.112	18	0.094	23	0.071	28	0.043	33	-0.010	40	0.030	46		
47	0.161	2	0.159	6	0.153	10	0.143	13	0.130	18	0.112	23	0.089	28	0.061	34	0.027	40	+0.013	47		
48	0.179	2	0.177	6	0.171	10	0.161	14	0.147	18	0.129	23	0.106	28	0.078	33	0.045	40	-0.005	48		
49	0.197	2	0.195	6	0.189	10	0.179	14	0.165	18	0.147	23	0.124	28	0.096	34	0.062	40	0.022	49		
+50	-0.215	2	-0.213	6	-0.207	10	-0.197	14	-0.183	18	-0.165	23	-0.142	28	-0.114	34	-0.080	40	-0.040	+50		
51	0.233	2	0.231	6	0.225	9	0.216	15	0.201	18	0.183	23	0.160	29	0.131	33	0.098	41	0.057	51		
52	0.252	2	0.250	6	0.244	10	0.234	14	0.220	19	0.201	23	0.178	29	0.149	34	0.115	40	0.075	52		
53	0.270	2	0.268	6	0.262	10	0.252	14	0.238	19	0.219	23	0.196	29	0.167	34	0.133	41	0.092	53		
54	0.289	2	0.287	6	0.281	10	0.271	14	0.257	19	0.238	24	0.214	29	0.185	34	0.151	41	0.110	54		
55	-0.308	2	0.306	6	0.300	10	0.290	15	0.275	19	0.256	24	0.232	29	0.203	35	0.168	40	0.128	55		
56	0.327	2	0.325	6	0.319	10	0.309	15	0.294	19	0.275	24	0.251	30	0.221	35	0.186	41	0.145	56		
57	0.347	2	0.345	7	0.338	10	0.328	15	0.313	19	0.294	25	0.269	29	0.240	35	0.205	42	0.163	57		
58	0.366	2	0.364	6	0.358	10	0.348	16	0.332	19	0.313	25	0.288	30	0.258	35	0.223	42	0.181	58		
59	0.386	2	0.384	6	0.378	11	0.367	15	0.352	20	0.332	25	0.307	30	-0.277	36	0.241	42	0.199	59		
+60	-0.406	2	-0.404	6	-0.398	11	-0.387	15	-0.372	20	-0.352	26	-0.326	30	-0.296	36	-0.260	42	-0.218	+60		
61	0.427	2	0.425	7	0.418	11	0.407	15	0.392	21	0.371	25	0.346	31	0.315	36	0.279	43	0.236	61		
62	0.448	2	0.446	7	0.439	11	0.428	16	0.412	21	0.391	26	0.365	31	0.334	36	0.298	44	0.254	62		
63	0.469	2	0.467	7	0.460	11	0.449	17	0.432	20	0.412	27	0.385	31	0.354	37	0.317	44	0.273	63		
64	0.490	2	0.488	7	0.481	11	0.470	17	0.453	21	0.432	26	0.406	32	0.374	38	0.336	44	0.292	64		
65	0.512	2	0.510	7	0.503	12	0.491	16	0.475	22	0.453	27	0.426	32	0.394	39	0.355	44	0.311	65		
66	0.535	3	0.532	7	0.525	12	0.513	17	0.496	22	0.474	27	0.447	33	0.414	39	0.375	45	0.330	66		
67	0.557	2	0.555	7	0.548	13	0.535	17	0.518	22	0.496	28	0.468	33	0.435	40	0.395	45	0.350	67		
68	0.580	2	0.578	7	0.571	13	0.558	18	0.540	22	0.518	29	0.489	34	0.455	39	0.416	47	0.369	68		
69	0.604	3	0.601	7	0.594	13	0.581	18	0.563	23	0.540	29	0.511	34	0.477	41	0.436	47	0.389	69		
+70	-0.628	3	-0.625	7	-0.618	13	-0.605	19	-0.586	23	-0.563	30	-0.533	35	-0.498	41	-0.457	47	-0.410	+70		
71	0.652	2	0.650	8	0.642	13	0.629	19	0.610	24	0.586	30	0.556	36	0.520	42	0.478	48	0.430	71		
72	0.678	3	0.675	8	0.667	13	0.654	20	0.634	24	0.610	31	0.579	36	0.543	43	0.500	49	0.451	72		
73	0.704	3	0.701	9	0.692	13	0.679	20	0.659	25	0.634	31	0.603	38	0.565	43	0.522	50	0.472	73		
74	0.730	3	0.727	8	0.719	14	0.705	21	0.684	26	0.658	31	0.627	38	0.589	45	0.544	50	0.494	74		
75	0.757	3	0.754	9	0.745	14	0.731	21	0.710	26	0.684	33	0.651	39	0.612	45	0.567	51	0.516	75		
76	0.785	3	0.782	9	0.773	15	0.758	21	0.737	27	0.710	34	0.676	39	0.637	46	0.591	53	0.538	76		
77	0.813	3	0.810	9	0.801	15	0.786	22	0.764	28	-0.736	34	0.702	40	0.662	48	0.614	53	0.561	77		
78	0.843	3	0.840	10	0.830	16	0.814	22	0.792	28	0.764	35	0.729	42	0.687	48	0.639	55	0.584	78		
79	0.873	3	0.870	10	0.860	16	0.844	23	0.821	29	0.792	36	0.756	43	0.713	49	0.664	57	0.607	79		
+80	-0.904	3	-0.901	10	-0.891	17	-0.874	23	-0.851	31	-0.820	37	-0.783	43	-0.740	51	-0.689	57	-0.632	+80		
81	0.937	4	0.933	10	0.923	17	0.906	25	0.881	31	0.850	38	0.812	45	0.767	52	0.715	59	0.656	81		
82	0.970	4	0.966	10	0.956	18	0.938	25	0.913	32	0.881	40	0.841	46	0.795	53	0.742	61	0.681	82		
83	1.004	3	1.001	11	0.990	19	0.971	26	0.945	33	0.912	40	0.872	48	0.824	55	0.769	62	0.707	83		
84	1.040	4	1.036	11	1.025	19	1.006	27	0.979	34	0.945	42	0.903	49	0.854	57	0.797	63	0.734	84		
85	1.077	4	1.073	12	1.061	20	1.041	27	1.014	36	0.978	43	0.935	51	0.884	58	0.826	65	0.761	85		
86	1.116	5	1.111	12	1.099	21	1.078	28	1.050	37	1.013	45	0.968	52	0.916	60	0.856	67	0.789	86		
87	1.156	5	1.151	13	1.138	21	1.117	30	1.087	38	1.049	46	1.003	55	0.948	61	0.887	69	0.818	87		
88	1.197	4	1.193	14	1.179	22	1.157	31	1.126	40	1.086	48	1.038	56	0.982	64	0.918	71	0.847	88		
89	1.241	5	1.236	14	1.222	23	1.199	33	1.166	41	1.125	50	1.075	58	1.017	66	0.951	73	0.878	89		
+90	-1.286	5	-1.281	15	-1.266	24	-1.242	34	-1.208	43	-1.165	51	-1.114	61	-1.053	68	-0.985	76	-0.909	+90		
δ	$s-\alpha$	0°	$d.$	5°	$d.$	10°	$d.$	15°	$d.$	20°	$d.$	25°	$d.$	30°	$d.$	35°	$d.$	40°	$d.$	45°	$s-\alpha$	δ

TABLE VI.

Δ

Δ is independent of the sign of $(s - \alpha)$.

$s - \alpha$	Δ	$s - \alpha$	Δ	$s - \alpha$	Δ	$s - \alpha$	Δ
0°	37 52	45	47 43	90°	90° 0'	135°	132° 17'
1	37 53	46	48 14	91	91 17	136	132 46
2	37 53			92	92 34		
3	37 55	47	48 45	93	93 51	137	133 14
		48	49 17			138	133 42
4	37 57	49	49 51	94	95 8	139	134 9
5	37 59	50	50 26	95	96 24	140	134 34
6	38	51	51 51	96	97 39	141	134 59
		52	51 38			142	135 23
7	38 4	53	52 16	97	98 54	143	135 46
8	38 8			98	100 9		
9	38 13	54	52 55	99	101 22	144	136 8
10	38 18	55	53 35	100	102 35	145	136 29
11	38 23	56	54 17	101	103 47	146	136 50
12	38 29			102	104 58		
13	38 36	57	55 0	103	106 8	147	137 10
		58	55 44			148	137 29
14	38 43	59	56 29	104	107 17	149	137 47
15	38 50	60	57 16	105	108 24	150	138 4
16	38 58	61	58 4	106	109 31	151	138 21
		62	58 53			152	138 37
17	39 7	63	59 44	107	110 36	153	138 53
18	39 16			108	111 40		
19	39 26	64	60 36	109	112 43	154	139 8
20	39 36	65	61 29	110	113 44	155	139 22
21	39 47	66	62 23	111	114 44	156	139 35
22	39 59			112	115 43		
23	40 12	67	63 20	113	116 40	157	139 48
		68	64 17			158	140 1
24	40 25	69	65 16	114	117 37	159	140 13
25	40 38	70	66 16	115	118 31	160	140 24
26	40 52	71	67 17	116	119 24	161	140 34
		72	68 20			162	140 44
27	41 7	73	69 24	117	120 16	163	140 53
28	41 23			118	121 7		
29	41 39	74	70 29	119	121 56	164	141 2
30	41 56	75	71 36	120	122 44	165	141 10
31	42 13	76	72 43	121	123 31	166	141 17
32	42 31			122	124 16		
33	42 50	77	73 52	123	125 0	167	141 24
		78	75 2			168	141 31
34	43 10	79	76 13	124	125 43	169	141 37
35	43 31	80	77 25	125	126 25	170	141 42
36	43 52	81	78 38	126	127 5	171	141 47
		82	79 51			172	141 52
37	44 14	83	81 6	127	127 44	173	141 56
38	44 37			128	128 22		
39	45 1	84	82 21	129	128 59	174	141 59
40	45 26	85	83 36	130	129 34	175	142 1
41	45 51	86	84 52	131	130 9	176	142 3
42	46 18			132	130 43		
43	46 46	87	86 9	133	131 15	177	142 5
		88	87 26			178	142 7
44	47 14	89	88 43	134	131 46	179	142 7
45	47 43	90	90 0	135	132 17	180	142 8

$$\tan \Delta = \tan \varphi \sec (s - \alpha).$$

TABLE VII.

$X \cos (\Delta - \delta)$

$X \cos (\Delta - \delta)$ has sign of $(s - \alpha)$. It is therefore positive when center of plate is west of meridian.

$s - \alpha$	$\log X \cos (\Delta - \delta)$	$X \cos (\Delta - \delta)$	$s - \alpha$	$s - \alpha$	$\log X \cos (\Delta - \delta)$	$X \cos (\Delta - \delta)$	$s - \alpha$
0°	-∞	0.000	180°	45°	9.8278	0.673	135°
1	8.139	14	179	46	9.8387	690	134
2	8.440	28	178				
3	8.616	41	177	47	9.8494	707	133
				48	9.8599	725	132
4	8.741	55	176	49	9.8702	742	131
5	8.839	0.069	175	50	9.8804	0.759	130
6	8.918	83	174	51	9.8903	777	129
				52	9.9001	795	128
7	8.985	97	173	53	9.9096	812	127
8	9.0435	110	172				
9	9.0949	124	171	54	9.9190	830	126
10	9.1411	0.138	170	55	9.9282	0.848	125
11	9.1830	152	169	56	9.9372	866	124
12	9.2211	166	168				
13	9.2563	180	167	57	9.9461	884	123
				58	9.9547	901	122
14	9.2890	195	166	59	9.9633	919	121
15	9.3195	0.209	165	60	9.9716	0.937	120
16	9.3482	223	164	61	9.9796	.954	119
				62	9.9875	.972	118
17	9.3751	237	163	63	9.9952	.989	117
18	9.4006	252	162				
19	9.4248	266	161	64	0.0028	1.007	116
20	9.4478	0.280	160	65	0.0103	1.024	115
21	9.4698	294	159	66	0.0175	1.041	114
22	9.4908	309	158				
23	9.5109	324	157	67	0.0243	1.058	113
				68	0.0310	1.074	112
24	9.5302	339	156	69	0.0375	1.090	111
25	9.5488	0.354	155	70	0.0438	1.106	110
26	9.5668	369	154	71	0.0498	1.122	109
				72	0.0555	1.136	108
27	9.5842	384	153	73	0.0610	1.151	107
28	9.6009	399	152				
29	9.6172	414	151	74	0.0663	1.165	106
30	9.6330	0.430	150	75	0.0713	1.178	105
31	9.6484	446	149	76	0.0760	1.192	104
32	9.6633	461	148				
33	9.6778	477	147	77	0.0804	1.203	103
				78	0.0845	1.215	102
34	9.6919	492	146	79	0.0883	1.225	101
35	9.7057	0.508	145	80	0.0920	1.236	100
36	9.7192	524	144	81	0.0952	1.245	99
				82	0.0981	1.254	98
37	9.7323	540	143	83	0.1007	1.260	97
38	9.7452	556	142				
39	9.7578	572	141	84	0.1029	1.266	96
40	9.7700	0.589	140	85	0.1048	1.273	95
41	9.7821	606	139	86	0.1064	1.278	94
42	9.7938	623	138				
43	9.8054	640	137	87	0.1076	1.281	93
				88	0.1085	1.284	92
44	9.8167	656	136	89	0.1091	1.286	91
45	9.8278	0.673	135	90	0.1093	1.286	90

$$X \cos (\Delta - \delta) = \tan (s - \alpha) \cos \Delta.$$

TABLE VIII.

FOR CONVERSION OF DEGREES
INTO RADIANS.

$k \begin{Bmatrix} x \\ y \end{Bmatrix}$	$k \begin{Bmatrix} x \\ y \end{Bmatrix}$	
	57.296	
0.0	R	
	0.00000	174
.1	174	175
.2	349	175
.3	524	174
.4	698	175
.5	873	174
.6	1047	175
.7	1222	174
.8	1396	175
.9	1571	174
1.0	0.01745	175
1.1	1920	174
1.2	2094	175
1.3	2269	174
1.4	2443	175
1.5	2618	175
1.6	2793	174
1.7	2967	175
1.8	3142	174
1.9	3316	175
2.0	0.03491	174
2.1	3665	175
2.2	3840	174
2.3	4014	175
2.4	4189	174
2.5	4363	175
2.6	4538	174
2.7	4712	175
2.8	4887	174
2.9	5061	175
3.0	0.05236	175
3.1	5411	174
3.2	5585	175
3.3	5760	174
3.4	5934	175
3.5	6109	174
3.6	6283	175
3.7	6458	174
3.8	6632	175
3.9	6807	174
4.0	0.06981	175
4.1	7156	174
4.2	7330	175
4.3	7505	174
4.4	7679	175
4.5	7854	175
4.6	8029	174
4.7	8203	174
4.8	8377	175
4.9	8552	175
5.0	0.08727	175
$k \begin{Bmatrix} x \\ y \end{Bmatrix}$	$k \begin{Bmatrix} x \\ y \end{Bmatrix}$	
	57.296	

TABLE IX.

Giving $x^2 + y^2$ with arguments x and y .

The quantities are expressed in radians. The tabulated values are given in units of the fourth decimal place.

x	y	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10
0.00	0	1	4	9	16	25	36	49	64	81	100	
1	1	2	5	10	17	26	37	50	65	82	101	
2	4	5	8	13	20	29	40	53	68	85	104	
3	9	10	13	18	25	34	45	58	73	90	109	
4	16	17	20	25	32	41	52	65	80	97	116	
0.05	25	26	29	34	41	50	61	74	89	106	125	
6	36	37	40	45	52	61	72	85	100	117	136	
7	49	50	53	58	65	74	85	98	113	130	149	
8	64	65	68	73	80	89	100	113	128	145	164	
9	81	82	85	90	97	106	117	130	145	162	181	
0.10	100	101	104	109	116	125	136	149	164	181	200	

TABLE X.

D

$xX + yY$	D	$xX + yY$	D	$xX + yY$	D	$xX + yY$	D
- .00	+ .0000	- 25	+ .3333	+ .00	- .0000	+ 25	- .2000
1	.0101	26	.3514	1	.0099	26	.2064
2	.0204	27	.3699	2	.0196	27	.2126
3	.0309	28	.3889	3	.0291	28	.2188
4	.0416	29	.4084	4	.0385	29	.2248
5	.0526	- .30	+ .4286	5	.0476	+ .30	- .2308
6	.0638	31	.4493	6	.0566	31	.2366
7	.0753	32	.4706	7	.0654	32	.2424
8	.0870	33	.4925	8	.0741	33	.2481
9	.0989	34	.5152	9	.0826	34	.2537
- .10	+ .1111	35	.5385	+ .10	- .0909	35	.2593
11	.1236	36	.5625	11	.0991	36	.2647
12	.1364	37	.5873	12	.1071	37	.2701
13	.1494	38	.6129	13	.1150	38	.2754
14	.1628	39	.6393	14	.1228	39	.2806
15	.1765	- .40	+ .6667	15	.1304	+ .40	- .2857
16	.1905	41	.6949	16	.1379	41	.2908
17	.2048	42	.7241	17	.1453	42	.2958
18	.2195	43	.7544	18	.1525	43	.3007
19	.2346	44	.7857	19	.1597	44	.3056
- .20	+ .2500	45	.8182	+ .20	- .1667	45	.3103
21	.2658	46	.8519	21	.1736	46	.3151
22	.2820	47	.8868	22	.1803	47	.3197
23	.2987	48	.9231	23	.1870	48	.3243
24	.3158	49	.9608	24	.1936	49	.3289
25	.3333	- .50	+ 1.0000	25	.2000	+ .50	- .3333
$xX + yY$	D	$xX + yY$	D	$xX + yY$	D	$xX + yY$	D

$$D = \frac{1}{1 + xX + yY} - 1$$

TABLE XI.

FOR ZENITH DISTANCE OF CENTER OF PLATE.

$$Z = \tan^{-1} \sqrt{X^2 + Y^2}$$

X and Y are expressed in units of radius, Z in degrees.

$X \backslash Y$	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0
R													
0.0	0°	27°	45°	56°	63.4	68.2	71.6	74.1	75.97	77.47	78.69	79.70	80.54
0.5	27.	35.	48.	58.	64.1	68.6	71.8	74.2	76.07	77.55	78.75	79.74	80.57
1.0	45.	48.	55.	61.0	65.9	69.6	72.4	74.6	76.37	77.76	78.90	79.86	80.67
1.5	56.	58.	61.0	64.8	68.2	71.1	73.4	75.29	76.82	78.10	79.16	80.05	
2.0	63.4	64.1	65.9	68.2	70.5	72.7	74.5	76.07	77.40	78.52	79.48	80.30	
2.5	68.2	68.6	69.6	71.1	72.7	74.2	75.63	76.91	78.03	79.00	79.86		
3.0	71.6	71.8	72.4	73.4	74.5	75.63	76.74	77.76	78.69	79.53	80.27		
3.5	74.1	74.2	74.6	75.29	76.07	76.91	77.76	78.58	79.34	80.05			
4.0	75.97	76.07	76.37	76.82	77.40	78.03	78.69	79.34	79.97				
4.5	77.47	77.55	77.76	78.10	78.52	79.00	79.53	80.05	80.57				
5.0	78.69	78.75	78.90	79.16	79.48	79.86	80.27						
5.5	79.70	79.74	79.86	80.05	80.30	80.60							
6.0	80.54	80.57	80.67										

Table for Transformation of Decimals of a Degree to Minutes and Seconds of Arc or Time, and the Converse.

°	'	"	m	s
.1	6	0.	0	24.
.2	12	0.		48.
.3	18	0.	1	12.
.4	24	0.	1	36.
.5	30	0.	2	0.
.6	36	0.	2	24.
.7	42	0.	2	48.
.8	48	0.	3	12.
.9	54	0.	3	36.
.01	0	36.		2.4
.02	1	12.		4.8
.03	1	48.		7.2
.04	2	24.		9.6
.05	3	0.		12.0
.06	3	36.		14.4
.07	4	12.		16.8
.08	4	48.		19.2
.09	5	24.		21.6
.001		3.6		
.002		7.2		
.003		10.8		
.004		14.4		
.005		18.0		
.006		21.6		
.007		25.2		
.008		28.8		
.009		32.4		

10	.1666667
20	.3333333
30	.5000000
40	.6666667
50	.8333333
1	.0166667
2	.0333333
3	.0500000
4	.0666667
5	.0833333
6	.1000000
7	.1166667
8	.1333333
9	.1500000
10	.0027778
20	.0055556
30	.0083333
40	.0111111
50	.0138889
1	.0002778
2	.0005556
3	.0008333
4	.0011111
5	.0013889
6	.0016667
7	.0019444
8	.0022222
9	.0025000

m	°
1	.2500000
2	.5000000
3	.7500000
s	
10	.0416667
20	.0833333
30	.1250000
40	.1666667
50	.2083333
s	
1	.0041667
2	.0083333
3	.0125000
4	.0166667
5	.0208333
6	.0250000
7	.0291667
8	.0333333
9	.0375000

INVESTIGATION OF THE REPSOLD MEASURING APPARATUS.

BY BURT L. NEWKIRK,
BERKELEY ASTRONOMICAL DEPARTMENT.



PROPORTIONAL PARTS.

7	8	9	11	12	13	14	15	16	17	18	19
1.7	1.8	1.9	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.4	1.6	1.8	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8
2.1	2.4	2.7	3.3	3.6	3.9	4.2	4.5	4.8	5.1	5.4	5.7
2.8	3.2	3.6	4.4	4.8	5.2	5.6	6.0	6.4	6.8	7.2	7.6
3.5	4.0	4.5	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5
4.2	4.8	5.4	6.6	7.2	7.8	8.4	9.0	9.6	10.2	10.8	11.4
4.9	5.6	6.3	7.7	8.4	9.1	9.8	10.5	11.2	11.9	12.6	13.3
5.6	6.4	7.2	8.8	9.6	10.4	11.2	12.0	12.8	13.6	14.4	15.2
6.3	7.2	8.1	9.9	10.8	11.7	12.6	13.5	14.4	15.3	16.2	17.1
27	28	29	31	32	33	34	35	36	37	38	39
2.7	2.8	2.9	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
5.4	5.6	5.8	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8
8.1	8.4	8.7	9.3	9.6	9.9	10.2	10.5	10.8	11.1	11.4	11.7
10.8	11.2	11.6	12.4	12.8	13.2	13.6	14.0	14.4	14.8	15.2	15.6
13.5	14.0	14.5	15.5	16.0	16.5	17.0	17.5	18.0	18.5	19.0	19.5
16.2	16.8	17.4	18.6	19.2	19.8	20.4	21.0	21.6	22.2	22.8	23.4
19.9	19.6	20.3	21.7	22.4	23.1	23.8	24.5	25.2	25.9	26.6	27.3
21.6	22.4	23.2	24.8	25.6	26.4	27.2	28.0	28.8	29.6	30.4	31.2
24.3	25.2	26.1	27.9	28.8	29.7	30.6	31.5	32.4	33.3	34.2	35.1
47	48	49	51	52	53	54	55	56	57	58	59
4.7	4.8	4.9	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9
9.4	9.6	9.8	10.2	10.4	10.6	10.8	11.0	11.2	11.4	11.6	11.8
14.1	14.4	14.7	15.3	15.6	15.9	16.2	16.5	16.8	17.1	17.4	17.7
18.8	19.2	19.6	20.4	20.8	21.2	21.6	22.0	22.4	22.8	23.2	23.6
23.5	24.0	24.5	25.5	26.0	26.5	27.0	27.5	28.0	28.5	29.0	29.5
28.2	28.8	29.4	30.6	31.2	31.8	32.4	33.0	33.6	34.2	34.8	35.4
32.9	33.6	34.3	35.7	36.4	37.1	37.8	38.5	39.2	39.9	40.6	41.3
37.6	38.4	39.2	40.8	41.6	42.4	43.2	44.0	44.8	45.6	46.4	47.2
42.3	43.2	44.1	45.9	46.8	47.7	48.6	49.5	50.4	51.3	52.2	53.1
67	68	69	71	72	73	74	75	76	77	78	79
6.7	6.8	6.9	7.1	7.2	7.3	7.4	7.5	7.6	7.7	7.8	7.9
13.4	13.6	13.8	14.2	14.4	14.6	14.8	15.0	15.2	15.4	15.6	15.8
20.1	20.4	20.7	21.3	21.6	21.9	22.2	22.5	22.8	23.1	23.4	23.7
26.8	27.2	27.6	28.4	28.8	29.2	29.6	30.0	30.4	30.8	31.2	31.6
32.5	34.0	34.5	35.5	36.0	36.5	37.0	37.5	38.0	38.5	39.0	39.5
38.2	40.8	41.4	42.6	43.2	43.8	44.4	45.0	45.6	46.2	46.8	47.4
43.9	47.6	48.3	49.7	50.4	51.1	51.8	52.5	53.2	53.9	54.6	55.3
49.6	54.4	55.2	56.8	57.6	58.4	59.2	60.0	60.8	61.6	62.4	63.2
55.3	61.2	62.1	63.9	64.8	65.7	66.6	67.5	68.4	69.3	70.2	71.1
87	88	89	91	92	93	94	95	96	97	98	99
8.7	8.8	8.9	9.1	9.2	9.3	9.4	9.5	9.6	9.7	9.8	9.9
17.4	17.6	17.8	18.2	18.4	18.6	18.8	19.0	19.2	19.4	19.6	19.8
26.1	26.4	26.7	27.3	27.6	27.9	28.2	28.5	28.8	29.1	29.4	29.7
34.8	35.2	35.6	36.4	36.8	37.2	37.6	38.0	38.4	38.8	39.2	39.6
43.5	44.0	44.5	45.5	46.0	46.5	47.0	47.5	48.0	48.5	49.0	49.5
52.2	52.8	53.4	54.6	55.2	55.8	56.4	57.0	57.6	58.2	58.8	59.4
60.9	61.6	62.3	63.7	64.4	65.1	65.8	66.5	67.2	67.9	68.6	69.3
69.6	70.4	71.2	72.8	73.6	74.4	75.2	76.0	76.8	77.6	78.4	79.2
78.3	79.2	80.1	81.9	82.8	83.7	84.6	85.5	86.4	87.3	88.2	89.1

S.

of
8ff
is

te;

on,

is

si-

ice

he

on

nd

he

on

he

he

ry

to

of

w.

on

er-

ght



PROPORTIONAL PARTS.

7	8	9	11	12	13	14	15	16	17	18	19
.7	.8	.9	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.4	1.6	1.8	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8
2.1	2.4	2.7	3.3	3.6	3.9	4.2	4.5	4.8	5.1	5.4	5.7
2.8	3.2	3.6	4.4	4.8	5.2	5.6	6.0	6.4	6.8	7.2	7.6
3.5	4.0	4.5	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5
4.2	4.8	5.4	6.6	7.2	7.8	8.4	9.0	9.6	10.2	10.8	11.4
4.9	5.6	6.3	7.7	8.4	9.1	9.8	10.5	11.2	11.9	12.6	13.3
5.6	6.4	7.2	8.8	9.6	10.4	11.2	12.0	12.8	13.6	14.4	15.2
6.3	7.2	8.1	9.9	10.8	11.7	12.6	13.5	14.4	15.3	16.2	17.1
27	28	29	31	32	33	34	35	36	37	38	39
2.7	2.8	2.9	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
5.4	5.6	5.8	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8
8.1	8.4	8.7	9.3	9.6	9.9	10.2	10.5	10.8	11.1	11.4	11.7
10.8	11.2	11.6	12.4	12.8	13.2	13.6	14.0	14.4	14.8	15.2	15.6
13.5	14.0	14.5	15.5	16.0	16.5	17.0	17.5	18.0	18.5	19.0	19.5
16.2	16.8	17.4	18.6	19.2	19.8	20.4	21.0	21.6	22.2	22.8	23.4
19.9	19.6	20.3	21.7	22.4	23.1	23.8	24.5	25.2	25.9	26.6	27.3
22.6	22.4	23.2	24.8	25.6	26.4	27.2	28.0	28.8	29.6	30.4	31.2
25.3	25.2	26.1	27.9	28.8	29.7	30.6	31.5	32.4	33.3	34.2	35.1
47	48	49	51	52	53	54	55	56	57	58	59
4.7	4.8	4.9	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9
9.4	9.6	9.8	10.2	10.4	10.6	10.8	11.0	11.2	11.4	11.6	11.8
14.1	14.4	14.7	15.3	15.6	15.9	16.2	16.5	16.8	17.1	17.4	17.7
18.8	19.2	19.6	20.4	20.8	21.2	21.6	22.0	22.4	22.8	23.2	23.6
23.5	24.0	24.5	25.5	26.0	26.5	27.0	27.5	28.0	28.5	29.0	29.5
28.2	28.8	29.4	30.6	31.2	31.8	32.4	33.0	33.6	34.2	34.8	35.4
39.9	33.6	34.3	35.7	36.4	37.1	37.8	38.5	39.2	39.9	40.6	41.3
47.6	38.4	39.2	40.8	41.6	42.4	43.2	44.0	44.8	45.6	46.4	47.2
55.3	43.2	44.1	45.9	46.8	47.7	48.6	49.5	50.4	51.3	52.2	53.1
67	68	69	71	72	73	74	75	76	77	78	79
6.7	6.8	6.9	7.1	7.2	7.3	7.4	7.5	7.6	7.7	7.8	7.9
13.4	13.6	13.8	14.2	14.4	14.6	14.8	15.0	15.2	15.4	15.6	15.8
20.1	20.4	20.7	21.3	21.6	21.9	22.2	22.5	22.8	23.1	23.4	23.7
26.8	27.2	27.6	28.4	28.8	29.2	29.6	30.0	30.4	30.8	31.2	31.6
33.5	34.0	34.5	35.5	36.0	36.5	37.0	37.5	38.0	38.5	39.0	39.5
40.2	40.8	41.4	42.6	43.2	43.8	44.4	45.0	45.6	46.2	46.8	47.4
46.9	47.6	48.3	49.7	50.4	51.1	51.8	52.5	53.2	53.9	54.6	55.3
53.6	54.4	55.2	56.8	57.6	58.4	59.2	60.0	60.8	61.6	62.4	63.2
60.3	61.2	62.1	63.9	64.8	65.7	66.6	67.5	68.4	69.3	70.2	71.1
87	88	89	91	92	93	94	95	96	97	98	99
8.7	8.8	8.9	9.1	9.2	9.3	9.4	9.5	9.6	9.7	9.8	9.9
17.4	17.6	17.8	18.2	18.4	18.6	18.8	19.0	19.2	19.4	19.6	19.8
26.1	26.4	26.7	27.3	27.6	27.9	28.2	28.5	28.8	29.1	29.4	29.7
34.8	35.2	35.6	36.4	36.8	37.2	37.6	38.0	38.4	38.8	39.2	39.6
43.5	44.0	44.5	45.5	46.0	46.5	47.0	47.5	48.0	48.5	49.0	49.5
52.2	52.8	53.4	54.6	55.2	55.8	56.4	57.0	57.6	58.2	58.8	59.4
60.9	61.6	62.3	63.7	64.4	65.1	65.8	66.5	67.2	67.9	68.6	69.3
69.6	70.4	71.2	72.8	73.6	74.4	75.2	76.0	76.8	77.6	78.4	79.2
78.3	79.2	80.1	81.9	82.8	83.7	84.6	85.5	86.4	87.3	88.2	89.1

S.

of
8ff
his

te;

on,

is

si-
ice

he
on

nd
he

on
he
he

ry
to
of

w.
on
er-
ght

INVESTIGATION OF THE REPSOLD MEASURING APPARATUS.

Doc. Indexed

By BURT L. NEWKIRK.

In the winter of 1903-4 the Students' Observatory received from the firm of Repsold Sons of Hamburg a measuring apparatus of the type described page 148ff SCHEINER'S "*Photographie der Gestirne*." The following investigations of this instrument have been made:

1. Division errors of the scales;
2. Irregularities of the micrometer screw used in measuring the X coördinate;
3. Straightness of the bar which guides the movable (X) microscope;
4. Straightness of the cylinder which guides the plate carriage in its motion, determining the Y axis;
5. Perpendicularity of these two directions of motion;
6. Parallelism of the plate with the two scales;
7. Distortion of field of objective of X microscope.

The following remarks will throw light upon the order of accuracy which it is desirable to attain in each of these investigations.

The apparatus is so constructed that it may be used to determine either position angle and distance or rectangular coördinates. Position angle and distance measures can be advantageously used in determining the parallax of any star, the line determined by the image of the star under investigation and the companion star being made as nearly as possible parallel to the direction of the X scale, and the distance directly measured, the position angle being read off roughly for the purpose of future identification, and the computation of refraction and aberration corrections. For this purpose it is desirable to know the corrections for the division lines of the X scale and the corrections due to irregularities of the micrometer screw to within a few ten-thousandths of a mm. It is also necessary to ascertain that the field of the object glass of the microscope is not subject to any distortion which would affect by so much as two or three ten-thousandths of a mm., measurements of distances up to 1 mm. made with the micrometer screw.

Measures of position angle might be carefully made so as (in combination with the distance measures) fully to determine the position of one star with reference to another, or with reference to the center of rotation of the plate, the right

ascension and declination of which would be determined in the reduction. The position angle measures would however not be as thoroughly independent of changes in relative position and length of various parts of the instrument, due to temperature variation or mechanical strains and inaccuracies of construction, as are the measures of distance, and would, therefore, be less trustworthy. Positions determined by measurements of angle and distance would be subject to the same degree of inaccuracy as the measures of position angle. The division errors of the circle, by means of which the position angles are measured, and the irregularities of the micrometer screws used in making the settings on the circle have not been investigated, the accuracy of their construction being presumed to be commensurate with the accuracy obtainable in the measures of position angle. Measures made in this way would be subject to errors of altogether inadmissible magnitude if the bar which carries the movable microscope were not made with particular care to insure its freedom from curvature. The straightness of this bar has therefore been tested and found to be free from curvature or irregularities large enough to introduce errors of a thousandth of a mm. in a position determined from measures of position angle and distance.

Rectangular coördinates can be measured either by setting on a star and reading off both coördinates on the X and Y scales respectively, or by measuring the X coördinate only, in the first position of the plate, and then rotating the plate through 90° so that the Y coördinate also may be determined by comparison with the X scale.

The latter method recommended by BAKHUYZEN (SCHEINER *Photographie der Gestirne*) is considered more reliable, the former being open to objections similar to those attached to the measurements of position angle. In the measurement of rectangular coördinates by the latter method, the limit of negligible error might be placed at about one thousandth of a mm. This necessitates in addition to the determination of scale corrections and corrections due to the irregularities of the micrometer, an investigation of the straightness of the cylinder which guides the plate carriage in its motion in the direction of the Y coördinate, and the effect of temperature variation upon the straightness of this cylinder.

The former of these two methods can be used when extreme accuracy is of secondary importance. For this purpose it is desirable to know the corrections for the division lines of both scales to within a few thousandths of a mm., to ascertain that the two scales are approximately parallel to the plate and perpendicular to each other and that the two directions of motion (of the movable microscope determining the X axis, and of the plate carriage determining the Y axis) be straight and approximately perpendicular. The investigation shows that errors introduced into the measures by maladjustment in the above-mentioned particulars will not exceed five thousandths of a mm.

DIVISION ERRORS OF THE TWO SCALES.

The two scales used in measuring X and Y coördinates are designated respectively as N. E. K. 10-1903 and N. E. K. 11-1903. I shall refer to them as "scale 10" and "scale 11." The division errors of these two scales were investigated and tabulated by the *Normal Aichungs Institute of Berlin*. Values were given to thousandths of a millimeter, accompanied by a note to the effect that they

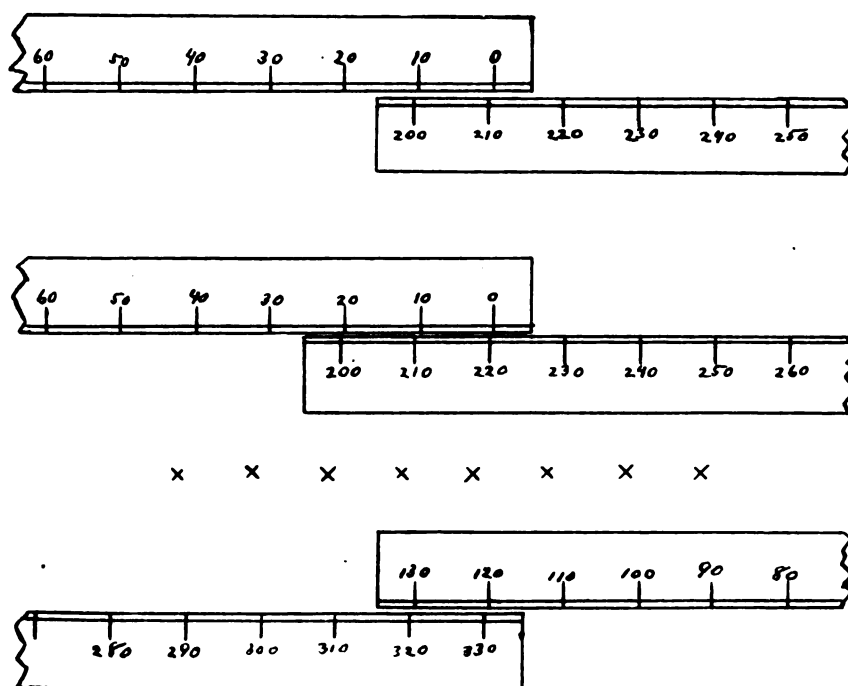


FIG. 1.

were probably reliable to within about .0015 mm. Inasmuch as *single settings* made with ordinary care upon a scale line differ from each other on the average by an amount considerably less than .0015 mm., it seemed to me that a more accurate determination of the division errors was essential if the instrument were to be used in work requiring the highest accuracy of which it is capable. I have therefore made an investigation of the division errors of some of the lines in the middle of each scale and tabulated the results to ten-thousandths of a mm. The tabulated values are probably reliable in the case of scale 10 to within three or four units of the last place. The tabulated corrections for lines near the center of scale 11 seem to be about five ten-thousandths of a mm. too small. It would not be difficult to remove this gradually accumulated error but it does not seem desirable to do so at the present time. See page 149.

The method employed was used by GILL in investigating the scales of the Cape heliometer and the scales of his Repsold measuring apparatus. The two scales are each approximately 130 mm. long and are made as nearly alike as possible except for the numbering of the division lines. The divisions of scale 10 are numbered from 1 to 130, and those of scale 11 from 200 to 330. The first step was to determine the correction for each tenth division line on both scales. By the correction corresponding to any division line, is meant that fraction of the unit of length which it is necessary to add to the number of the division line (less 200 in the case of scale 11) in order to obtain the distance of that line from the zero line of the scale, the unit of length being one one-hundred-and-thirtieth of the interval between the first and last lines of the scale. For example, the tabulated correction for line 263 of scale 11 is +.0196, and its distance from the zero line is therefore $\frac{63.0196}{130}$ of the distance between lines 200 and 330. These corrections for each tenth division line of both scales were deduced from a comparison of each 10 mm. interval of scale 10 with each 10 mm. interval of scale 11, the two scales being placed side by side, on the stage designed for the *X* scale, in the successive positions indicated in Figure 1.

Eight settings were made for the determination of the micrometer reading of a line. Let a_1 represent the micrometer reading of line 0 of scale 10 and b_1 the reading of line 210 of scale 11, the two readings having been taken without moving the micrometer carriage in the mean time. Then, $b_1 - a_1 = c_1$ represents the distance of the line 210 to the left, let us say, of the line 0. If now the microscope carriage be moved through an interval of approximately 10 mm., and a similar series of readings be taken on lines 10 and 200, designating the corresponding quantities by the subscript 2, the quantity $c_2 - c_1 = d$ represents the amount by which the interval 200-210 of scale 11 is larger than the interval 0-10 of scale 10. The comparison of each of the thirteen 10 mm. intervals of scale 10 with each 10 mm. interval of scale 11 yields 13^2 quantities such as d . Let these be designated by double subscripts, the first subscript referring to the interval of scale 10, the second to the interval of scale 11.

$d_{1\ 1}$	$d_{2\ 1}$	$d_{3\ 1}$	$d_{13\ 1}$
$d_{1\ 2}$	$d_{2\ 2}$	$d_{3\ 2}$	$d_{13\ 2}$
$d_{1\ 3}$	$d_{2\ 3}$	$d_{3\ 3}$	$d_{13\ 3}$
*	*	*	*
$d_{1\ 13}$	$d_{2\ 13}$	$d_{3\ 13}$	$d_{13\ 13}$

Now let $D'_1, D'_2, \dots, D'_{13}$ represent the means of the quantities in the columns. D'_1 is the amount by which the interval 0-10 of scale 10 is shorter than one-thirteenth of the total length of scale 11. $\Sigma D'_i$ represents the total amount by which scale 10 is shorter than scale 11. The amounts by which the successive ten mm. intervals of scale 10 are shorter than one one-thirteenth of its own total length are obtained by applying with the opposite sign $\frac{1}{13} \Sigma D'_i$ to each D' .

Let: $D_1 = D'_1 - \frac{1}{13} \Sigma D'_i$, $D_2 = D'_2 - \frac{1}{13} \Sigma D'_i$, etc.

The correction for the n th line of scale 10 as defined above is:

Correction to n th line = $-(D_1 + D_2 + D_3 \dots + D_n)$. The correction to the thirteenth line is $-\Sigma D_i = 0$.

To obtain the corrections for the lines at 10 mm. intervals of scale 11, represent the means of the quantities in the rows by \mathcal{A}'_i .

Let $\mathcal{A}_1 = \mathcal{A}'_1 - \frac{1}{13} \Sigma \mathcal{A}'_i$, etc.

The \mathcal{A} 's are the amounts by which the successive intervals of scale 11 are longer than one-thirteenth of the total length of the scale. We have, therefore,

Correction to n th line = $\mathcal{A}_1 + \mathcal{A}_2 + \dots + \mathcal{A}_n$.

The observations were made with the high power microscope, 10 revolutions of the micrometer being equivalent to one scale division. The readings were made to thousandths of a revolution so that the quantities c were obtained directly in units approximately equal to .0001 of a scale division. Settings were made upon the scale line on that side of the line running the length of the scale which is nearest the other scale in the positions indicated (see figure) and the attention was fixed upon that portion of the scale line about one-twentieth mm. distant from the long line. The intervals c measured by the micrometer were small and nearly uniform, so that the quantities d were also small and the error of run was negligible throughout most of the investigation. The same portion of the micrometer was used in measuring all the intervals c in any one position of the scales so that the irregularities of pitch or eccentricity of the head would not affect the results.

The following is an array of the quantities d with the values of the D 's and \mathcal{A} 's. When d is positive the interval of scale 11 is larger.

TABLE I.
 d_{ij}

Interval of Sc. 11. Interval of Sc. 10.	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100	100-110	110-120	120-130	Σ	$\Sigma/13 = \mathcal{A}'_1$
200-210	+ .04001	-.00250	+ .00057	-.00817	+ .00052	+ .00034	-.00041	-.00117	-.00108	+ .00077	-.00110	-.00202	+ .00046	+ .02622	+ .00202
210-220	4065	- 168	+ 199	- 671	+ 246	+ 189	+ 48	+ 97	+ 11	+ 148	+ 30	- 5	+ 268	+ .04457	+ .00343
220-230	4208	- 29	+ 244	- 642	+ 290	+ 361	+ 138	+ 132	+ 55	+ 162	+ 17	- 62	+ 166	+ .05040	+ .00387
230-240	3975	- 246	+ 121	- 774	+ 28	+ 32	- 179	- 96	- 120	+ 8	- 142	- 181	+ 87	+ .02513	+ .00193
240-250	4174	- 31	+ 243	- 634	+ 296	+ 311	+ 164	+ 67	+ 14	+ 205	+ 31	+ 12	+ 197	+ .05049	+ .00388
250-260	4078	- 18	+ 201	- 602	+ 205	+ 325	+ 60	+ 77	- 13	+ 119	- 64	- 42	+ 135	+ .04461	+ .00343
260-270	4088	+ 9	+ 226	- 627	+ 185	+ 277	+ 72	+ 66	+ 66	+ 174	+ 34	- 7	+ 254	+ .04817	+ .00371
270-280	3760	- 382	- 109	- 963	- 70	- 37	- 233	- 306	- 273	- 141	- 266	- 434	- 78	+ .00468	+ .00036
280-290	4033	- 81	+ 139	- 736	+ 33	+ 178	+ 23	- 41	+ 104	+ 86	- 63	- 203	+ 84	+ .03348	+ .00257
290-300	4711	+ 560	+ 870	- 1	+ 859	+ 977	+ 707	+ 718	+ 685	+ 870	+ 673	+ 652	+ 902	+ .13183	+ .01014
300-310	3926	- 185	+ 51	- 847	- 11	+ 68	- 19	- 151	- 242	+ 28	- 174	- 236	+ 47	+ .02255	+ .00173
310-320	4094	- 96	+ 291	- 738	+ 240	+ 274	+ 5	+ 52	00	+ 138	- 34	- 102	+ 227	+ .04351	+ .00335
320-330	0056	- 4151	- 3802	- 4728	- 3811	- 3697	- 3990	- 4025	- 4012	- 3891	- 4029	- 4137	- 3849	- .48066	- .03698
Σ	+ .49169	-.05068	-.01269	-.12780	-.01458	-.00708	-.03245	-.03527	-.04041	-.02017	-.04097	-.04947	-.01514		
$\Sigma/13 = D'_1$	+ .03782	-.00390	-.00098	-.00983	-.00112	-.00054	-.00250	-.00271	-.00311	-.00155	-.00315	-.00381	-.00117		

$$\Sigma D_i = \Sigma \mathcal{A}'_i = + .00345$$

$$\frac{\Sigma D'_i}{13} = + .000265$$

The unit is 1 mm.

TABLE II.
Table for check see p. 146.

Interval of Sc. 11. Interval of Sc. 10.	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100	100-110	110-120	120-130
200-210	+ .03958	- .00214	+ .00078	- .00807	+ .00064	+ .00122	- .00074	- .00095	- .00135	+ .00021	- .00139	- .00205	+ .00059
210-220	4099 -	73 +	219 -	666 +	205 +	263 +	67 +	46 +	6 +	62 +	2 -	64 +	200
220-230	4143 -	29 +	263 -	622 +	249 +	307 +	111 +	90 +	50 +	206 +	46 -	20 +	244
230-240	3949 -	223 +	69 -	816 +	55 +	113 -	83 -	104 -	144 +	12 -	148 -	214 +	50
240-250	4144 -	28 +	264 -	621 +	250 +	308 +	112 +	91 +	51 +	207 +	47 -	19 +	245
250-260	4099 -	73 +	219 -	666 +	205 +	263 +	67 +	46 +	6 +	162 +	2 -	64 +	200
260-270	4127 -	45 +	247 -	638 +	233 +	291 +	95 +	74 +	34 +	190 +	30 -	36 +	228
270-280	3792 -	380 -	88 -	973 -	102 -	44 -	240 -	261 -	301 -	145 -	305 -	371 -	107
280-290	4013 -	159 +	133 -	752 +	119 +	177 -	19 -	40 -	80 +	76 -	84 -	150 +	114
290-300	4770 +	598 +	890 +	5 +	876 +	934 +	738 +	717 +	677 +	833 +	673 +	607 +	871
300-310	3929 -	243 +	49 -	836 +	35 +	93 -	103 -	124 -	164 -	8 -	168 -	234 +	30
310-320	4091 -	81 +	211 -	674 +	197 +	255 +	59 +	38 -	2 +	54 -	6 -	72 +	192
320-330	+ 58 -	4114 -	3822 -	4707 -	3836 -	3778 -	3974 -	3995 -	4035 -	3879 -	4039 -	4105 -	3841

The unit is 1 mm.

The D 's and the Δ 's with the determination of the corrections to the scale lines follow.

D_1	+ .03755 ₅	D_6	- .00080 ₅	D_{11}	- .00341 ₅	Δ_1	+ .00175 ₅	Δ_6	+ .00316 ₅	Δ_{11}	+ .00146 ₅
D_2	- .00416 ₅	D_7	- .00276 ₅	D_{12}	- .00407 ₅	Δ_2	+ .00316 ₅	Δ_7	+ .00344 ₅	Δ_{12}	+ .00308 ₅
D_3	- .00124 ₅	D_8	- .00297 ₅	D_{13}	- .00143 ₅	Δ_3	+ .00360 ₅	Δ_8	+ .00009 ₅	Δ_{13}	- .03724 ₅
D_4	- .01009 ₅	D_9	- .00337 ₅			Δ_4	+ .00166 ₅	Δ_9	+ .00230 ₅		
D_5	- .00138 ₅	D_{10}	- .00181 ₅			Δ_5	+ .00361 ₅	Δ_{10}	+ .00987 ₅		

CORRECTIONS FOR LINES OF SCALE 10.

Line.	Cor.	Line	Cor.	Line.	Cor.
10	- 375.6	60	- 198.7	110	- 55.2
20	- 334.0	70	- 171.0	120	- 14.4
30	- 321.5	80	- 141.3	130	.0
40	- 220.6	90	- 107.5		
50	- 206.7	100	- 89.4		

CORRECTIONS FOR LINES OF SCALE 11.

Line.	Cor.	Line.	Cor.	Line.	Cor.
210	+ 17.6	260	+ 169.7	310	+ 341.6
220	+ 49.2	270	+ 204.2	320	+ 372.4
230	+ 85.3	280	+ 205.1	330	.0
240	+ 101.9	290	+ 228.2		
250	+ 138.1	300	+ 326.9		

The unit is one ten-thousandth of a scale interval.

These tables of corrections form the basis of the following determination of corrections for intermediate lines.

When corrections for the lines at the beginning and end of any 10 mm. interval are known, corrections for the intervening lines are as follows.

Correction for n th line = Correction for first line

$$- \left\{ \left[D_1 + \frac{(D)}{10} \right] + \left[D_2 + \frac{(D)}{10} \right] + \dots + \left[D_n + \frac{(D)}{10} \right] \right\},$$

where (D) = amount by which the 10 mm. interval is smaller than one-thirteenth of the whole scale, D_1, D_2 , etc. = amounts by which successive mm. intervals are smaller than one-tenth of the whole interval.

Since $D_n = D'_n - \frac{1}{10} \Sigma D'$.

Correction for n th line = Correction for first line

$$- \left\{ \left[D'_1 + \frac{(D) - \Sigma D'_1}{10} \right] + \left[D'_2 + \frac{(D) - \Sigma D'_1}{10} \right] + \dots + \left[D'_n + \frac{(D) - \Sigma D'_1}{10} \right] \right\}$$

The corresponding formula for the correction of a line of scale 11 is:

Correction for n th line = Correction for first line

$$+ \left\{ \left[\Delta'_1 + \frac{(\Delta) - \sum \Delta'_i}{10} \right] + \left[\Delta'_2 + \frac{(\Delta) - \sum \Delta'_i}{10} \right] + \dots \text{etc.} \right\}$$

where (Δ) represents the amount by which the 10 mm. interval is larger than one-thirteenth of the whole scale.

Corrections were next determined for lines 35 and 95 of scale 10 and for lines 235 and 295 of scale 11, and these were made the basis for a determination of corrections for the intervening lines at intervals of 10 mm. The corrections for lines 35 and 295 were obtained by comparing the intervals 30-35-40 with the intervals 290-295-300 of the other scale and also by a comparison of the intervals 20-35-50 with the intervals 280-295-300. Similar comparisons were used to determine the corrections for lines 95 and 235. The adopted values are means of three investigations.

First adopted values of correction for line: 35 = -310.4

95 = -105.2

235 = +105.0

295 = +315.7

A second series of comparisons, made some months later, yielded the following results:

Correction for line: 35 = -314.0

95 = -106.2

235 = +105.5

295 = +313.8

The largest discrepancy is 3.6 ten-thousandths of a mm.

There is a very serious objection to the plan of using the corrections for the lines at 10 mm. intervals between lines 35 and 95 in conjunction with the corrections previously obtained for the lines 30, 40, 50, etc., as the basis for the determination of corrections for the lines within the successive 5 mm. intervals. This objection arises from the fact that the error of the determination of scale correction by this method is cumulative toward the center of the portion of the scale under investigation. Thus we should expect the error of the first determination to accumulate in the neighborhood of lines 60 and 70 and the error of the second determination to accumulate toward the line 65. If the two cumulative errors have opposite sign, the table of corrections would show the interval 60-65 to be considerably shorter than it really is, and the interval 65-70 correspondingly too long or vice versa. This is a serious matter in the measurement of short distances in which cumulative error of scale correction ought to be eliminated.

To guard against such error in the assumed lengths of the 5 mm. intervals two other determinations were made of corrections for lines at 5 mm. intervals

from 35 to 95 of scale 10, and from 235 to 295 of scale 11. In the first of these, corrections were determined for lines at 5 mm. intervals from 40-60 and from 60-90 of scale 10, and from 240-260 and from 260-290 of scale 11. In the second, corrections were determined for lines at 5 mm. intervals from 35 to 60 and from 70 to 95 of scale 10, and from 235 to 260 and 270 to 295 of scale 11. The method ordinarily used in determining corrections for any lines within an interval of the scale bases these corrections upon the corrections already determined for the lines at the beginning and end of that interval. In this case the method was so modified as to base the work upon the corrections obtained in the first investigation for the lines 40, 50, 60, 70, 80, 90 of scale 10 and lines 240, 250, 260, 270, 280, 290 of scale 11 and upon these lines only. The way in which this was accomplished may best be explained by reference to a particular example. In determining the corrections for the lines from 35 to 60 inclusive the D 's were:

Interval:	35-40;	40-45;	45-50;	50-55;	55-60
D'_1	-62.4	+11.1	+22.3	+18.5	+18.6
Assumed correction for line 35			-310.8	+5x	
Assumed correction for line 60			-198.7	+5y	
(D)			-112.1	-5(y-x)	
$\Sigma D'_1$			+8.1		
(D) - $\Sigma D'_1$			-120.2	-5(y-x)	
$\frac{(D) - \Sigma D'_1}{5}$			-24.0	-(y-x)	

Interval:	35-40;	40-45;	45-50;	50-55;	55-60	
D'_i	- 62.4	+ 11.1	+ 22.3	+ 18.5	+ 18.6	
$D'_i + \frac{(D) - \Sigma D'_i}{5}$	$\begin{bmatrix} - 86.4 \\ -y + x \end{bmatrix}$	$\begin{bmatrix} - 12.9 \\ -y + x \end{bmatrix}$	$\begin{bmatrix} - 1.7 \\ -y + x \end{bmatrix}$	$\begin{bmatrix} - 5.5 \\ -y + x \end{bmatrix}$	$\begin{bmatrix} - 5.4 \\ -y + x \end{bmatrix}$	
Line:	35	40	45	50	55	60
Correction to Line:	$\begin{bmatrix} - 310.8 \\ + 5x \end{bmatrix}$	$\begin{bmatrix} - 224.4 \\ + y + 4x \end{bmatrix}$	$\begin{bmatrix} - 211.5 \\ + 2y + 3x \end{bmatrix}$	$\begin{bmatrix} - 209.8 \\ + 3y + 2x \end{bmatrix}$	$\begin{bmatrix} - 204.3 \\ + 4y + x \end{bmatrix}$	$\begin{bmatrix} - 198.9 \\ + 5y \end{bmatrix}$

The corrections given in the table for lines 40, 50, 60 were:

Line 40	-220.5
Line 50	-206.7
Line 60	-198.7

This leads to the equations:

$$\begin{aligned}
 y + 4x &= +224.4 - 220.5 = +3.9 \\
 3y + 2x &= +209.8 - 206.7 = +3.1 \\
 5y &= +198.9 - 198.7 = +.2
 \end{aligned}$$

Taking the sum of the second and third minus the first, and the sum of the first and second minus the third, and solving, we have

$$\begin{aligned} 7y - 2x &= - .6 \\ y - 6x &= - 6.8 \\ x &= + 1.18 \\ y &= + .25 \end{aligned}$$

Giving as the result of the determination:

Line	35	40	45	50	55	60
Correction . .	- 304.9	- 219.4	- 207.5	- 206.7	- 202.1	- 197.7

The final table of corrections for lines at 5 mm. intervals, between lines 30 and 100 of scale 10 and lines 230 and 300 of scale 11, is obtained by taking the average of the results of the three determinations just described. These results with their averages are inserted here in tabular form. These corrections, with those for the lines 30, 100, 230, 300, given page 140, form the basis of the determination of corrections for the intervening lines at intervals of 1 mm.

SCALE 10.

Line. Cor.	35	40	45	50	55	60	65	70	75	80	85	90	95
	- 310.7	- 220.5 - 219.7	- 213.5 - 209.0	- 206.7 - 208.4	- 206.3 - 200.6	- 198.7 - 197.9	- 188.7 - 187.6	- 171.0 - 171.3	- 159.4 - 161.0	- 141.3 - 143.6	- 135.6 - 131.5	- 107.5 - 105.1	- 105.2 - 104.0
	- 304.9	- 219.4	- 207.5	- 206.7	- 202.1	- 197.7		- 169.3	- 158.4	- 141.4	- 134.0	- 105.7	- 104.0
Aver.	- 307.8	- 219.8	- 210.0	- 207.2	- 203.0	- 198.2	- 188.2	- 170.5	- 159.6	- 142.1	- 133.7	- 106.1	- 104.6

SCALE 11.

Line. Cor.	235	240	245	250	255	260	265	270	275	280	285	290	295
	+ 105.0	+ 101.9 + 103.9	+ 130.0 + 133.2	+ 138.1 + 136.5	+ 162.3 + 159.1	+ 169.7 + 170.3	+ 195.5 + 195.4	+ 204.2 + 203.6	+ 202.6	+ 205.1	+ 215.8	+ 228.2	+ 315.7
	+ 101.0	+ 104.0	+ 133.6	+ 138.3	+ 159.3	+ 168.3		+ 204.4 + 201.2	+ 207.2 + 202.2	+ 205.8 + 205.9	+ 221.3 + 217.2	+ 228.5 + 225.9	+ 315.5
Aver.	+ 103.0	+ 103.3	+ 132.3	+ 137.7	+ 160.2	+ 169.5	+ 195.4	+ 203.4	+ 204.0	+ 205.7	+ 217.8	+ 227.6	+ 315.6

The unit is one ten-thousandth of a scale interval.

I insert here, as an example, the array of d's for the interval 45-50 of scale 10 and 281-5 of scale 11, with the check array as described on page 146, and the determination of the corrections to the lines of the two scales.

LINES 46-50, 281-5.

Interval.	45-6	46-7	47-8	48-9	49-50	
280-1	- 52	+ 32	+ 97	- 73	+ 39	Cor. for line 45 - 2100
281-2	+ 22	+ 15	+ 48	- 79	+ 41	Cor. for line 50 - 2072
282-3	- 23	+ 21	+ 64	- 79	+ 87	(D) - 28
283-4	- 138	- 77	+ 20	- 215	- 22	$\Sigma D'_i$ + 83
284-5	+ 133	+ 157	+ 167	+ 22	+ 212	
Σ	- 58	+ 148	+ 396	- 424	+ 357	(D) - $\Sigma D'_i$ 5) - 111
$D'_i = \Sigma/5$	- 12	+ 30	+ 79	- 85	+ 71	5 - 22
$D'_i + (D) - \Sigma D'_i$	- 34	+ 8	+ 57	- 107	+ 49	Sum = + 83
Cor.	- 206.6	- 207.4	- 213.1	- 202.4	- 207.3	

Int.	45-6	46-7	47-8	48-9	49-50	
280-1	- 20	+ 22	+ 71	- 93	+ 63	Cor. for line 280 + 2057
281-2	- 20	- 22	+ 71	- 93	+ 63	Cor. for line 285 + 2178
282-3	- 15	+ 27	+ 76	- 88	+ 68	(D) + 121
283-4	- 115	- 73	- 24	- 188	- 32	$\Sigma D'_i$ + 83
284-5	+ 109	+ 151	+ 200	+ 36	+ 192	
						(D) - $\Sigma D'_i$ 5) + 38
						5 + 8

Interval.	280-1	281-2	282-3	283-4	284-5	
45-6	- 52	+ 22	- 23	- 138	+ 133	
46-7	+ 32	+ 15	+ 21	- 77	+ 157	
47-8	+ 97	+ 48	+ 64	+ 20	+ 167	
48-9	- 73	- 79	- 79	- 215	+ 22	
49-50	+ 39	+ 41	+ 87	- 22	+ 212	
Σ	+ 43	+ 47	+ 70	- 432	+ 691	
$D'_i = \Sigma/5$	+ 9	+ 9	+ 14	- 86	+ 138	
$D'_i + (D) - \Sigma D'_i$	+ 17	+ 17	+ 22	- 78	+ 146	Sum = + 84
Cor.	+ 207.4	+ 209.1	+ 211.3	+ 203.5	+ 218.1	

Following are the tables of corrections so far as they have been determined:

CORRECTIONS FOR SCALE 10.

Subtract the correction from observed reading.
The unit is one ten-thousandth of a scale division.

[illegible]

CORRECTIONS FOR SCALE 11.

Add the corrections to the observed reading.
The unit is one ten-thousandth of a scale division.

[illegible]

METHODS OF CHECKING WORK AND ACCURACY OF DETERMINATION.

The quantities D_1, D_2, D_3 , etc., are the amounts by which the successive 10 mm. intervals of scale 10 are found to be smaller than one-thirteenth of that scale. The quantities $\mathcal{D}'_1, \mathcal{D}'_2, \mathcal{D}'_3$, etc., are the amounts by which the successive 10 mm. intervals of scale 11 are found to be larger than one-thirteenth of scale 10. $D_i + \mathcal{D}'_j (=d_{ij})$ is therefore the amount by which the i th interval of scale 10 is smaller than the j th interval of scale 11, according to our determination. This affords a convenient method of checking the observation and reduction as far as the formation of the table of values of the d 's, given page 139. The table for check (p. 140) is formed by adding each \mathcal{D}' to each D . Disagreement between numbers in this table and corresponding numbers in the table of d 's is due to error in observation or reduction. The unit employed being approximately one mm., it is seen that no single d is in error by as much as one thousandth of a mm. When this check was first applied, several larger discrepancies were found. After investigation, one value of d was discarded, being affected with a large error, the cause which could not be determined. This and five other intervals were recompared, the mean of the original and newly determined values of d being taken in the other five cases.

Another check is given by the relation:

$$\sum D' = \sum \mathcal{D}'.$$

Similar checks were applied in the determination of corrections for the lines at intervals of 10 mm. from line 35 to line 95 and from line 265 to line 295, and in the investigation of the intervening lines in each of the 5 mm. intervals.

Several different tests have been applied with a view to determining the extent to which the final results can be trusted, and all point to the conclusion previously stated that they are reliable to within three or four ten-thousandths of a mm. in the case of scale 10 and only slightly less reliable in the case of scale 11. The values of D at the foot of Table I, page 139, are the arithmetic means of thirteen quantities each. Inasmuch as the thirteen values of each column result from a comparison of one of the intervals of scale 10 with thirteen different intervals of scale 11 it would not do to regard the differences between one of the D 's and the thirteen quantities of which it is the average as the residuals from which to obtain a probable error of the value of D . However, the differences between the quantities in the body of Table I and the corresponding quantities of Table II may be

regarded as residuals due to errors of observation. This would be rigorously true if the values of the D 's and the \mathcal{D} 's used in building up Table II were absolutely correct. They are undoubtedly correct to within a few ten-thousandths of a mm., so that the errors with which they are affected are negligible as compared with the error of a single comparison. An estimate of the probable error of one of the D 's may therefore be obtained by applying the formula:

$$p. e. = \frac{1}{\sqrt{3}} \sqrt{\frac{\sum v^2}{n(n-1)}}$$

v representing the difference between corresponding values in Tables I and II. This yields for the following intervals taken at random:

Interval	$p. e.$
230-240	0.84 ten-thousandths of a mm.
260-270	0.55
290-300	1.05

The probable error of one of the D 's is therefore about eight-tenths of a ten-thousandth of a mm.

The probable error of $\sum D'_i$ would be $\sqrt{13} \times (0.00008)^2 \text{ mm.} = .0003 \text{ mm.}$ $\sum D'_i$ is the difference in length of the two scales, which can be directly observed. A comparison with the directly observed value yields the following:

Computed value = $\sum D'_i$	= +0.0034 mm. (scale 11 longer).
Directly observed	= 0.0036 mm. (scale 11 longer).

Thus the actual difference between the computed and observed values is somewhat less than the probable error of the computed value.

The values of the corrections are obtained in such a way that the accumulated error is zero at the extremities of the interval. The probable errors are therefore largest for the lines near the center of the interval and are symmetrical on both sides of the center. The largest actual errors are therefore to be expected in the neighborhood of lines 65 and 265. I have made an independent determination of the corrections for lines 65 and 265 by comparing one half of each scale with each half of the other scale. The mean of two separate determinations yields the following results:

	Correction to line.	
	65	265
Independent determination	-.0190	+.0193 mm.
Tabulated value	-.0188	+.0195

In the investigation of the lines at intervals of 1 mm. in groups of five, only four settings were made on each line instead of eight, as in the case of the lines at intervals of 10 mm. the corrections for which form the basis of the subsequent work. I have found values of the probable errors of the D 's used in determining

corrections for lines in the group 45-50 by the method employed in finding the probable errors of the D 's in the investigation of the lines at 10 mm. intervals. They are as follows:

	D_1	D_2	D_3	D_4	D_5	
<i>p. e.</i>	0.7	0.3	0.8	0.6	0.7	unit = 0.0001 mm.,

the average being 0.6. The probable accumulated error in any 5 mm. interval is therefore less than one ten-thousandth of a mm.

As a final and most searching test of the order of accuracy of the tabulated corrections, I have made an essentially independent determination of the correction for certain lines in the center of each scale. Corrections were first determined for lines at intervals of 26 mm., *i. e.*, one-fifth of the scale length, using the method of comparing each 26 mm. interval of one scale with each 26 mm. interval of the other scale as described above. In order to guard against accumulated error four independent determinations were made and the averages taken. The results are as follows:

SCALE 10.					SCALE 11.				
Line.	26	52	78	104	Line.	226	252	278	304
	336.3	206.9	153.1	80.9		81.3	145.7	210.5	343.6
	333.6	206.7	151.9	79.0		79.3	145.8	214.5	342.8
	332.9	212.7	151.8	83.9		82.2	148.4	211.3	340.6
	330.8	211.1	150.0	83.8		78.8	141.2	209.2	341.7
Ave.	333.4	209.4	151.7	81.9	Ave.	80.4	145.3	211.4	342.2

Corrections for the lines 77 and 277 were derived from the corrections for lines 78 and 278 by comparing the interval 77-78 with each one mm. interval from line 230 to line 235 of scale 11, and the interval 277-8 with each 1 mm. interval between lines 30 and 35 of scale 10. In thus determining the amounts by which the intervals 77-78 and 277-8 are larger or smaller than one one-hundred-and-thirtieth of their respective scales, use is made of the previously determined value of the amount by which the 5 mm. intervals used in the comparison differ from one twenty-sixth of the length of their respective scales. Inasmuch, however, as any error of the previous determination (probably not exceeding three ten-thousandths of a mm.) enters into the present determination divided by five, this circumstance does not vitiate the essential independence of this test, of the previous determinations of scale corrections. The resulting values of the corrections for lines 77 and 277 are:

Correction for line 77 = - 145.

Correction for line 277 = + 217.

Corrections were next determined by the ordinary method, described on page 140, for lines at 5 mm. intervals from 52 to 77 and from 252 to 277, basing the determi-

nation on the values given by this independent investigation of the corrections for lines 52, 77, 252, 277. The comparison of the tabulated values of scale corrections with the values obtained by this independent determination follows.

SCALE 10.

Comparison of tabulated with independently determined corrections.

Line.	52	57	62	67	72	77	78
Tabulated . .	- 208	- 194	- 197	- 202	- 172	- 146	- 151
Independent . .	- 209	- 196	- 196	- 205	- 168	- 145	- 152
T - I	+ 1	+ 2	- 1	+ 3	- 4	- 1	+ 1

SCALE 11.

Line.	252	257	262	267	272	277	278
Tabulated . .	+ 142	+ 177	+ 172	+ 200	+ 214	+ 210	+ 205
Independent . .	+ 145	+ 181	+ 180	+ 207	+ 222	+ 217	+ 211
T - I	- 3	- 4	- 8	- 7	- 8	- 7	- 6

The unit is one ten-thousandth of a scale division.

In the case of scale 10, the discrepancies do not exceed four ten-thousandths of a mm., of which a part may reasonably be supposed to be due to error in each of the two values. A constant correction of six ten-thousandths of a mm. to the tabulated values of the corrections for scale 11 would make the agreement better than it is in the case of scale 10. This accumulated error in the tabulated values of the corrections for scale 11, amounting in the maximum to about six ten-thousandths of a mm., being constant throughout an interval of 26 mm., would not affect the measurement of distances shorter than 26 mm. if the center of the scale were used. Further investigation might be made to determine the amount of this error affecting the tabulated corrections for lines outside the interval 252-278 and corrections applied to the tabulated quantities. This does not seem worth while, however, in view of the fact that the errors of the present tabulated values are entirely negligible in most investigations in which the measuring apparatus is to be employed, and measurements requiring the highest degree of accuracy are made dependent upon scale 10 alone, which is to be used as the *X* scale. In fact, the investigation of the *Y* scale would have been omitted altogether were it not for the fact that to include it in the investigation required no more observations and only a little more numerical work in reduction than would have been required for the investigation of the *X* scale alone.

This final test therefore confirms the indications of probable error given by various tests mentioned at the beginning of this section: Namely, that the tabulated corrections for scale 10 are probably reliable to within three or four thousandths of a mm.

At various times during this investigation, the scales were carefully cleaned, notably before the final test just described.

The accumulation of particles of dust about any scale line might cause an error in the determination of the correction for that line but would not produce such a cumulative error as seems to affect the tabulated corrections for scale 11.

Comparisons of the total lengths of the two scales at various times while the scales were under investigation aroused a suspicion that changes had occurred in the length of one or both, of the order of magnitude of the changes in length of the pendula used in determinations of the force of gravity. If such a change were due to a uniform expansion or contraction of the metal throughout its entire length, the values of the tabulated corrections would be unaffected, since each interval would be changed in the same proportion as the whole scale which is the unit. (See definition of "Correction for Scale Line," p. 138.) If, however, the change in the length of the scale were due to a local expansion or contraction of the metal, the effect would be to produce an apparent accumulated error such as appears in the case of scale 11. It should be remarked in this connection that the independent determination of the correction for line 265, mentioned on page 147, was made about July 1, 1904, soon after the determination of corrections for lines at intervals of 10 mm. throughout the scales. This measure indicates that the tabulated correction was at that time certainly not too small. The final check which seems to indicate that the tabulated values of the corrections for lines near the middle of scale 11 are all too small was made about a year later, and it is during this time that the relative length of the two scales may have changed.

Temperature changes, if not too rapid, do not affect this determination, the two scales being made of the same material and having the same coefficient of expansion.

IRREGULARITIES OF THE MICROMETER USED IN MEASURING THE X COÖRDINATE. The method employed in this investigation is that described in BRÜNNOW'S *Sphärsche Astronomie* (p. 426 vierte Auflage, Part 7, Par. 5). The two scales were placed side by side so that division line 50 of scale 10 was about one-tenth mm. ($=\frac{1}{5}$ rev.) distant from line 276 of scale 11. This interval was then measured with each succeeding portion of the screw corresponding to one-tenth mm. from 10.0 to 13.0, *i. e.*, through three revolutions. The measures were then repeated in the reverse order. The means of the two measures of the interval made with the same portion of the screw were taken, there being thirty

such values. Brünnow designates these means by the letters $a', a'', \dots a^n$ and sets.

$$a = \frac{a' + a'' + \dots + a^n}{n}$$

The corrections which must be applied to the readings of the micrometer head are then

Reading	Correction
10.0	0
10.1	$a - a'$
10.2	$2a - a' - a''$
10.3	$3a - a' - a'' - a'''$
.	.
.	.
.	.
13.0	$30a - a' - a'' - \dots - a^n = 0$

Eight series of observations were made and combined by averaging the resulting corrections into two sets of four each. These values of the corrections were plotted on coördinate paper and two curves constructed through the plotted points. This was done to determine whether the smaller waves that characterize the curves are real or accidental and therefore to be disregarded in making up a

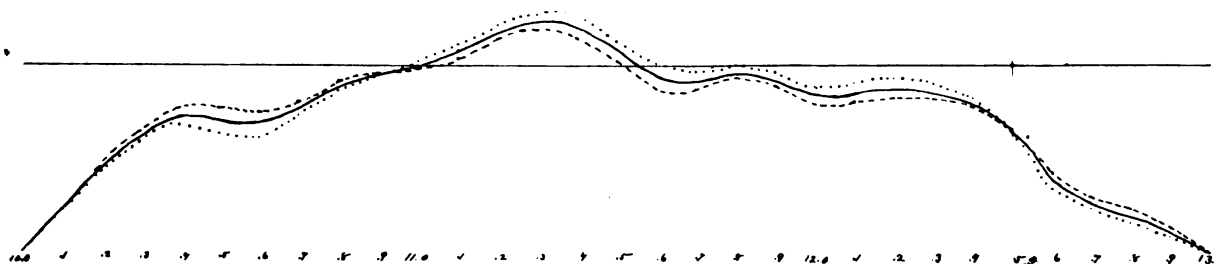


FIG. 2.

table of corrections. The two curves were found to agree surprisingly well, the waves corresponding without exception. See Figure 2. A third curve was drawn representing the mean of the eight determinations and a table of corrections read off from this curve. It is proposed to make measurements of star plates in such a way that the reading on the scale line shall lie about 11.0. The table of corrections has been so constructed, therefore, that the correction is zero for the micrometer reading 11.0. Below are given the observed corrections and the table constructed from them with the help of the plotted curve. The unit is one ten-thousandth mm.

Unit = 1 ten-thousandth mm.

Mic. Reading.	Observed Cor.		Table of Cor.
	Ave. 1st 4	Ave. 2d 4	
10.0	0	0	- 23
.1			- 18
.2	+ 11	+ 14	- 14
.3			- 10
.4	+ 16	+ 18	- 6
.5			- 7
.6	+ 14	+ 18	- 8
.7			- 6
.8	+ 21	+ 22	- 2
.9			- 1
11.0	+ 29	+ 23	0
.1			+ 2
.2			+ 4
.3	+ 29	+ 28	+ 6
.4	+ 30	+ 26	+ 5
.5			+ 2

Mic. Reading.	Observed Cor.		Table of Cor.
	Ave. 1st 4	Ave. 2d 4	
11.5			+ 2
.6	+ 23	+ 20	- 2
.7			- 2
.8	+ 23	+ 22	- 1
.9			- 2
12.0	+ 21	+ 18	- 4
.1			- 4
.2	+ 22	+ 20	- 3
.3			- 3
.4	+ 19	+ 18	- 4
.5			- 8
.6	+ 7	+ 9	- 14
.7	+ 4	+ 5	- 17
.8			- 19
.9			- 21
13.0	0	0	- 24

STRAIGHTNESS OF THE GUIDING BAR OF THE MOVABLE MICROSCOPE. The high power microscope (used ordinarily in measuring the I' coördinate) which gives an enlargement of about 38 diameters was placed in the movable microscope carriage, and the X scale, designated as N. E. K. 10 was placed on its stage below. Microscope and scale were so adjusted that the double hairs of the micrometer were parallel to the direction of the line which runs the entire length of the scale, cutting the millimeter division lines at right angles, and this line was nearly parallel with the direction of the bar (carrying the microscope) whose straightness was to be tested. Settings were made upon the long scale line at its points of intersection with the millimeter division lines at intervals of 13 mm. throughout its entire length, the scale being in the position "numbers down" as seen through the microscope. The scale was then turned end for end so as to be in the position "numbers up" and the settings were repeated. The table below gives in columns two and three the means of the readings. The number of settings combined in each mean varies from eight to twelve. From each of these readings I have subtracted the reading taken at the division 130 and then applied a further correction proportional to the distance of the division where the setting was made from division 130, to remove the effect of the inclination of the line to the direction of the bar. The readings so corrected are given in columns four and five. Column six is the sum of these two columns. The values of column six are thus freed from the effect of symmetrical curvature of the line upon which the settings were made. The unit throughout is the ten-thousandth of a millimeter. The values of column six show no regular variation

such as would be due to a curvature of the bar under investigation and they may be regarded as due to accidental errors of the settings.

Results of investigation of straightness of bar carrying X microscope.

Div.	Settings.		Cor. for Incl.		Sum.
	Nos. Down.	Nos. Up.	Nos. Down.	Nos. Up.	
0	530	582	+ 6	- 10	- 4
13	510	596	- 11	+ 12	+ 1
26	497	591	- 18	+ 15	- 3
39	506	569	- 9	0	- 9
52	517	575	+ 6	+ 14	+ 20
65	522	537	+ 15	- 16	- 1
78	500	540	- 4	- 6	- 10
91	506	530	+ 5	- 8	- 3
104	505	532	+ 7	+ 2	+ 9
117	498	520	+ 4	- 3	+ 1
130	491	515	0	0	0

The unit is .0001 mm.

IRREGULARITIES OR CURVATURE OF CYLINDER WHICH GUIDES THE PLATE CARRIAGE IN ITS MOTION IN THE DIRECTION OF THE Y COÖRDINATE. The cylinder is of steel and has evidently been ground into its present shape. It is 34 cm. long and 3.3 cm. in diameter, and therefore very rigid. Slight local irregularities such as dents or kinks of any considerable size are not to be expected. The plate carriage bears upon the cylinder by means of two Y supports 21 cm. apart and each in contact with the cylinder along two lines for a distance of 1 cm. The effect of scratches or very slight local irregularities upon the motion of the plate carriage would be reduced by the contact of the supports along these lines.

A defect much more to be feared is curvature of the bar, by which its middle portion is displaced laterally with respect to the two ends. This might arise if the bar were in a state of pressure, due either to faulty construction or unequal temperature expansion. Let us suppose that the axis of the cylinder is an arc, the projection of which on the plane of the plate is the circular ark ab (See Figure 3) with center at C . Let c and d represent the two points of the cylinder at which the two supports of the plate carriage are in contact with it. As the plate carriage moves along the guiding cylinder in the general direction of the Y axis, the line cd remaining fixed as regards the plate, every point of the plate instead of suffering a rectilinear displacement, as it should if the cylinder were straight, moves in an arc of a circle whose center is at C . The points that fall successively under the cross lines of the microscope as the plate carriage is moved in the Y direction lie on such a circular arc, and all measured X coördinates are measured along radii of this arc. (See definition of coördinate system, next paragraph.)

Let the origin of a rectangular coördinate system on the plate be some arbitrarily chosen point which remains approximately fixed when the plate is rotated, and let the Y axis be a line tangent at this point to an arc of the circle with radius R and center at C . It is assumed that the axis determined by the direction of motion of the X microscope is perpendicular to this line. Inasmuch as only small values of X are to be measured in this investigation, this condition need be only approxi-

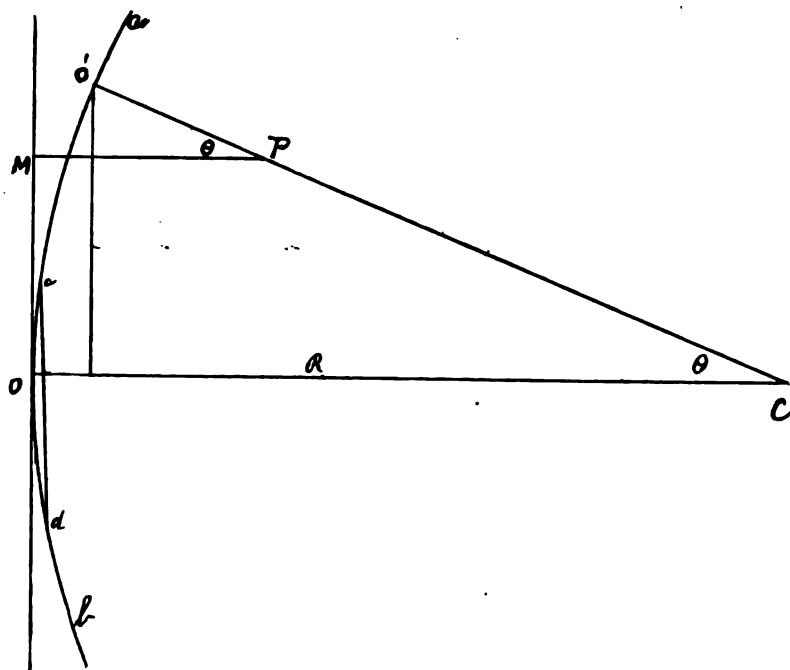


FIG. 3.

mately fulfilled. Let P represent any point on the plate, and aOb the locus of points of the plate which fall successively under the cross-hairs of the microscope when the plate is moved in the Y direction. Then if:

$$MP = \xi, OM = \eta$$

$$O'P = X = \text{measured coördinate}$$

$$\theta = \text{angle } OCP$$

$$OC = R \text{ (regarded as negative if } C \text{ lies to the left of } O)$$

we have

$$\xi = X \cos \theta + R \text{vers } \theta$$

(ξ and X being positive or negative respectively according as P lies to the right or left of the lines OM and OO').

If the plate be rotated through $180^\circ + \varphi$ and the new coördinates of P be denoted by the subscript π

$$\begin{aligned} \xi_\pi &= -\xi + a\xi + b\eta \\ &= X_\pi \cos \theta_\pi + R_\pi \text{vers } \theta_\pi \end{aligned}$$

we have, approximately:

$$R = R_{\pi}, \theta = -\theta_{\pi} = \frac{\eta}{R} \text{ (since } OM = OM_{\pi} = OO' \text{ approximately)}$$

$$\xi + \xi_{\pi} = X \cos \theta + X_{\pi} \cos \theta + 2 R \text{ vers } \theta$$

$$= X + X_{\pi} - \frac{1}{2} (X + X_{\pi} - 2 R) \left(\frac{\eta}{R} \right)^2$$

$$\xi - \xi_{\pi} = X - X_{\pi} - \frac{1}{2} (X - X_{\pi}) \left(\frac{\eta}{R} \right)^2$$

$$X + X_{\pi} = a \xi + b \eta + \frac{1}{2} (X + X_{\pi} - 2 R) \left(\frac{\eta}{R} \right)^2$$

$$X - X_{\pi} - 2 \xi = -a \xi - b \eta + \frac{1}{2} (X - X_{\pi}) \left(\frac{\eta}{R} \right)^2$$

a and b are small if φ is made small, and if we choose points for which ξ is small, the term $a \xi$ may be dropped. R will amount to some hundreds of meters. For points for which ξ is small, we write, therefore:

$$X + X_{\pi} = b \eta - \frac{1}{R} \eta^2$$

and determine R with the help of this equation by observing the X coördinates of points near the η axis at the upper and lower edges of the plate. The quantity

$$X - X_{\pi} - 2 \xi = -a \xi - b \eta + \frac{1}{2} (X - X_{\pi}) \left(\frac{\eta}{R} \right)^2$$

represents the error in the assumption ordinarily made in reduction that

$$\frac{X - X_{\pi}}{2} = \xi$$

and that part of the error of the ξ coördinate which is due to curvature of the cylinder is:

$$\frac{1}{2} (X - X_{\pi}) \left(\frac{\eta}{R} \right)^2$$

The investigation for the determination of curvature was made as follows: With a needle, well defined marks were made in the gelatine of a photographic plate near the η axis and at the top, middle, and bottom of the plate, and at two intermediate stations. The X coördinates of nine or fifteen such points in groups of three were accurately measured in two positions of the plate, differing in orientation by 180° . The Y coördinates were read off to the nearest tenth millimeter. The measures of the group of three points at the middle of the plate were used to determine the scale reading of the zero of the X coördinates, and the groups at the top and bottom of the plate yielded the absolute terms of the two equations of conditions necessary to determine b and $1/R$ in the above

expression for $X + X_{\pi}$. Four determinations of curvature were made at widely different temperatures. The results are:

Thermometer—F.	R	
52	937 m	Center to right
59	555 m	Center to right
62	> 1,000 m	Center to left
80	> 1,000 m	Center to left.

The last two determinations are probably much more trustworthy than the other two, greater care having been taken to secure sharply defined marks for the pointings.

Substituting $R = 500$ m. in the above expression the error in the ξ coördinate due to such curvature of the cylinder is found to be altogether negligible.

PERPENDICULARITY OF THE TWO DIRECTIONS OF MOTION.

If observations are to be made according to the second method as described on page 136, the four constant method of reduction being used, it is essential that the directions of motions which define the X and Y axes be perpendicular to each other. The following plan was adopted for the accomplishment of this adjustment. Two crosses were scratched in the gelatine of a plate about 10 mm. apart. The plate was then placed in the machine and oriented in such a way that the two crosses fell successively under the cross-hairs of the microscope when the plate was moved in the Y direction. The plate was then turned through 90° (which angle was measured with the circle) and the arm which carries the movable microscope adjusted so that the two crosses fell successively under the cross-hairs when the microscope was moved along its track, *i. e.* in the direction of the X axis. After this adjustment had been made, its accuracy was tested by taking circle readings in the two positions of the plate. The two directions were found to be perpendicular to each other within about $20''$. The maximum error introduced into any measured coördinate by the lack of perpendicularity of the directions of motion defining the two axes is, for the X coördinate: $Y \sin 20''$, or for the Y coördinate: $X \sin 20''$. The scale being 130 mm. long, the maximum value of X or Y is 65 mm., so that the greatest error due to this source would be $.0001 \times 65$ mm. = 0.006 mm. The use of the six constant method of reduction would eliminate errors due to this cause.

PARALLELISM OF SCALES TO PLATE.

It is only necessary that the two scales make equal angles with the plate so that the scale values may be the same in both coördinates. This has been tested by measuring the coördinates of well defined points on a plate in two positions, differing by 90° in orientation, so that the same distances were compared with both scales. The result indicates that a distance of a hundred mm. when measured as an X coördinate yields a value

larger by about .0058 mm. than the value obtained when the same distance is measured as a *Y* coördinate. Two determinations were made, nine distances being used in each. Following are the results in the form *X* measure—*Y* measure.

Dist. No.	<i>X</i> Measure - <i>Y</i> Measure.	
	1st Deter.	2d Deter.
1	+ .0080	+ .0099
2	+ 90	+ 87
3	+ 96	+ 73
4	+ 80	+ 121
5	+ 90	+ 109
6	+ 96	+ 95
7	+ 51	+ 118
8	+ 59	+ 116
9	+ 65	+ 92
Mean	+ 79	+ 101
Average of two determinations .0090 mm		

Applying a correction of — .0023 for difference in length of scales, and reducing to a value corresponding to 100 mm. of distance (the actual measured distance was about 116 mm. in every case) we have the value .0058 mm. given above.

DISTORTION OF THE FIELD OF THE OBJECT GLASS OF THE *X* MICROSCOPE. The micrometer is to be used only in the measurement of distances less than 1 mm. Consequently, little is to be feared from distortion of the field of the objective. A short investigation of the matter, however, has been made. The two scales were placed side by side on the stage and so that the division lines of the one scale about bisected the intervals of the other scale. The interval between line 278 of scale 11 and line 14 of scale 10 was then measured with different portions of the micrometer screw, thus utilizing different portions of the field of the objective along the line of a diameter of the field. The results are given below and indicate that no serious distortion exists but that the pitch of the screw varies, which indication is borne out by the results of the more extended investigation, pages 150–152.

r = distance in mm. from part of field corresponding to 11 revolutions;

D = measured distance;

v is expressed in ten-thousandths of a mm.

<i>r</i>	<i>D</i>	<i>v</i>
— 2.0	0.5001 mm.	— 7
— 1.5	6	— 2
— 1.0	6	— 2
— .5	0	— 8
.0	12	+ 4
+ .5	11	+ 3
+ 1.0	8	0
+ 1.5	18	+ 10
+ 2.0	14	+ 6
Mean	0.5008	

UNIVERSITY OF CALIFORNIA PUBLICATIONS.
PUBLICATIONS OF THE LICK OBSERVATORY, VOL. VII, PART 6.

ON ASTRONOMICAL REFRACTION.

Des. 100-300

BY RUSSELL TRACY CRAWFORD,
ASSISTANT PROFESSOR OF PRACTICAL ASTRONOMY,
BERKELEY ASTRONOMICAL DEPARTMENT.

PREFACE.

That the tables for computing refractions need modification has long been a recognized fact. It has been evident from observations made at the Lick Observatory that, not only are Bessel's refractions too large, but that the Pulkowa refractions are also. Consequently, in March, 1899, Dr. KEELER, late Director of the Lick Observatory, suggested that I should undertake to reduce a series of observations that had been made for refraction by Professor SCHAEBERLE, to discuss them completely, and to construct therefrom refraction tables for the Lick Observatory. This was to be done for a dissertation in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the University of California.

After consultation with him, and with his advice and consent, it was decided to forego the reduction of Professor SCHAEBERLE'S observations and to investigate the Constant of Refraction from observations to be made by myself, and to be reduced according to the method hereinafter set forth.

RUSSELL TRACY CRAWFORD.

Berkeley Astronomical Department,
March, 1907.

CONTENTS.

	PAGE.
PREFACE.	
I. INTRODUCTION - - - - -	163-169
1. General Statement regarding Observations - - - - -	163
2. Meteorological Observations and Conditions - - - - -	163
3. Plan for Observing - - - - -	165
II. OBSERVATIONS - - - - -	169-171
1. List - - - - -	169
2. Details of Observations - - - - -	170
III. REDUCTION OF OBSERVATIONS - - - - -	171-198
1. Explanation of Reduction - - - - -	171
2. Computation of Refractions - - - - -	174-195
IV. RESULTS - - - - -	198
V. REFRACTION TABLES - - - - -	202-216

INTRODUCTION.

1. The observations for this investigation were made with the 6.4-inch Meridian Circle of the Lick Observatory. A detailed account of this instrument and the room in which it is housed has been given by Astronomer TUCKER in Volume IV of these *Publications*, 1900.

Two independent sets were made under different meteorological conditions; one during the summer months June and July, the other during the autumn months October and November. The summer series comprises observations of 31 stars on seventeen nights, the autumn 43 stars on ten nights. Other stars were also observed for the purpose of determining bisection error. The detailed reduction of the summer series has already been published as No. 8, Vol. I, Third Series "*Proceedings of the California Academy of Sciences*." This will not be duplicated, but the reduction of the autumn series is here given in full. The final solution is based upon both series.

2. *Meteorological Observations and Conditions.* To make quite sure of the condition of the atmosphere at any time during the observations, the thermometers were read, on the average, three times an hour (at nearly equal intervals); and the barometer was observed every hour. The reading of the wet-bulb thermometer was also taken when the dry was read. The relative humidity has not been introduced into the reductions, but it was thought desirable to have it for possible future reductions.

The barometer, Green 2839, hangs on the north wall of the observing room. The least reading of its vernier is one two-hundredths of an inch. The dry and the wet-bulb thermometers (F.) hang in the air-space between the north walls. The dry-bulb thermometer, used to indicate the external temperature, is Green 494. This thermometer has been calibrated at the Yale Observatory. The corrections which have been applied to all the readings have been taken from the following table sent from Yale Observatory:

t (F.)	Cor.
0	0
0	+ 0.1
32	--- 0.2
52	- 0.1
72	--- 0.2
112	--- 0.1

The table which follows contains the *uncorrected* temperatures (t), the readings of the attached thermometer (T), of the barometer (B), and the times at which they were taken. The readings of the wet-bulb thermometer are not given. The unit of B is one two-hundredths of an inch.

TABLE OF BAROMETER AND THERMOMETER READINGS.

Sid. T	October 17.			October 24.			October 26.			October 31.			November 1.		
h	t	T	B	t	T	B	t	T	B	t	T	B	t	T	B
23.9	53.6	53	5155	45.6	45	5187	50.8	51½	5194	44.8	45	5185	51.5	53	5181
0.3	53.0			46.0			50.9			45.8			51.6		
0.7	53.5			46.6			50.9			46.3			51.9		
1.0	53.0	53	5153	46.3	46	5186	50.2	51	5192	46.0	45	5184	51.7	53	5181
1.4	52.8			46.5			50.2			45.8			51.6		
1.9	52.4	53	5153	46.1	46	5187	50.7	51	5190	46.0	45	5183	52.0	53	5181
2.3	51.9			46.1			50.8			46.5			52.0		
2.7	52.0			45.9			50.2			50.8			51.6		
3.0	52.9	52	5151	46.0	46	5187	50.0	51	5189	51.7	47	5177	51.5	53	5180
3.5	53.2			46.1			49.8			52.8			51.6		
3.8	53.0			45.9			49.0			50.9	48½	5185	51.0		
4.3	53.0			45.4			49.5						50.5		
4.6	53.0	53	5148	45.0	46	5186	49.9	50½	5185				50.0	52	5180

Sid. T	November 6.			November 22.			November 23.			December 1.			December 6.		
h	t	T	B	t	T	B	t	T	B	t	T	B	t	T	B
23.9	47.0	47	5183	40.5	41	5158	44.0	44	5172	56.2	54	5186	49.0	46½	5167
0.3	47.1			41.1			44.2			55.4			49.9		
0.7	47.1			41.5			44.9			56.0			49.9		
1.0	47.4	47½	5182	41.5	41	5160	45.0	45	5175	55.6	54½	5185	49.8	47½	5167
1.4	48.2			41.2			45.3			55.5			50.0		
1.9	48.0	48½	5183	42.0	40½	5160	45.5	45½	5178	56.3	54½	5185	50.8	48	5166
2.3	47.8			42.0			45.8			55.8			50.8		
2.7	48.0			41.8			46.5			56.0			50.4		
3.0	48.5	48½	5183	41.2	40½	5164	45.4	46	5178	55.9	54½	5185	50.2	48½	5161
3.5	48.9			41.0			47.0			55.6			50.0		
3.8	48.9			41.6			47.5			55.3			49.5		
4.3	48.8			42.3			47.0			55.0			49.3		
4.6	48.5	49	5185	42.8	41	5163	47.0	47½	5178	54.7	54½	5184	49.1	50	5159

The maximum temperature of the two series was 74°.0, July 3; the minimum was 40°.5, November 22, giving a range of 33°.5. The total range of the barometer for the two series was one-quarter of an inch.

Besides the regular thermometers in the air-space between the north walls, three other thermometers were suspended from the ceiling of the observing room. All three were swung under the observing slit, near the plane of the meridian. One was directly over the instrument, and three or four feet from the ceiling. The other two were hung, one north and one south, about halfway between the instrument and the north and south walls respectively, and at such a distance above the floor that the plane of the axis of the instrument and the line of sight

of the telescope, pointed at about 83° zenith distance (north and south respectively), would intersect the thermometers near their bulbs.

Before being thus placed, these thermometers were compared with Green 494, so that their readings could be reduced for comparison with those of the external thermometer (Green 494).

During the course of an evening's observations these three thermometers were read just after reading the regular thermometer. The average difference between the inside and the outside thermometers was found to be the same for all three, and is $0^\circ.3$ (F.) for the summer series, and even less for the autumn. It is nearly always the case (in this hemisphere) that the southern part of a room is a trifle warmer than the northern. But this is not the case on Mount Hamilton. The temperature of the air inside is, on the average, very uniform and but very little ($0^\circ.3$) warmer than the air outside. In his "*Untersuchung über die Astronomische Refraction u. s. w.*," Dr. BAUSCHINGER notes that the southern part of his observing room in Munich was warmer than the northern, and that at night the average difference between the inside and the outside temperatures is $1^\circ.3$ (C.). From his investigations, he concludes that the temperature of the air *within* the observing room should be taken into account.

Because of this difficulty, many observers have seriously considered the idea of mounting their instruments under a movable house, so that when at work the instrument may be entirely out of doors. But this would needlessly endanger the instrument. To accomplish the same purpose, the Meridian Circle house built at Kiel is constructed in the shape of a cylinder whose axis coincides with the axis of the instrument. This is undoubtedly the best form of construction.

For the efficiency of the Meridian Circle house on Mount Hamilton, the difference between the inside and the outside thermometers can speak. As has been noted, the average difference (in the sense Inside - Outside) is $+ 0^\circ.3$ (F.) for the summer series, and even less for the autumn. For the first series the *maximum* difference observed is $+ 1^\circ.1$ (F.). It may be noted further that this observation was made a few minutes before the sun had set. *This maximum is less than half the average at Munich.* For the other series the maximum difference is $+ 2^\circ.1$ (F.), which is less than the average at Munich. There was one still larger difference, viz., $- 3^\circ.7$ (F.), which may be discarded for it occurred on a poor night, immediately after observing had been stopped because of clouds and poor "seeing." The hot wave, which caused the outside temperature to rise suddenly, undoubtedly destroyed the "seeing." It is safe to assume, therefore, that after the observing room has been completely opened for an hour and a half, the temperature inside is practically the same as it is outside.

3. *Plan for Observing.* The method of determining the refractions may be stated as being a quasi converse to TALCOTT'S method of determining latitude.

Instead of eliminating the refractions to get the latitude, the method is to determine the refractions by eliminating the latitude, as follows:

Let

z_s = the zenith distance of a southern star,
 z_n = the zenith distance of a northern star,
 z'_s = the apparent zenith distance of the southern star,
 z'_n = the apparent zenith distance of the northern star,
 δ_s = the declination of the southern star,
 δ_n = the declination of the northern star,
 r_s = the refraction of the southern star,
 r_n = the refraction of the northern star,
 φ = the latitude of the Meridian Circle.

Then

$$\delta_n = \varphi + z_n = \varphi + (z'_n + r_n) \quad (1)$$

$$\delta_s = \varphi - z_s = \varphi - (z'_s + r_s) \quad (2)$$

$$\delta_n - \delta_s = z'_s + z'_n + r_s + r_n \quad (3)$$

Let

$$A = \delta_n - \delta_s \quad (4)$$

$$B = z'_s + z'_n \quad (5)$$

Then

$$A = B + r_s + r_n \quad (6)$$

or

$$r_s + r_n = A - B \quad (7)$$

If now, the southern and northern zenith distances were the same, and if, at the times of observing them, the conditions of the atmosphere were the same, the two refractions would be the same, *i. e.*

$$r_s = r_n.$$

In this case we have

$$2r = A - B \quad (I)$$

In practice these ideal conditions are only approximately satisfied. We therefore proceed as follows:

From (7) we have

$$2r_s - r_s + r_n = A - B \quad (8)$$

whence

$$2r_s = (A - B) + (r_s - r_n)$$

and

$$r_s = \frac{1}{2} (A - B) + \frac{1}{2} (r_s - r_n) \quad (II)$$

also

$$r_n = \frac{1}{2} (A - B) + \frac{1}{2} (r_n - r_s)$$

In case the northern star is at lower culmination we shall have:

$$\delta_n = 180^\circ - z_n - \varphi \quad (9)$$

$$\delta_s = \varphi - z_s \quad (10)$$

$$\delta_n + \delta_s = 180^\circ - z_n - z_s \quad (11)$$

$$= 180^\circ - [z'_n + r_n + z'_s + r_s] \quad (12)$$

Hence

$$r_n + r_s = 180^\circ - [z'_n + z'_s] - [\delta_n + \delta_s] \quad (13)$$

and

$$2r_s = 180^\circ - [z'_n + z'_s] - [\delta_n + \delta_s] + [r_s - r_n] \quad (14)$$

Calling

$$A' = \delta_n + \delta_s \quad (15)$$

and since

$$B = z'_s + z'_n \quad (5)$$

we have

$$r_s = 90^\circ - \frac{1}{2} [A' + B] + \frac{1}{2} [r_s - r_n] \quad (III)$$

and

$$r_n = 90^\circ - \frac{1}{2} [A' + B] + \frac{1}{2} [r_n - r_s]$$

In order to obtain the refractions from (II) and (III) it is necessary to know the declinations of the stars, their apparent zenith distances (or rather the sums of the zenith distances of the pairs of north and south stars), and the differences between the refractions of the pairs. The stars chosen for this work are all fundamental, and in a first approximation their declinations are to be considered absolute. The list of stars, given later, has been taken from Professor NEWCOMB'S "*Catalogue of Fundamental Stars for 1875 and 1900, reduced to an absolute System.*" The apparent zenith distances, or the sums of the zenith distances of the several pairs, are obtained from the Meridian Circle observations; and the differences in the refractions are found by computing the refractions from some standard table. In this work the Pulkowa tables have been used. The term $\frac{1}{2} (r_s - r_n)$ being of the nature of a differential refraction, any error in the constant of refraction of the table used will have practically no effect upon this difference. The more nearly ideal conditions (*i. e.*, when $r_s = r_n$) are approached, of course, the better the determination of the refractions will be.

This method has both its advantages and its disadvantages. Among the former, the most important are: first, the complete elimination of the latitude and hence also of its variation; second, the elimination of the nadir, since $(z'_s + z'_n)$ is nothing more nor less than the difference between the circle readings, and is therefore independent of the zenith point; third, there is no wait of twelve hours or of six months in order to observe the same star at both culminations, as is usually done; fourth, only one half of any error in an observed zenith distance enters into the observed refraction; and fifth, the simplicity of the reductions.

The greatest disadvantage in the method lies in the fact that the declinations of the stars have to be considered known. But by taking fundamental stars, such as those whose places are given by Professor NEWCOMB'S new Fundamental Catalogue, and by taking a large number of these stars, this difficulty will be nearly completely eliminated. It should be noted that the stars observed at great northern zenith distances are observed at lower culmination, and that their positions have been determined from observations made at upper culmination when the effect of refraction is small; and that the stars observed at great southern zenith distances are those whose positions have been determined at observatories in the southern hemisphere where they culminated near the zenith and where again the effects of refraction are small.

Having now the new refractions, the correction to the constant of the table used (Pulkowa) is found from the following equation [eq. (701) pg. 672, Vol. 1, CHAUVENET, "*Spherical and Practical Astronomy*"]:

$$dr = A da + B d\beta,$$

where

$$A = \frac{r}{a}$$

and

$$B = \sin^2 z \sqrt{\frac{2}{\beta}} \left(\frac{dQ}{d\beta} - \frac{Q}{2\beta} \right)$$

For the Lick Observatory, whose altitude is 4,209 feet and where the mean annual pressure is less than 26 inches, an investigation into the effect of the higher powers of $\Delta\beta$ involved in the factor $\beta = \frac{b}{B} = 1 + \frac{b-B}{B} = 1 + \frac{\Delta b}{B}$ (in BESSEL's notation for r) was necessary. In his memoir, "*Untersuchungen über die Constitution der Atmosphäre und die Strahlenbrechung in Derselben*," St. Petersburg, 1866, GYLDEN has neglected the squares and higher powers of $\frac{\Delta b}{B}$, since for places at low altitudes $\frac{\Delta b}{B}$ is a very small quantity. This investigation was made by Professor COMSTOCK (Vol. I, "*Publications of the Lick Observatory*"). From his investigation the conclusion is drawn that "the Pulkowa Refraction Tables may be used for atmospheric pressures as low as 25 inches without taking into account the squares and higher powers of Δb , and the quantities so neglected will not be sensible at zenith distances less than 80° ." The minimum reading of the barometer during these observations was 25.72 inches, so that in these reductions no modification of the factor of the refraction depending upon the barometer need be made.

This question having been disposed of, the assumption is here made that all of the error in the refractions is due to an error in the constant of refraction. This amounts to assuming the constant β to be correct or that $d\beta = 0$. The equation above then reduces to the very simple expression

$$dr = A da = \frac{r}{a} da;$$

hence

$$\frac{da}{a} = \frac{dr}{r},$$

or

$$d \log a = d \log r.$$

Having $d \log r$ from the reductions, we thus have $d \log a$, and hence da .

The assumption would perhaps seem somewhat risky for stars whose zenith distances are greater than 80° . But at the conclusion of the reductions, the value of $d \log a$ deducted from such stars was found to fit in very well with those deducted from the other stars. Furthermore, down to 85° zenith distance the "seeing" was very good. In consequence of these facts it was decided to take into account all the stars observed. The zenith distances of the stars in the list for the summer series range from $21^\circ 21'$ to $89^\circ 12'$, and for the other series from $15^\circ 46'$ to $87^\circ 49'$ (apparent).

Below 85° zenith distance the quality of the "seeing" decreases quite rapidly. This can be seen from the following table of average weights. These were derived from the probable errors of the individual determinations of $d \log a$.

Z. D.	Av. Wt.	Z. D.	Av. Wt.
20° to 30°	2.0	70° to 80°	11.8
50° to 60°	7.5	80° to 85°	14.8
60° to 70°	7.5	below 85°	3.6

The small weight for the small zenith distance is due to the fact that in the expression for da the refraction occurs in the denominator. The small weight for the stars at zenith distances greater than 85° is, of course, due to uncertainties in observing at such low altitudes.

OBSERVATIONS.

1. *List.*—The following list of 43 stars was observed on ten nights, from October 17 to December 6, 1899, inclusive, and have been reduced according to the plan outlined in the preceding section.

The numbers of the stars are those of NEWCOMB'S "*Catalogue of Fundamental Stars for 1875 and 1900, reduced to an Absolute System.*"

No.	α (1900).			δ (1900).		
	h	m	s	$^\circ$	'	"
5	0	4	20	— 46	17	57.15
9	0	6	39	— 35	41	34.42
788	12	25	17	+ 58	57	21.47
793	12	29	13	+ 70	20	22.03
38	0	36	36	— 46	38	2.99
49		43	30	+ 7	2	27.19
58	0	53	47	— 29	53	52.78
66	1	1	37	— 47	15	16.03
76		8	30	+ 7	2	47.77
79		12	38	+ 3	5	16.48
83		18	52	+ 67	36	29.32
89		22	33	+ 88	46	26.61
96		30	31	+ 72	31	49.56
103		34	56	+ 67	32	14.25
108	1	40	7	+ 8	39	16.19
885	14	1	41	+ 64	51	13.52
136	2	6	38	+ 66	3	20.75
139		8	30	— 31	11	34.97
149		20	49	+ 66	57	10.47
153	2	22	50	+ 8	0	42.99
911	14	27	44	+ 76	8	26.20
161	2	33	8	+ 21	31	44.60
174		39	22	— 14	16	55.59
179	2	44	54	— 32	49	33.16
944	14	51	0	+ 74	33	51.05
187	2	54	28	— 40	42	19.13
190	2	57	33	+ 53	6	53.92
202	3	7	49	— 29	22	52.49
210	3	15	56	— 43	27	7.78
976	— 15	20	53	+ 72	11	23.33
216	3	22	37	— 41	59	15.22
223		29	36	— 50	43	4.76
235	3	38	27	— 10	6	6.83
1002	15	45	8	+ 62	54	30.82
1014	15	55	25	+ 55	1	56.23
1019	16	0	1	+ 58	49	56.19
264	4	5	6	+ 85	17	29.06
270	4	10	41	— 42	32	27.86
1034	16	13	40	+ 76	7	45.78
1045		20	25	+ 75	59	9.05
1050	16	22	38	+ 61	44	25.78
290	4	31	40	— 30	46	1.53
297	4	37	20	— 42	3	17.36

2. *Details of Observations.*—A night's program consisted in observing the above list, together with three nadirs, one before, one during, and one after the observing of the stars. As has been pointed out, the nadirs are not necessary for the refraction determinations, but were taken for the reduction of the latitude, which is a problem practically inseparable from the main one undertaken here.

No transits were observed during these observations, the whole attention being devoted to the observations for zenith distance. The telescope was set to the nearest 2' and not disturbed until the observation had been completed. The bisection was made (with but a very few exceptions) at the central transit wire, by means of the declination micrometer. For the sake of uniformity every star was bisected but once during its transit. Because of unavoidable circumstances a few of the stars had passed the meridian before the bisection could have been made. In these cases the readings have been reduced to the meridian.

For the position of the circle four microscopes were read. Settings were made upon two scratches under every microscope. The circle microscopes were usually read after the star had been bisected. In a few cases, because of a following star culminating very soon, the microscopes were read before the bisection. In such cases the position of the circle was quickly checked after the bisection.

The correction for runs for a night was obtained from all of the microscope readings of the night. This correction has been applied to all of the observations. Its values for the several nights of observing are given in the following table:

Date.		R.	Date.		R.
October	17 . . .	+ 0".10	November	6 . . .	+ 0".08
	24 . . .	+ 0 .10		22 . . .	+ 0 .08
	26 . . .	+ 0 .09		23 . . .	+ 0 .10
	31 . . .	+ 0 .07	December	1 . . .	+ 0 .11
November	1 . . .	+ 0 .08		6 . . .	+ 0 .11

These corrections were applied to the circle readings to reduce them to the mean position of the two scratches: so that for a reading of 0" the correction is + R, for 60" it is 0, and for 120" it is - R.

In the few cases where the bisections were made a little late the reductions to the meridian were computed from the formula,

$$\delta = \delta' - \frac{\sin^2 \frac{1}{2} (\tau - m)}{\sin 1''} \sin 2\delta'$$

The horizontal flexure in this instrument is very small. In his work published in Volume IV of these *Publications*, Astronomer TUCKER adopts the correction 0".1 sin Z. D., which was determined from a series of observations extending over two and a half years. In this work the mean of three determinations of

flexure being less than the probable error of a single determination, the flexure is considered zero. The mean of the values of one revolution of the declination micrometer, determined at the same time, is $48''.11$. The value adopted is $48''.10$.

For the computation of the preliminary refractions (called r' in the reductions) the Pulkowa tables have been used. The reductions for the barometer, for the attached, and for the external thermometers were taken from Volume I of these *Publications*.

At the time these reductions were made the graduation errors of the 1° divisions of the fixed circle had been determined by Astronomer TUCKER. His results are given in Volume IV of these *Publications*. He says there, in part: "The probable error of a reading upon four divisions of the fixed circle due to graduation may be adopted as $\pm 0''.15$. * * * * There is some evidence of periodic character in the errors, and it may be assumed, in absence of further data, that the probable error due to errors of graduation is not diminished by reading upon two adjoining divisions under each microscope. * * * * The largest error measured is $0''.7$ for the mean of four divisions." In view of this no corrections have been applied for division error.

Three nadirs were observed every night. The changes during a night were usually very small. The following table gives the means of the three determinations on the several nights:

Date.	Nadir $134^\circ 57'$	t	Date.	Nadir $134^\circ 57'$	t
October 17 . .	$19''.87$	53°	November 6 . .	$22''.95$	48^c
24 . .	$21''.71$	46	22 . .	$24''.02$	42
26 . .	$21''.20$	50	23 . .	$23''.40$	46
31 . .	$22''.52$	47	December 1 . .	$22''.46$	56
November 1 . .	$22''.12$	51	6 . .	$23''.07$	50

All of the observations were taken with the fixed circle west. Had more time been available the instrument would have been reversed.

Weights, ranging from 5, the highest, to 1 (occasionally $\frac{1}{2}$), the lowest, were arbitrarily assigned to all the observations. Judgment on a weight was formed from the steadiness of the image during the observation. These weights have been applied all through the reductions.

REDUCTION OF OBSERVATIONS.

1. *Explanation of Reduction.*—The first thing done on the reductions was to take the means of the microscope readings and to apply the micrometer corrections, giving the circle readings (called C' in the tables following). The means of the microscopes were checked by taking the difference of every microscope reading from the mean of the four. If the sums of these differences for the

two opposite pairs of microscopes were the same, the mean was correct. The corrections for the micrometers were checked by duplicating this part of the work.

From the readings C' the quantity B [equations (II) and (III)] is obtained. The terms A and A' of these equations are obtained from the declinations.

The declinations have been reduced to 1899.0 by means of the data furnished in NEWCOMB'S Catalogue. The reductions to apparent places were computed by using the Besselian Star Numbers from the *American Ephemeris*. The factors a', b', c' and d' were computed from the *American Ephemeris* data. The reductions to apparent places for the first night (October 17) were computed by means of the Independent Star Numbers also. The places for the remaining nights were checked by differences. The apparent declinations are placed in the columns δ of the tables given later.

The following table exhibits the stars' approximate zenith distances and the stars with which they are grouped in the reductions for the refractions:

Star No.	Z. D. South.	Z. D. North.	Grouped with Star No.	Star No.	Z. D. South.	Z. D. North.	Grouped with Star No.
5 . . .	83° 31'	. . .	{ 788		{ 58
9 . . .	72 59	. . .	{ 1019	944	68 4	{ 139
			{ 793				{ 179
788	83 36	{ 5				{ 202
			{ 38	187 . . .	77 59	. . .	{ 290
			{ 66	190	15 46	{ 885
793	72 17	{ 9				{ 161
			{ 179	202 . . .	66 41	. . .	{ 911
38 . . .	83 51	. . .	{ 788				{ 944
			{ 1019				{ 1034
49 . . .	30 17	. . .	{ 83	210 . . .	80 43	. . .	{ 1045
			{ 103				{ 1002
58 . . .	67 12	. . .	{ 911	976	70 26	{ 1050
			{ 944				{ 139
66 . . .	84 28	. . .	{ 788	216 . . .	79 15	. . .	{ 179
			{ 1019	223 . . .	87 49	. . .	{ 1002
76 . . .	30 17	. . .	{ 83	235 . . .	47 25	. . .	{ 1050
79 . . .	34 14	. . .	{ 103				{ 1014
			{ 96				{ 264
83	30 16	{ 49	1002	79 40	{ 210
			{ 76				{ 216
89	51 25	{ 174				{ 270
96	35 11	{ 79	1014	87 24	{ 297
			{ 49				{ 223
103	30 11	{ 76	1019	83 43	{ 5
			{ 136				{ 38
108 . . .	28 41	. . .	{ 149	264	47 56	{ 66
885	77 45	{ 187				{ 235
136	28 43	{ 108	270 . . .	79 48	. . .	{ 1002
			{ 153				{ 1050
139 . . .	68 30	. . .	{ 911	1034	66 30	{ 202
			{ 944				{ 290
149	29 36	{ 976	1045	66 38	{ 202
			{ 108				{ 290
153 . . .	29 20	. . .	{ 153				{ 210
			{ 136	1050	80 50	{ 216
			{ 149				{ 270
911	66 29	{ 58				{ 297
			{ 139	290 . . .	68 4	. . .	{ 944
161 . . .	15 48	. . .	{ 202				{ 1034
174 . . .	51 36	. . .	{ 190	297 . . .	79 19	. . .	{ 1045
			{ 89				{ 1002
179 . . .	70 8	. . .	{ 793				{ 1050
			{ 944				
			{ 976				

The following tables show the reductions for the new refractions. The column ρ contains the means of the weights assigned to the stars. The other columns have already been explained. In the grouping of the pairs on the several dates the southern star is written first and the northern star below it. The numbers of the stars given at the top of each table are arranged in the same order. The pairs which have their northern stars at upper culmination are placed in the first few tables. It will be noticed that the headings for these are slightly different from those containing the lower culmination stars.

Because of very bad "seeing," or of other circumstances, some of the stars were not observed on some nights. In such cases blanks appear after the dates. No observations have been rejected.

Stars No. $\left\{ \begin{array}{l} 49 \\ 88 \end{array} \right.$ 2. *Computation of Refractions.*

Date.	δ	A B	C'	r'	$\frac{r' - r_n}{2}$	$\frac{A - B}{2}$	r	$\log r$	p
1899.									
Oct. 17	$+ 7^{\circ} 2' 37.32''$ $+ 67 36 36.23$	$60^{\circ} 33' 58.91''$ $60 33 0.37$	$345^{\circ} 14' 38.89''$ $284 41 38.52$	$0' 28.95''$ $0 28.95$	0.00 ..	$0' 29.27''$..	$0' 29.27''$ $0 29.27$	1.46642 1.46642	4 4
24	37.45 38.55	$34 1.10$ $33 0.91$	39.64 38.73	29.55 29.53	$+0.01$..	$0 30.09$..	$0 30.10$ $0 30.08$	1.47857 1.47828	$1\frac{1}{2}$ $1\frac{1}{2}$
26	37.50 39.26	1.76 3.02	39.68 36.66	29.33 29.32	0.00 ..	$0 29.37$..	$0 29.37$ $0 29.37$	1.46790 1.46790	4 4
31	37.37 40.72	3.35 4.45	41.00 36.55	29.56 29.55	0.00 ..	$0 29.45$..	$0 29.45$ $0 29.45$	1.46909 1.46909	$2\frac{1}{2}$ $2\frac{1}{2}$
Nov. 1	37.34 40.99	3.65 5.51	40.98 35.47	29.19 29.18	0.00 ..	$0 29.07$..	$0 29.07$ $0 29.07$	1.46345 1.46345	$3\frac{1}{2}$ $3\frac{1}{2}$
6	37.34 42.54	5.20 6.15	41.26 35.11	29.49 29.42	$+0.03$..	$0 29.52$..	$0 29.56$ $0 29.49$	1.47070 1.46967	$4\frac{1}{2}$ $4\frac{1}{2}$
22	37.01 47.03	10.02 10.29	42.90 32.61	29.72 29.70	$+0.01$..	$0 29.86$..	$0 29.87$ $0 29.85$	1.47524 1.47494	$2\frac{1}{2}$ $2\frac{1}{2}$
23	36.97 47.28	10.31 10.79	42.04 31.25	29.58 29.55	$+0.01$..	$0 29.76$..	$0 29.78$ $0 29.75$	1.47392 1.47349	2 2
Dec. 1	36.45 48.91	11.46 13.64	41.93 28.29	28.98 28.98	0.00 ..	$0 28.91$..	$0 28.91$ $0 28.91$	1.46105 1.46105	3 3
Dec. 6	$+ 7^{\circ} 2' 36.35''$ $+ 67 36 50.07$	$60^{\circ} 34' 13.72''$ $60 33 14.65$	$345^{\circ} 14' 42.11''$ $284 41 27.46$	$0 29.25$ $0 29.22$	$+0.01$..	$0 29.53$..	$0 29.55$ $0 29.52$	1.47056 1.47012	3 3

Stars No. $\left\{ \begin{array}{l} 49 \\ 108 \end{array} \right.$

Date.	δ	A B	C'	r'	$\frac{1}{2}(r' - r_n)$	$\frac{1}{2}(A - B)$	r	$\log r$	p
1899.									
Oct. 17	$+ 7^{\circ} 2' 37.32''$ $+ 67 32 19.85$	$60^{\circ} 29' 42.53''$ $60 28 43.75$	$345^{\circ} 14' 38.89''$ $284 45 55.14$	$0' 28.95''$ $0 28.88$	$+0.03$..	$0' 29.39''$..	$0' 29.42''$ $0 29.36$	1.46864 1.46776	4 4
24	37.45 22.15	44.70 44.56	39.64 55.08	29.55 29.45	$+0.05$..	$0 30.07$..	$0 30.12$ $0 30.02$	1.47885 1.47741	$1\frac{1}{2}$ $1\frac{1}{2}$
26	37.50 22.85	45.35 46.37	39.68 53.31	29.33 29.24	$+0.04$..	$0 29.49$..	$0 29.53$ $0 29.45$	1.47026 1.46909	4 4
31	37.37 24.32	46.95 47.49	41.00 53.51	29.56 29.47	$+0.04$..	$0 29.73$..	$0 29.77$ $0 29.69$	1.47378 1.47261	$2\frac{1}{2}$ $2\frac{1}{2}$
Nov. 1	37.34 24.59	47.25 48.87	40.98 52.11	29.19 29.10	$+0.04$..	$0 29.19$..	$0 29.23$ $0 29.15$	1.46583 1.46464	$3\frac{1}{2}$ $3\frac{1}{2}$
6	37.34 26.16	48.82 49.39	41.26 51.87	29.49 29.32	$+0.08$..	$0 29.71$..	$0 29.79$ $0 29.63$	1.47407 1.47173	$4\frac{1}{2}$ $4\frac{1}{2}$
22	37.01 30.75	53.74 54.15	42.90 48.75	29.72 29.61	$+0.05$..	$0 29.79$..	$0 29.84$ $0 29.74$	1.47480 1.47334	3 3
23	36.97 31.02	54.05 54.22	42.04 47.82	29.58 29.47	$+0.05$..	$0 29.91$..	$0 29.96$ $0 29.86$	1.47654 1.47509	2 2
Dec. 1	36.45 32.75	56.30 57.26	41.93 44.67	28.98 28.88	$+0.05$..	$0 29.52$..	$0 29.57$ $0 29.47$	1.47085 1.46938	3 3
Dec. 6	$+ 7^{\circ} 2' 36.35''$ $+ 67 32 33.97$	$60^{\circ} 29' 57.62''$ $60 28 58.28$	$345^{\circ} 14' 42.11''$ $284 45 43.83$	$0 29.25$ $0 29.11$	$+0.07$..	$0 29.67$..	$0 29.74$ $0 29.60$	1.47334 1.47129	3 3

Stars No. $\begin{cases} 76 \\ 88 \end{cases}$

Date.	δ	A B	C'	r'	$\frac{r'_1 - r'_n}{2}$	$\frac{A - B}{2}$	r	log r	p
1899.									
Oct. 17	$+ 7^{\circ} 2' 57.52''$ $+ 67 36 36.23$	$60^{\circ} 33' 38.71''$ $60 32 40.26$	$345^{\circ} 14' 18.78''$ $284 41 38.52$	$0' 28.97''$ $0 28.95$	$+0.01$..	$0' 29.22''$..	$0' 29.23''$ $0 29.21$	1.46583 1.46553	4 4
24	57.63 38.55	40.92 41.58	20.31 38.73	29.55 29.53	$+0.01$..	$0 29.67$..	$0 29.68$ $0 29.66$	1.47246 1.47217	1 1
26	57.68 39.26	41.58 43.03	19.69 36.66	29.35 29.32	$+0.01$..	$0 29.27$..	$0 29.28$ $0 29.26$	1.46657 1.46627	4 4
31	57.54 40.72	43.18 43.49	20.04 36.55	29.57 29.55	$+0.01$..	$0 29.84$..	$0 29.85$ $0 29.83$	1.47494 1.47465	2 2
Nov. 1	57.50 40.99	43.49 45.41	20.88 35.47	29.20 29.18	$+0.01$..	$0 29.04$..	$0 29.05$ $0 29.03$	1.46315 1.46285	$3\frac{1}{2}$ $3\frac{1}{2}$
6	57.48 42.54	45.06 46.33	21.44 35.11	29.46 29.42	$+0.02$..	$0 29.36$..	$0 29.38$ $0 29.34$	1.46805 1.46746	4 4
22	57.13 47.03	49.90 50.11	22.72 32.61	29.72 29.70	$+0.01$..	$0 29.89$..	$0 29.90$ $0 29.88$	1.47567 1.47538	2 2
23	57.10 47.28	50.18 50.60	21.85 31.25	29.58 29.55	$+0.01$..	$0 29.79$..	$0 29.80$ $0 29.78$	1.47422 1.47392	2 2
Dec. 1	56.56 48.91	52.35 53.61	21.90 28.29	28.99 28.98	$+0.01$..	$0 29.37$..	$0 29.38$ $0 29.36$	1.46805 1.46776	3 3
Dec. 6	$+ 7^{\circ} 2' 56.45''$ $+ 67 36 50.07$	$60^{\circ} 33' 53.62''$ $60 32 55.38$	$345^{\circ} 14' 22.84''$ $284 41 27.46$	$0 29.24$ $0 29.22$	$+0.01$..	$0 29.12$..	$0 29.13$ $0 29.11$	1.46434 1.46404	3 3

Stars No. $\begin{cases} 76 \\ 108 \end{cases}$

Date.	δ	A B	C'	r'	$\frac{r'_1 - r'_n}{2}$	$\frac{A - B}{2}$	r	log r	p
1899.									
Oct. 17	$+ 7^{\circ} 2' 57.52''$ $+ 67 32 19.85$	$60^{\circ} 29' 22.33''$ $60 28 23.64$	$345^{\circ} 14' 18.78''$ $284 45 55.14$	$0' 28.97''$ $0 28.88$	$+0.04$..	$0' 29.33''$..	$0' 29.37''$ $0 29.29$	1.46790 1.46672	4 4
24	57.63 22.15	24.52 25.23	20.31 55.08	29.55 29.45	$+0.05$..	$0 29.64$..	$0 29.69$ $0 29.59$	1.47261 1.47114	1 1
26	57.68 22.85	25.17 26.38	19.69 53.31	29.35 29.24	$+0.05$..	$0 29.39$..	$0 29.44$ $0 29.34$	1.46894 1.46746	4 4
31	57.54 24.32	26.78 26.53	20.04 53.51	29.57 29.47	$+0.05$..	$0 30.12$..	$0 30.17$ $0 30.07$	1.47958 1.47813	2 2
Nov. 1	57.50 24.59	27.09 28.77	20.88 52.11	29.20 29.10	$+0.05$..	$0 29.16$..	$0 29.21$ $0 29.11$	1.46553 1.46404	$3\frac{1}{2}$ $3\frac{1}{2}$
6	57.48 26.16	28.68 29.57	21.44 51.87	29.46 29.32	$+0.07$..	$0 29.55$..	$0 29.62$ $0 29.48$	1.47159 1.46953	4 4
22	57.13 30.75	33.62 33.97	22.72 48.75	29.72 29.61	$+0.05$..	$0 29.82$..	$0 29.87$ $0 29.77$	1.47524 1.47378	$2\frac{1}{2}$ $2\frac{1}{2}$
23	57.10 31.02	33.92 34.03	21.85 47.82	29.58 29.47	$+0.05$..	$0 29.94$..	$0 29.99$ $0 29.89$	1.47698 1.47553	2 2
Dec. 1	56.56 32.75	36.19 37.23	21.90 44.67	28.99 28.88	$+0.05$..	$0 29.48$..	$0 29.53$ $0 29.43$	1.47026 1.46879	3 3
Dec. 6	$+ 7^{\circ} 2' 56.45''$ $+ 67 32 33.97$	$60^{\circ} 29' 37.52''$ $60 28 39.01$	$345^{\circ} 14' 22.84''$ $284 45 43.83$	$0 29.24$ $0 29.11$	$+0.06$..	$0 29.25$..	$0 29.31$ $0 29.19$	1.46702 1.46523	3 3

Stars No. $\left\{ \begin{array}{l} 79 \\ 96 \end{array} \right.$

Date.	δ	A B	C'	r'	$\frac{1}{2}(r'_n - r'_n)$	$\frac{1}{2}(A - B)$	r	log r	p
1899.									
Oct. 17	+ 3° 5' 26.02 + 72 31 54.96	69° 26' 28.94 69 25 20.56	349° 11' 45.74 279 46 25.18	0' 33.77 0 34.97	-0.60 ..	0' 34.19 ..	0' 33.59 0 34.79	1.52621 1.54145	4 4
24	25.95 57.37	31.42 20.91	46.93 26.02	34.45 35.67	-0.61 ..	35.25 ..	0 34.64 0 35.86	1.53958 1.55461	1 1
26	25.96 58.10	32.14 22.06	46.01 23.95	34.21 35.41	-0.60 ..	35.04 ..	0 34.44 0 35.64	1.53706 1.55194	4 4
31	25.70 59.65	33 95 23.82	48.10 24.28	34.47 35.70	-0.61 ..	35.06 ..	0 34 45 0 35.67	1.53719 1.55230	2 2
Nov. 1	5 25.64 31 59.94	34 30 24.81	47.57 22.76	34.03 35.24	-0.60 ..	34.74 ..	0 34.14 0 35.34	1.53326 1.54827	3½ 3½
6	5 25.50 32 1.60	36.10 25.65	47.84 22.19	34.33 35.52	-0.60 ..	35.22 ..	0 34.62 0 35.82	1.53933 1.55413	4 4
22	24.81 6.46	41.65 29.81	49.30 19.49	34.64 35.88	-0.62 ..	35.92 ..	0 35.30 0 36.54	1.54777 1.56277	2½ 2½
23	24.76 6.74	41.98 31.46	49.33 17.87	34.47 35.69	-0.61 ..	35.26 ..	0 34.65 0 35.87	1.53970 1.55473	2 2
Dec. 1	24.08 8.59	44.51 35.83	49.94 14.11	33.80 35.00	-0.60 ..	34.34 ..	0 33.74 0 34.94	1.52815 1.54332	3 3
Dec. 6	+ 3 5 23.93 + 72 32 9.88	69 26 45.95 69 25 36.11	349 11 49.70 279 46 13.59	0 34.08 0 35.27	-0.60 ..	34.92 ..	0 34.32 0 35.52	1.53555 1.55047	3 3

Stars No. $\left\{ \begin{array}{l} 174 \\ 89 \end{array} \right.$

Date.	δ	A B	C'	r'	$\frac{1}{2}(r'_n - r'_n)$	$\frac{1}{2}(A - B)$	r	log r	p
1899.									
Oct. 17	- 14° 16' 45.47 + 88 46 30.76	103° 3' 16.23 103 1 11.02	366° 33' 28.62 263 32 17.60	1' 2.62 1 2.11	+0.25 ..	1' 2.60 ..	1' 2.85 1 2.35	1.79831 1.79484	4 4
24	46.92 33.42	20.34 12.58	29.63 17.05	3.85 3.34	+0.25 ..	3.88 ..	1 4.13 1 3.63	1.80706 1.80366	1½ 1½
26	47.14 34.21	21.35 15.28	30.52 15.24	3.29 2.91	+0.19 ..	3.03 ..	1 3.22 1 2.84	1.80085 1.79824	3 3
31	47.92 35.96	23.88 16.75	31.90 15.15	3.21 3.41	-0.10 ..	3.56 ..	1 3.46 1 3.66	1.80250 1.80387	1½ 1½
Nov. 1	48.11 36.29	24.40 19.63	32.56 12.93	3.01 2.60	+0.20 ..	2.38 ..	1 2.58 1 2.18	1.79644 1.79365	4 4
6	48.90 38.15	27.05 20.57	33.77 13.20	3.53 3.10	+0.21 ..	3 24 ..	1 3.45 1 3.03	1.80243 1.79955	4½ 4½
22	51.39 43.65	35.04 27.95	36.71 8.76	4.13 3.74	+0.20 ..	3.54 ..	1 3.74 1 3.34	1.80441 1.80168	3½ 3½
23	51.55 43.99	35.54 27.82	36.49 8.67	3.68 3.38	+0.15 ..	3.86 ..	1 4.01 1 3.71	1.80625 1.80421	2 2
Dec. 1	53.06 46.16	39.22 34.61	38.13 3 52	2.52 2.17	+0.17 ..	2.30 ..	1 2.47 1 2.13	1.79567 1.79330	3½ 3½
Dec. 6	- 14 16 53.71 + 88 46 47.66	103 3 41.37 103 1 35.60	366 33 38.14 263 32 2.54	1 2.96 1 2.67	+0.15 ..	1 2.88 ..	1 3.03 1 2.73	1.79955 1.79748	3 3

Stars No. $\begin{cases} 108 \\ 136 \end{cases}$

Date.	δ	A B	C'	r'	$\frac{1}{2}(r'_a - r'_n)$	$\frac{1}{2}(A - B)$	r	log r	p
1899.									
Oct. 17	+ 8° 39' 25.41 + 66 3 23.89	57° 23' 58.48 57 23 3.79	343° 37' 52.54 286 14 48.75	0' 27.15 27.22	-0.03 ..	0' 27.34 ..	0' 27.31 0 27.37	1.43632 1.43727	4 4
24	25.52 26.11	24 0.59 23 4.79	54.01 49.22	27.69 27.74	-0.02 ..	27.90 ..	27.88 27.92	1.44529 1.44592	1 1
26	25.60 26.77	1.17 5.99	52.87 46.88	27.47 27.49	-0.01 ..	27.59 ..	27.58 27.60	1.44059 1.44091	3 3
31	25.51 28.23	2.72 7.72	54.73 47.01	27.70 27.71	0.00 ..	27.50 ..	27.50 27.51	1.43933 1.43949	2½ 2½
Nov. 1	25.46 28.49	3.03 8.13	53.87 45.74	27.34 27.37	-0.01 ..	27.45 ..	27.44 27.46	1.43838 1.43870	3½ 3½
6	25.45 30.02	4.57 8.94	54.69 45.75	27.57 27.62	-0.02 ..	27.81 ..	27.79 27.83	1.44389 1.44451	4 4
22	25.18 34.67	9.49 12.92	56.01 43.09	27.83 27.85	-0.01 ..	28.28 ..	28.27 28.29	1.45133 1.45163	3 3
23	25.16 34.96	9.80 13.49	55.13 41.64	27.69 27.72	-0.01 ..	28.15 ..	28.14 28.16	1.44932 1.44963	2½ 2½
Dec. 1	24.66 36.78	12.12 16.97	55.44 38.47	27.14 27.18	-0.02 ..	27.57 ..	27.55 27.59	1.44012 1.44075	3 3
Dec. 6	+ 8° 39' 24.60 + 66 3 38.09	57° 24' 13.49 57 23 17.91	343° 37' 55.38 286 14 37.47	0' 27.35 0 27.37	-0.01 ..	0 27.79 ..	0 27.78 0 27.80	1.44373 1.44404	3 3

Stars No. $\begin{cases} 108 \\ 149 \end{cases}$

Date.	δ	A B	C'	r'	$\frac{1}{2}(r'_a - r'_n)$	$\frac{1}{2}(A - B)$	r	log r	p
1899.									
Oct. 17	+ 8° 39' 25.41 + 66 57 12.31	58° 17' 46.90 58 16 50.49	343° 37' 52.54 285 21 2.05	0' 27.15 0 28.25	-0.55 ..	0' 28.20 ..	0' 27.65 0 28.75	1.44170 1.45864	4 4
24	25.52 14.49	48.97 51.90	54.01 2.11	27.69 28.79	-0.55 ..	28.53 ..	27.98 29.08	1.44685 1.46359	1½ 1½
26	25.60 15.18	49.58 52.56	37 52.87 21 0.31	27.47 28.53	-0.53 ..	28.51 ..	27.98 29.04	1.44685 1.46300	3 3
31	25.51 16.65	51.14 55.12	37 54.73 20 59.61	27.70 28.70	-0.50 ..	28.01 ..	27.51 28.51	1.43949 1.45500	1½ 1½
Nov. 1	25.46 16.91	51.45 55.68	53.87 53.19	27.34 28.41	-0.53 ..	27.88 ..	27.35 28.41	1.43696 1.45347	3½ 3½
6	25.45 18.44	17 52.99 16 56.26	54.69 58.43	27.57 28.66	-0.54 ..	28.36 ..	27.82 28.90	1.44436 1.46090	4 4
22	25.18 23.18	17 58.00 17 0.53	56.01 55.48	27.83 28.90	-0.53 ..	28.73 ..	28.20 29.26	1.45025 1.46627	3 3
23	25.16 23.48	17 58.32 17 1.26	55.13 53.87	27.69 28.75	-0.53 ..	28.53 ..	28.00 29.06	1.44716 1.46330	1½ 1½
Dec. 1	24.66 25.38	18 0.72 17 5.38	55.44 50.06	27.14 28.21	-0.53 ..	27.67 ..	27.14 28.21	1.43361 1.45040	3 3
Dec. 6	+ 8° 39' 24.60 + 66 57 26.74	58° 18' 2.14 58 17 5.51	343° 37' 55.38 285 20 49.87	0' 27.35 0 28.39	-0.52 ..	0 28.31 ..	0 27.79 0 28.83	1.44389 1.45984	3 3

Stars No. $\begin{cases} 153 \\ 186 \end{cases}$

Date.	δ	A B	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$\frac{1}{2}(A - B)$	r	log r	p
1899. Oct. 17									
24	$+ 8^{\circ} 0' 51.43$ $+ 66 \quad 3 \quad 26.11$	$58^{\circ} 2' 34.68$ $58 \quad 1 \quad 37.97$	$344^{\circ} 16' 27.19$ $286 \quad 14 \quad 49.22$	$0' 28.45$ $0 \quad 27.74$	$+ 0.35$..	$0' 28.35$..	$0' 28.70$ $0 \quad 28.00$	1.45788 1.44716	$1\frac{1}{2}$ $1\frac{1}{2}$
26	51.46 26.77	35.31 39.92	26.80 46.88	28.20 27.49	$+ 0.35$..	27.69 ..	28.04 27.34	1.44778 1.43680	2 2
31	51.34 28.23	36.89 41.15	28.16 47.01	28.33 27.71	$+ 0.31$..	27.87 ..	28.18 27.56	1.44994 1.44028	2 2
Nov. 1	51.29 28.49	37.20 42.19	27.93 45.74	28.07 27.37	$+ 0.35$..	27.50 ..	27.85 27.15	1.44483 1.43377	3 3
6	51.16 30.02	38.86 42.83	28.58 45.75	28.33 27.62	$+ 0.35$..	28.01 ..	28.36 27.66	1.45271 1.44185	$4\frac{1}{2}$ $4\frac{1}{2}$
Nov. 22	50.75 34.67	43.92 46.88	29.97 43.09	28.57 27.85	$+ 0.36$..	28.52 ..	28.88 28.16	1.46060 1.44963	3 3
23	50.73 34.96	44.23 47.81	29.45 41.64	28.41 27.72	$+ 0.34$..	28.21 ..	28.55 27.87	1.45561 1.44514	2 2
Dec. 1	50.17 36.78	46.61 50.99	29.46 38.47	27.88 27.18	$+ 0.35$..	27.81 ..	28.16 27.46	1.44963 1.43870	3 3
Dec. 6	$+ 8^{\circ} 0' 50.05$ $+ 66 \quad 3 \quad 38.09$	$58^{\circ} 2' 48.04$ $58 \quad 1 \quad 52.56$	$344^{\circ} 16' 30.03$ $286 \quad 14 \quad 37.47$	$0' 28.06$ $0 \quad 27.37$	$+ 0.34$..	$0' 27.74$..	$0' 28.08$ $0 \quad 27.40$	1.44840 1.43775	3 3

 Stars No. $\begin{cases} 153 \\ 149 \end{cases}$

Date.	δ	A B	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$\frac{1}{2}(A - B)$	r	log r	p
1899. Oct. 17									
24	$+ 8^{\circ} 0' 51.43$ $+ 66 \quad 57 \quad 14.49$	$58^{\circ} 56' 23.06$ $58 \quad 55 \quad 25.08$	$344^{\circ} 16' 27.19$ $285 \quad 21 \quad 2.11$	$0' 28.45$ $0 \quad 28.79$	$- 0.17$..	$0' 28.99$..	$0' 28.82$ $0 \quad 29.16$	1.45969 1.46479	2 2
26	51.46 15.18	23.72 26.49	16 26.80 21 0.31	28.20 28.53	$- 0.16$..	28.61 ..	28.45 28.77	1.45408 1.45894	2 2
31	51.34 16.65	25.31 28.55	16 28.16 20 59.61	28.33 28.70	$- 0.18$..	28.38 ..	28.20 28.56	1.45025 1.45576	1 1
Nov. 1	51.29 16.91	25.62 29.74	27.93 58.19	28.07 28.41	$- 0.17$..	27.94 ..	27.77 28.11	1.44358 1.44886	3 3
6	51.16 18.44	27.28 30.15	28.58 58.43	28.33 28.66	$- 0.16$..	28.56 ..	28.40 28.72	1.45332 1.45818	$4\frac{1}{2}$ $4\frac{1}{2}$
22	50.75 23.18	32.43 34.49	29.97 55.48	28.57 28.90	$- 0.16$..	28.97 ..	28.81 29.13	1.45954 1.46434	3 3
23	50.73 23.48	32.75 35.58	29.45 53.87	28.41 28.75	$- 0.17$..	28.58 ..	28.41 28.75	1.45347 1.45864	1 1
Dec. 1	50.17 25.38	35.21 39.40	29.46 50.06	27.88 28.21	$- 0.16$..	27.90 ..	27.74 28.06	1.44311 1.44809	3 3
Dec. 6	$+ 8^{\circ} 0' 50.05$ $+ 66 \quad 57 \quad 26.74$	$58^{\circ} 56' 36.69$ $58 \quad 55 \quad 40.16$	$344^{\circ} 16' 30.03$ $285 \quad 20 \quad 49.87$	$0' 28.06$ $0 \quad 28.39$	$- 0.16$..	28.26 ..	$0' 28.10$ $0 \quad 28.42$	1.44871 1.45362	3 3

Stars No. $\left\{ \begin{array}{l} 161 \\ 190 \end{array} \right.$

Date.	δ	A B	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$\frac{1}{2}(A - B)$	r	log r	P
1899.									
Oct. 17	+21° 31' 51.51 +53 6 54.97	31° 35' 3.46 31 34 34.35	330° 45' 39.84 299 11 5.49	0' 14" 07 0 14.02	+0.02 ..	0' 14.55 ..	0' 14.57 0 14.53	1.16346 1.16227	4 4
24	52.05 56.60	4.55 34.91	40.12 5 21	14 35 14.31	+0.02 ..	14.82 ..	14.84 14.80	1.17143 1.17026	1½ 1½
26	52.27 57.14	4.87 35.45	39.94 4.49	14.22 14.20	+0.01 ..	14.71 ..	14.72 14.70	1.16791 1.16732	3 3
31	52.55 58.28	5.73 34.84	41.32 6.48	14.24 14.13	+0.05 ..	15.44 ..	15.49 15.39	1.19005 1.18724	1½ 1½
Nov. 1	52.58 58.47	5.89 36.57	39.73 3.16	14 15 14.13	+0.01 ..	14.16 ..	14.17 14.15	1.15137 1.15076	3 3
6	31 52.88 6 59.63	6.75 37.19	40.54 3.35	14.28 14.23	+0.02 ..	14.78 ..	14.80 14.76	1.17026 1.16909	4½ 4½
22	31 53.72 7 3.33	9.61 39.55	45 41.30 11 1.75	14.40 14.40	0.00 ..	15.03 ..	15.03 15.03	1.17696 1.17696	3½ 3½
23	53.78 3.58	9.80 40.87	45 40.50 10 59.63	14.31 14.30	0.00 ..	14.46 ..	14.46 14.46	1.16017 1.16017	2 2
Dec. 1	53.83 5.06	11.23 41.97	39.09 57.12	14.05 14.02	+0.01 ..	14.63 ..	14.64 14.62	1.16554 1.16495	3 3
Dec. 6	+21 31 54.07 +53 7 6.15	31 35 12.08 31 34 43.47	330 45 39.15 299 10 55.68	0 14.15 0 14.12	+0.01 ..	0 14.30 ..	0 14.31 0 14.29	1.15564 1.15503	2½ 2½

Stars No. $\left\{ \begin{array}{l} 235 \\ 264 \end{array} \right.$

Date.	δ	A B	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$\frac{1}{2}(A - B)$	r	log r	P
1899.									
Oct. 17	-10° 5' 57.29 +85 17 19.51	95° 23' 16.80 95 21 27.77	362° 22' 48.68 267 1 20.91	0' 53" 90 0 54.86	-0.48 ..	0' 54.51 ..	0' 54.03 0 54.99	1.73263 1.74028	3 3
24	58.16 21.37	19.53 27.69	49 86 22.17	55.09 56.11	-0.51 ..	55.92 ..	55.41 56.43	1.74359 1.75151	2 2
26	58.31 22.03	20.34 28.90	50.00 21.10	54.69 55.67	-0.49 ..	55.72 ..	55.23 56.21	1.74128 1.74981	3 3
31									
Nov. 1	59.19 23.79	22.98 32.70	52.01 19.31	54.41 55.45	-0.52 ..	55.14 ..	54.62 55.66	1.73735 1.74544	3½ 3½
6	5 59.94 17 25.28	25.22 34.38	53.24 18.86	54.73 55.72	-0.49 ..	55.42 ..	54.93 55.91	1.73981 1.74749	4 4
22	6 2.34 17 30.62	32.96 40.17	55.74 15.57	55.39 56.31	-0.46 ..	56.39 ..	55.93 56.85	1.74764 1.75473	4½ 4½
23	2.49 30.99	33.48 42.64	56.00 13.36	54.85 55.84	-0.49 ..	55.42 ..	54.93 55.91	1.73981 1.74749	2 2
Dec. 1	3.96 33.53	37 49 47.51	57.08 9.57	54.01 55.01	-0.50 ..	54.99 ..	54.49 55.49	1.73632 1.74421	3 3
Dec. 6	-10 6 4.65 +85 17 35.31	95 23 39.96 95 21 49.77	362 22 58.54 267 1 8.77	0 54.39 0 55.40	-0.50 ..	0 55.09 ..	0 54.59 0 55.59	1.73711 1.74500	3½ 3½

Stars No. $\left\{ \begin{array}{c} 5 \\ 788 \text{ l. c.} \end{array} \right.$

Date.	δ	A' B	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	log r	p
1899.									
Oct. 17	$-46^\circ 17' 55''.65$ $+58 57 17.38$	$12^\circ 39' 21''.73$ $167 7 12.73$	$398^\circ 29' 1''.41$ $231 21 48.68$	$6' 44''.57$ $6 48.36$	$-1''.89$..	$6' 42''.77$..	$6' 40''.88$ $6 44.66$	2.60301 2.60709	4 4
24	57.20 14.83	39 17.63 6 58.23	28 56.09 21 57.86	53.94 57.36	-1.71 ..	52.07 ..	50.36 53.78	2.61316 2.61677	2 2
26	57.63 14.07	39 16.44 7 5.64	57.70 52.06	49.86 53.38	-1.76 ..	48.96 ..	47.20 50.72	2.60981 2.61355	4 4
31	58.86 12.46	39 13.60 6 58.03	28 55.12 21 57.09	54.30 57.41	-1.55 ..	54.18 ..	52.63 55.73	2.61556 2.61881	2 2
Nov. 1	17 59.11 57 12.15	39 13.04 7 13.41	29 1.52 21 48.11	48.24 51.75	-1.75 ..	46.77 ..	45.02 48.52	2.60748 2.61121	3 3
6	18 0.06 57 10.36	10.30 2.95	28 55.65 21 52.70	52.49 56.11	-1.81 ..	53.37 ..	51.56 55.18	2.61443 2.61824	3 3
22	2.86 5.08	2.22 4.36	58.52 54.16	56.52 59.88	-1.68 ..	56.71 ..	55.03 58.39	2.61808 2.62158	4 4
23	3.01 4.78	39 1.77 7 12.75	28 57.07 21 44.32	54.46 58.22	-1.88 ..	52.74 ..	50.86 54.62	2.61369 2.61765	1 $\frac{1}{2}$ 1 $\frac{1}{2}$
Dec. 1	4.28 2.67	38 58.39 7 34.17	29 11.22 21 37.05	45.15 49.05	-1.95 ..	43.72 ..	41.77 45.67	2.60398 2.60817	3 3
Dec. 6	$-46^\circ 18' 4.67$ $+58 57 1.29$	$12^\circ 38' 56''.62$ $167 7 28.37$	$398^\circ 29' 6''.81$ $231 21 38.44$	$6' 49''.41$ $6 52.63$	-1.61 ..	$6' 47''.50$..	$6' 45''.89$ $6 49.11$	2.60841 2.61184	2 2

Stars No. $\left\{ \begin{array}{c} 5 \\ 1019 \text{ l. c.} \end{array} \right.$

Date.	δ	A' B	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	log r	p
1899.									
Oct. 17	$-46^\circ 17' 55''.65$ $+58 50 8.23$	$12^\circ 32' 12''.58$ $167 14 15.47$	$398^\circ 29' 1''.41$ $231 14 45.94$	$6' 44''.57$ $6 54.51$	$-4''.97$..	$6' 45''.97$..	$6' 41''.00$ $6 50.94$	2.60314 2.61378	3 $\frac{1}{2}$ 3 $\frac{1}{2}$
24	57.20 6.34	9.14 59.77	28 56.09 14 56.32	6 53.94 7 4.27	-5.16 ..	55.54 ..	6 50.38 7 0.70	2.61319 2.62397	2 $\frac{1}{2}$ 2 $\frac{1}{2}$
26	57.63 5.68	8.05 6.36	28 57.70 14 51.34	6 49.86 7 1.04	-5.59 ..	52.79 ..	6 47.20 6 58.38	2.60981 2.62157	2 $\frac{1}{2}$ 2 $\frac{1}{2}$
31									
Nov. 1	17 59.11 50 3.87	4.76 15.70	29 1.52 14 45.82	6 48.24 6 59.09	-5.42 ..	49.77 ..	6 44.35 6 55.19	2.60676 2.61825	3 $\frac{1}{2}$ 3 $\frac{1}{2}$
6	18 0.06 50 2.29	32 2.23 14 8.72	28 55.65 14 46.93	6 52.49 7 1.28	-4.39 ..	54.52 ..	6 50.13 6 58.91	2.61292 2.62212	2 $\frac{1}{2}$ 2 $\frac{1}{2}$
22	18 2.86 49 56.57	31 53.71 14 10.47	28 58.52 14 48.05	6 56.52 7 6.45	-4.96 ..	57.91 ..	6 52.95 7 2.87	2.61590 2.62621	4 4
23									
Dec. 1	4.28 53.40	49.12 35.28	29 11.22 14 35.94	6 45.15 6 55.64	-4.74 ..	47 80 ..	6 43.06 6 52.54	2.60537 2.61547	3 3
Dec. 6	$-46^\circ 18' 4.67$ $+58 49 51.45$	$12^\circ 31' 46''.78$ $167 14 32.84$	$398^\circ 29' 6''.81$ $231 14 33.97$	$6' 49''.41$ $6 59.11$	-4.85 ..	$6' 50''.19$..	$6' 45''.34$ $6 55.04$	2.60782 2.61809	2 2

Stars No. $\left\{ \begin{array}{l} 9 \\ 793 \text{ l. c.} \end{array} \right.$

Date.	δ	A' B	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	log r	p
1899.									
Oct. 17	$-35^\circ 41' 30.86$ $+70 20 16.36$	$34^\circ 38' 45.50$ $145 16 2.84$	$387^\circ 56' 37.38$ $242 40 34.54$	$2' 40.20$ $2 33.55$	$+3.32$..	$2' 35.83$..	$2' 39.15$ $2 32.51$	2.20181 2.18330	4 4
24	32.17 13.70	38 41.53 15 59.34	36.14 36.80	43.76 36.82	+3.47 ..	39.56 ..	43.03 36.09	2.21227 2.19337	2 2
26	32 51 12.92	38 40.41 16 3 38	37.30 33.92	42.25 35.44	+3.40 ..	38.10 ..	41.50 34.70	2.20817 2.18949	4 4
31	33.60 11.23	37.63 3.45	38.11 34.66	43.89 36.83	+3.53 ..	39.46 ..	42.99 35.93	2.21116 2.19297	3 3
Nov. 1	33.82 10.91	37.09 8.70	41.34 32.64	41.57 34.83	+3.37 ..	37.10 ..	40.47 33.73	2.20539 2.18676	3 3
6	34.66 9.05	34.39 7.80	40.50 32.70	43.18 36.40	+3.39 ..	38.90 ..	42.29 35.51	2.21029 2.19176	3 3
22	37.17 3.73	26.56 14.13	44.26 30.13	44.61 37.64	+3.48 ..	39.65 ..	43.13 36.17	2.21253 2.19360	4 4
23	37.32 3.43	26.11 14.11	42.33 28.22	43.87 36.97	+3.45 ..	39.89 ..	43.34 36.44	2.21209 2.19435	1½ 1½
Dec. 1	41 38 55 20 1.35	22.80 24.49	46.45 21.96	40.37 33.76	+3.30 ..	36.35 ..	39.65 33.05	2.20317 2.18483	3 3
Dec. 6	$-35^\circ 41' 38.95$ $+70 19 59.93$	$34^\circ 38' 20.98$ $145 16 24.09$	$387^\circ 56' 46.59$ $242 40 22.50$	$2' 41.91$ $2 35.07$	$+3.42$..	$2' 37.96$..	$2' 41.38$ $2 34.54$	2.20785 2.18804	3 3

Stars No. $\left\{ \begin{array}{l} 38 \\ 788 \text{ l. c.} \end{array} \right.$

Date.	δ	A' B	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	log r	p
1899.									
Oct. 17	$-46^\circ 37' 59.90$ $+58 57 17.38$	$12^\circ 19' 17.48$ $167 26 57.80$	$398^\circ 48' 46.48$ $231 21 48.68$	$7' 3.22$ $6 48.36$	$+7.43$..	$6' 52.36$..	$6' 59.79$ $6 44.93$	2.62303 2.60738	4 4
24	38 1.58 57 14.83	13.25 38.60	36.46 57.86	12.38 57.36	+7.51 ..	7 4.07 ..	7 11.58 6 56.56	2.63506 2.61968	2 2
26	2.03 14.07	12.04 49.89	41.95 52.06	8.69 53.38	+7.65 ..	6 59.03 ..	7 6.68 6 51.38	2.63010 2.61424	4 4
31	3.37 12.46	9.09 43.77	40.86 57.09	12.62 57.41	+7.60 ..	7 3.57 ..	7 11.17 6 55.97	2.63465 2.61806	3 3
Nov. 1	3.65 12.15	19 8.50 27 0.84	48.95 48.11	6.89 51.75	+7.57 ..	6 55.33 ..	7 2.90 6 47.76	2.62624 2.61040	3 3
6	4.74 10.36	19 5.62 26 49.19	41.89 52.70	11.45 56.11	+7.67 ..	7 2.59 ..	7 10.26 6 54.92	2.63373 2.61796	3 3
22	7.99 5.08	18 57.09 26 51.22	45.38 54.16	15.34 59 88	+7.73 ..	7 5.84 ..	7 13.57 6 58.11	2.63606 2.62129	3½ 3½
23	8.17 4.78	18 56.61 27 0.51	44.83 44.32	13.25 58.22	+7.51 ..	7 1.44 ..	7 8.95 6 53.93	2.63241 2.61693	2 2
Dec. 1	9.71 2.67	52.96 19.48	56.53 37.05	3.63 49.05	+7.29 ..	6 53.78 ..	7 1.07 6 46.49	2.62435 2.60905	3 3
Dec. 6	$-46^\circ 38' 10.27$ $+58 57 1.29$	$12^\circ 18' 51.02$ $167 27 14.06$	$398^\circ 48' 52.50$ $231 21 38.44$	$7' 7.84$ $6 52.63$	$+7.60$..	$6' 57.46$..	$7' 5.06$ $6 49.86$	2.62845 2.61264	2 2

Stars No. $\left\{ \begin{array}{l} 66 \\ 788 \text{ l. c.} \end{array} \right.$

Date.	δ	A' B	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	log r	p
1899.									
Oct. 17	$-47^\circ 15' 11.67$ $+58 57 17.38$	$11^\circ 42' 5.71$ $168 3 31.03$	$399^\circ 25' 19.71$ $231 21 48.68$	$7' 42.68$ $6 48.36$	$+27.16$..	$7' 11.63$..	$7' 38.79$ $6 44.47$	2.66161 2.60689	4 4
24	13.45 14.83	1.38 13.34	11.20 57.86	52.71 57.36	$+27.67$..	22.64 ..	50.31 54.97	2.67238 2.61802	$1\frac{1}{2}$ $1\frac{1}{2}$
26	13.94 14.07	42 0.13 3 23.32	15.38 52.06	49.11 53.38	$+27.86$..	18.27 ..	46.13 50.41	2.66851 2.61321	4 4
31	15.34 12.46	41 57.12 3 16.86	13.95 57.09	52.94 57.41	$+27.76$..	23.01 ..	50.77 55.25	2.67281 2.61831	2 2
Nov. 1	15.63 12.15	56.52 32.46	20.57 48.11	46.63 51.75	$+27.44$..	15.51 ..	42.95 48.07	2.66553 2.61073	3 3
6	16.84 10.36	53.52 26.56	19.26 52.70	51.34 56.11	$+27.61$..	19.96 ..	47.57 52.35	2.66985 2.61527	$3\frac{1}{2}$ $3\frac{1}{2}$
22									
23	20.63 4.78	44.15 33.45	17.77 44.32	53.34 58.22	$+27.56$..	21.15 ..	48.71 53.59	2.67090 2.61657	2 2
Dec. 1	22.38 2.67	40.29 51.04	28.09 37.05	43.22 49.05	$+27.08$..	14.33 ..	41.41 47.25	2.66409 2.60986	3 3
Dec. 6	$-47^\circ 15' 23.07$ $+58 57 1.29$	$11^\circ 41' 38.22$ $168 3 48.51$	$399^\circ 25' 26.95$ $231 21 38.44$	$7' 47.73$ $6 52.63$	$+27.55$..	$7' 16.63$..	$7' 44.18$ $6 49.08$	2.66669 2.61181	$2\frac{1}{2}$ $2\frac{1}{2}$

Stars No. $\left\{ \begin{array}{l} 179 \\ 793 \text{ l. c.} \end{array} \right.$

Date.	δ	A' B	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	log r	p
1899.									
Oct. 17	$-32^\circ 49' 23.05$ $+70 20 16.36$	$37^\circ 30' 53.31$ $142 24 17.99$	$385^\circ 4' 52.53$ $242 40 34.54$	$2' 16.28$ $2 33.55$	-8.63 ..	$2' 24.35$..	$2' 15.72$ $2 32.98$	2.13264 2.18464	4 4
24	24.64 13.70	49.06 17.08	53.88 36.80	19.03 36.82	-8.84 ..	26.93 ..	18.09 35.77	2.14016 2.19248	2 2
26	25.03 12.92	47.89 20.00	53.92 33.92	17.83 35.44	-8.80 ..	26.05 ..	17.25 34.85	2.13725 2.18991	4 4
31	26.29 11.23	44.94 23.99	58.65 34.66	17.40 36.83	-9.71 ..	25.53 ..	15.82 35.24	2.13296 2.19100	2 2
Nov. 1	26.57 10.91	44.34 24.91	57.55 32.64	17.19 34.83	-8.82 ..	25.37 ..	16.55 34.19	2.13529 2.18866	$3\frac{1}{2}$ $3\frac{1}{2}$
6	27.80 9.05	41.25 24.95	4 57.65 40 32.70	18.29 36.40	-9.05 ..	26.90 ..	17.85 35.95	2.13941 2.19298	$3\frac{1}{2}$ $3\frac{1}{2}$
22	31.67 3.73	32.16 31.86	5 1.99 40 30.13	19.68 37.64	-8.98 ..	27.99 ..	19.01 36.97	2.14305 2.19582	4 4
23	31.90 3.43	32.53 33.28	1.50 28.22	18.69 36.97	-9.14 ..	27.09 ..	17.95 36.23	2.13972 2.19376	2 2
Dec. 1	49 34.01 20 1.35	27.34 42.94	4.90 21.96	16.12 33.76	-8.82 ..	24.86 ..	16.04 33.68	2.13367 2.18662	$2\frac{1}{2}$ $2\frac{1}{2}$
Dec. 6	$-32^\circ 49' 35.02$ $+70 19 59.93$	$37^\circ 30' 24.91$ $142 24 43.74$	$385^\circ 5' 6.24$ $242 40 22.50$	$2' 17.11$ $2 35.07$	-8.98 ..	$2' 25.67$..	$2' 16.69$ $2 34.65$	2.13574 2.18934	3 3

Stars No. $\left\{ \begin{array}{l} 88 \\ 1019 \text{ l. c.} \end{array} \right.$

Date.	δ	$\begin{array}{c} A' \\ B \end{array}$	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	$\log r$	p
1899.									
Oct. 17	$-46^\circ 37' 59.90$ $+58 50 8.23$	$12^\circ 12' 8.33$ $167 34 0.54$	$398^\circ 48' 46.48$ $231 14 45.94$	$7' 3.22$ $6 54.51$	$+4.35$..	$6' 55.56$..	$6' 59.91$ $6 51.21$	2.62316 2.61406	$3\frac{1}{2}$ $3\frac{1}{2}$
24	$38 1.58$ $50 6.34$	$12 4.76$ $33 40.14$	36.46 56.32	$7 12.38$ $7 4.27$	$+4.05$..	$7 7.55$..	$7 11.60$ $7 3.50$	2.63508 2.62685	$2\frac{1}{2}$ $2\frac{1}{2}$
26	2.03 5.68	3.65 50.61	41.95 51.34	$7 8.69$ $7 1.04$	$+3.82$..	$7 2.87$..	$7 6.69$ $6 59.05$	2.63011 2.62227	$2\frac{1}{2}$ $2\frac{1}{2}$
31									
Nov. 1	3.65 3.87	$12 0.22$ $34 3.13$	48.95 45.82	$7 6.89$ $6 59.09$	$+3.90$..	$6 58.32$..	$7 2.22$ $6 54.42$	2.62554 2.61744	$3\frac{1}{2}$ $3\frac{1}{2}$
6	$38 4.74$ $50 2.29$	$11 58.55$ $33 54.96$	41.89 46.93	$7 11.45$ $7 1.28$	$+4.58$..	$7 3.24$..	$7 7.82$ $6 58.66$	2.63125 2.62186	$2\frac{1}{2}$ $2\frac{1}{2}$
22	$38 7.99$ $49 56.57$	$11 48.58$ $33 57.33$	45.38 48.05	$7 15.34$ $7 6.45$	$+4.49$..	$7 7.04$..	$7 11.53$ $7 2.55$	2.63501 2.62585	$3\frac{1}{2}$ $3\frac{1}{2}$
23									
Dec. 1	9.71 53.40	$11 43.69$ $34 20.59$	56.53 35.94	$7 3.63$ $6 55.64$	$+3.99$..	$6 57.86$..	$7 1.85$ $6 53.87$	2.62516 2.61686	3 3
Dec. 6	$-46 38 10.27$ $+58 49 51.45$	$12 11 41.18$ $167 34 18.53$	$398 48 52.50$ $231 14 33.97$	$7 7.84$ $6 59.11$	$+4.36$..	$7 0.14$..	$7 4.50$ $6 55.78$	2.62788 2.61897	$2\frac{1}{2}$ $2\frac{1}{2}$

Stars No. $\left\{ \begin{array}{l} 58 \\ 911 \text{ l.c.} \end{array} \right.$

Date.	δ	$\begin{array}{c} A' \\ B \end{array}$	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	$\log r$	p
1899.									
Oct. 17	$-29^\circ 53' 46.43$ $+76 8 29.76$	$46^\circ 14' 43.33$ $133 41 25.50$	$382^\circ 9' 34.42$ $248 28 8.92$	$1' 57.26$ $1 53.59$	$+1.83$..	$1' 55.58$..	$1' 57.41$ $1 53.75$	2.06971 2.05595	4 4
24	47.71 27.20	39.49 25.82	35.48 9.66	59.70 55.79	$+1.95$..	57.34 ..	59.29 55.39	2.07660 2.06217	2 2
26	48.06 26.39	38.33 27.05	34.92 7.87	58.83 54.73	$+2.05$..	57.31 ..	59.36 55.26	2.07686 2.06168	3 3
31	49.14 24.59	35.45 29.84	37.43 7.59	59.74 55.23	$+2.25$..	57.35 ..	59.60 55.10	2.07773 2.06108	$2\frac{1}{2}$ $2\frac{1}{2}$
Nov. 1	49.36 24.24	34.88 33.61	38.40 4.79	58.23 54.24	$+1.99$..	55.75 ..	57.74 53.76	2.07093 2.05599	3 3
6	50.28 22.34	32.06 33.26	39.01 5.75	$1 59.38$ $1 55.25$	$+2.06$..	57.34 ..	$1 59.40$ $1 55.28$	2.07700 2.06175	5 5
22	53.08 16.17	23.09 40.47	$9 42.66$ $28 2.19$	$2 0.37$ $1 56.29$	$+2.04$..	58.22 ..	$2 0.26$ $1 56.18$	2.08012 2.06513	$2\frac{1}{2}$ $2\frac{1}{2}$
23	53.25 15.78	22.53 42.50	$9 42.10$ $27 59.60$	$1 59.80$ $1 55.59$	$+2.10$..	57.48 ..	$1 59.58$ $1 55.38$	2.07766 2.06213	$2\frac{1}{2}$ $2\frac{1}{2}$
Dec. 1	54.72 13.07	18.35 50.00	44.23 54.23	57.41 53.43	$+1.99$..	55.82 ..	57.81 53.83	2.07119 2.05625	$3\frac{1}{2}$ $3\frac{1}{2}$
Dec. 6	$-29 53 55.28$ $+76 8 11.20$	$46 14 15.92$ $133 41 50.46$	$382 9 44.06$ $248 27 53.60$	$1 58.44$ $1 54.21$	$+2.11$..	$1 56.81$..	$1 58.92$ $1 54.70$	2.07525 2.05956	3 3

Stars No. $\left\{ \begin{array}{l} 58 \\ 944 \text{ l. c.} \end{array} \right.$

Date.	δ	$\begin{array}{c} A' \\ B \end{array}$	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	$\log r$	p
1899.									
Oct. 17	$-29^\circ 53' 46''.43$ $+74 33 56.81$	$44^\circ 40' 10''.38$ $135 15 48.51$	$382^\circ 9' 34''.42$ $246 53 45.91$	$1' 57''.26$ $2' 2.43$	$-2''.58$..	$2' 0''.55$..	$1' 57''.97$ $2' 3.13$	2.07177 2.09036	4 4
24	47.71 54.36	6.65 48.89	35.48 46.59	59.70 4.97	-2.63 ..	2.23 ..	59.60 4.86	2.07773 2.09642	2 2
26	48.06 53.56	5.50 51.04	34.92 43.88	58.83 3.92	-2.54 ..	1.73 ..	59.19 4.27	2.07624 2.09437	3 3
31	49.14 51.79	2.65 53.35	37.43 44.08	59.74 3.43	-2.34 ..	2.00 ..	59.66 4.34	2.07795 2.09461	2 2
Nov. 1	49.36 51.45	$40 2.09$ $15 57.46$	38.40 40.94	58.23 3.32	-2.54 ..	0.22 ..	57.68 2.76	2.07070 2.08906	$3\frac{1}{2}$ $3\frac{1}{2}$
6	50.28 49.58	$39 59.30$ $15 56.76$	39.01 42.25	59.38 4.29	-2.45 ..	1.97 ..	59.52 4.42	2.07744 2.09489	$4\frac{1}{2}$ $4\frac{1}{2}$
22	53.08 43.40	$39 50.32$ $16 3.54$	42.66 39.12	60.37 5.63	-2.63 ..	3.07 ..	60.44 5.70	2.08077 2.09933	3 3
23	53.25 43.00	49.75 5.76	42.10 36.34	59.80 4.75	-2.47 ..	2.24 ..	59.77 4.71	2.07835 2.09590	$2\frac{1}{2}$ $2\frac{1}{2}$
Dec. 1	54.72 40.22	45.50 12.97	44.23 31.26	57.41 2.37	-2.48 ..	0.76 ..	58.28 3.24	2.07291 2.09075	$3\frac{1}{2}$ $3\frac{1}{2}$
Dec. 6	$-29 53 55.28$ $+74 33 38.28$	$44 39 43.00$ $135 16 14.87$	$382 9 44.06$ $246 53 29.19$	$1' 58.44$ $2' 3.26$	-2.41 ..	$2' 1.06$..	$1' 58.65$ $2' 3.47$	2.07427 2.09156	3 3

 Stars No. $\left\{ \begin{array}{l} 66 \\ 1019 \text{ l. c.} \end{array} \right.$

Date.	δ	$\begin{array}{c} A' \\ B \end{array}$	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	$\log r$	p
1899.									
Oct. 17	$-47^\circ 15' 11''.67$ $+58 50 8.23$	$11^\circ 34' 56''.56$ $168 10 33.77$	$399^\circ 25' 19''.71$ $231 14 45.94$	$7' 42''.68$ $6' 54.51$	$+24''.08$..	$7' 14''.83$..	$7' 38''.91$ $6' 50.75$	2.66173 2.61358	$3\frac{1}{2}$ $3\frac{1}{2}$
24	13.45 6.34	52.89 14.88	11.20 56.32	$7' 52''.71$ $7' 4.27$	$+24.22$..	26.11 ..	$7' 50.33$ $7' 1.89$	2.67240 2.62520	2 2
26	13.94 5.68	51.74 24.04	15.38 51.34	$7' 49.11$ $7' 1.04$	$+24.03$..	22.11 ..	$7' 46.14$ $6' 58.08$	2.66852 2.62126	$2\frac{1}{2}$ $2\frac{1}{2}$
31									
Nov. 1	15.63 3.87	48.24 34.75	20.57 45.82	$7' 46.63$ $6' 59.09$	$+23.77$..	18.50 ..	$7' 42.27$ $6' 54.73$	2.66490 2.61777	$3\frac{1}{2}$ $3\frac{1}{2}$
6	$15 16.84$ $50 2.29$	45.45 32.33	19.26 46.93	$7' 51.34$ $7' 1.28$	$+25.03$..	21.11 ..	$7' 46.14$ $6' 56.08$	2.66852 2.61918	3 3
22									
23									
Dec. 1	$15 22.38$ $49 53.40$	31.02 52.15	28.09 35.94	$7' 43.22$ $6' 55.64$	$+23.79$..	18.41 ..	$7' 42.20$ $6' 54.62$	2.66483 2.61765	3 3
Dec. 6	$-47 15 23.07$ $+58 49 51.45$	$11 34 28.38$ $168 10 52.98$	$399 25 26.95$ $231 14 33.97$	$7' 47.73$ $6' 59.11$	$+24.31$..	$7' 19.32$..	$7' 43.63$ $6' 55.01$	2.66617 2.61806	3 3

Stars No. $\left\{ \begin{array}{l} 187 \\ 885 \text{ l. c.} \end{array} \right.$

Date.	δ	$\frac{A'}{B}$	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	log r	p
1899.									
Oct. 17	$-40^\circ 42' 8''.67$ $+64 51 16.10$	$24^\circ 9' 7''.43$ $155 43 24.47$	$392^\circ 56' 7''.84$ $237 12 43.37$	$3' 47''.46$ $3 43.35$	$+2.05$..	$3' 44''.05$..	$3' 46''.10$ $3 42.00$	2.35430 2.34635	4 4
24	10.47 13.53	3.06 18.65	6.69 48.04	52.26 47.82	$+2.22$..	49.14 ..	51.36 46.92	2.36429 2.35587	1 1
26	10.93 12.74	$9 1.81$ $43 23.32$	7.68 44.36	50.33 45.74	$+2.29$..	47.43 ..	49.72 45.14	2.36120 2.35245	3 3
31									
Nov. 1	12.64 10.60	$8 57.96$ $43 31.76$	12.11 40.35	49.25 44.73	$+2.26$..	45.14 ..	47.40 42.88	2.35679 2.34807	3 3
6	14.05 8.69	54.64 30.40	12.37 41.97	50.96 46.75	$+2.10$..	47.48 ..	49.58 45.38	2.36093 2.35291	4 4
22	$42 18.43$ $51 2.59$	44.16 34.90	14.38 39.48	53.63 48.77	$+2.43$..	50.47 ..	52.90 48.04	2.36717 2.35801	$3\frac{1}{2}$ $3\frac{1}{2}$
23									
Dec. 1	$42 21.04$ $50 59.54$	38.50 51.77	21.71 29.94	47.44 43.03	$+2.20$..	44.86 ..	47.06 42.66	2.35614 2.34764	$2\frac{1}{2}$ $2\frac{1}{2}$
Dec. 6	$-40 42 22.21$ $+64 50 57.70$	$24 8 35.49$ $155 43 52.10$	$392^\circ 56' 21.49$ $237 12 29.39$	$3 49.09$ $3 44.79$	$+2.15$..	$3 46.20$..	$3 48.35$ $3 44.05$	2.35860 2.35034	$2\frac{1}{2}$ $2\frac{1}{2}$

Stars No. $\left\{ \begin{array}{l} 189 \\ 911 \text{ l. c.} \end{array} \right.$

Date.	δ	$\frac{A'}{B}$	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	log r	p
1899.									
Oct. 17	$-31^\circ 11' 26''.07$ $+76 8 29.76$	$44^\circ 57' 3''.69$ $134 58 57.72$	$383^\circ 27' 6''.64$ $248 28 8.92$	$2' 5''.25$ $1 53.59$	$+5.83$..	$1' 59''.29$..	$2' 5''.12$ $1 53.46$	2.09733 2.05484	4 4
24	27.57 27.20	$56 59.83$ $58 57.03$	6.69 9.66	7.67 55.79	$+5.94$..	$2 1.57$..	7.51 55.63	2.10554 2.06307	2 2
26	27.96 26.39	$56 58.43$ $58 59.64$	7.51 7.87	6.50 54.73	$+5.88$..	0.96 ..	5.84 54.08	2.09982 2.05721	2 2
31	29.16 24.59	$56 55.43$ $59 2.61$	10.20 7.59	7.52 55.23	$+6.14$..	$2 0.98$..	7.12 54.84	2.10421 2.06009	2 2
Nov. 1	29.43 24.24	54.81 6.06	10.83 4.79	5.96 54.24	$+5.86$..	$1 59.56$..	5.42 53.70	2.09837 2.05576	3 3
6	30.55 22.34	51.79 5.69	11.44 5.75	7.10 55.25	$+5.92$..	$2 1.26$..	7.18 55.34	2.10442 2.06198	$4\frac{1}{2}$ $4\frac{1}{2}$
22	34.08 16.17	42.09 13.58	$27 15.77$ $28 2.19$	8.19 56.29	$+5.95$..	2.16 ..	8.11 56.21	2.10758 2.06524	3 3
23	34.29 15.78	41.49 15.20	$27 14.80$ $27 59.60$	7.57 55.59	$+5.99$..	$2 1.65$..	7.64 55.66	2.10599 2.06318	$1\frac{1}{2}$ $1\frac{1}{2}$
Dec. 1	36.19 13.07	36.88 23.60	17.83 54.23	5.06 53.43	$+5.81$..	$1 59.76$..	5.57 53.95	2.09889 2.05671	$3\frac{1}{2}$ $3\frac{1}{2}$
Dec. 6	$-31 11 37.05$ $+76 8 11.20$	$44 56 34.15$ $134 59 25.03$	$383^\circ 27' 18.63$ $248 27 53.60$	$2 5.94$ $1 54.21$	$+5.86$..	$2 0.41$..	$2 6.27$ $1 54.55$	2.10130 2.05899	3 3

Stars No. $\left\{ \begin{array}{l} 189 \\ 944 \text{ l. c.} \end{array} \right.$

Date.	δ	$\begin{array}{c} A' \\ B \end{array}$	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	$\log r$	p
1899.									
Oct. 17	$\begin{array}{r} -31^\circ 11' 26.07 \\ +74 33 56.81 \end{array}$	$\begin{array}{r} 43^\circ 22' 30.74 \\ 136 33 20.73 \end{array}$	$\begin{array}{r} 383^\circ 27' 6.64 \\ 246 53 45.91 \end{array}$	$\begin{array}{r} 2' 5.25 \\ 2 2.43 \end{array}$	$\begin{array}{r} +1.41 \\ \dots \end{array}$	$\begin{array}{r} 2' 4.26 \\ \dots \end{array}$	$\begin{array}{r} 2' 5.67 \\ 2 2.85 \end{array}$	$\begin{array}{r} 2.09923 \\ 2.08937 \end{array}$	$\begin{array}{r} 4 \\ 4 \end{array}$
24	$\begin{array}{r} 27.57 \\ 54.36 \end{array}$	$\begin{array}{r} 26.85 \\ 20.10 \end{array}$	$\begin{array}{r} 6.69 \\ 46.59 \end{array}$	$\begin{array}{r} 7.67 \\ 4.97 \end{array}$	$\begin{array}{r} +1.35 \\ \dots \end{array}$	$\begin{array}{r} 6.52 \\ \dots \end{array}$	$\begin{array}{r} 7.87 \\ 5.17 \end{array}$	$\begin{array}{r} 2.10677 \\ 2.09750 \end{array}$	$\begin{array}{r} 2 \\ 2 \end{array}$
26	$\begin{array}{r} 27.96 \\ 53.56 \end{array}$	$\begin{array}{r} 25.60 \\ 23.63 \end{array}$	$\begin{array}{r} 7.51 \\ 43.88 \end{array}$	$\begin{array}{r} 6.50 \\ 3.92 \end{array}$	$\begin{array}{r} +1.29 \\ \dots \end{array}$	$\begin{array}{r} 5.38 \\ \dots \end{array}$	$\begin{array}{r} 6.67 \\ 4.09 \end{array}$	$\begin{array}{r} 2.10267 \\ 2.09374 \end{array}$	$\begin{array}{r} 2 \\ 2 \end{array}$
31	$\begin{array}{r} 29.16 \\ 51.79 \end{array}$	$\begin{array}{r} 22.63 \\ 26.12 \end{array}$	$\begin{array}{r} 10.20 \\ 44.08 \end{array}$	$\begin{array}{r} 7.52 \\ 3.43 \end{array}$	$\begin{array}{r} +2.04 \\ \dots \end{array}$	$\begin{array}{r} 5.62 \\ \dots \end{array}$	$\begin{array}{r} 7.66 \\ 3.58 \end{array}$	$\begin{array}{r} 2.10605 \\ 2.09195 \end{array}$	$\begin{array}{r} 1\frac{1}{2} \\ 1\frac{1}{2} \end{array}$
Nov. 1	$\begin{array}{r} 29.43 \\ 51.45 \end{array}$	$\begin{array}{r} 22.02 \\ 29.89 \end{array}$	$\begin{array}{r} 10.83 \\ 40.94 \end{array}$	$\begin{array}{r} 5.96 \\ 3.32 \end{array}$	$\begin{array}{r} +1.32 \\ \dots \end{array}$	$\begin{array}{r} 4.04 \\ \dots \end{array}$	$\begin{array}{r} 5.36 \\ 2.72 \end{array}$	$\begin{array}{r} 2.09816 \\ 2.08891 \end{array}$	$\begin{array}{r} 3\frac{1}{2} \\ 3\frac{1}{2} \end{array}$
6	$\begin{array}{r} 30.55 \\ 49.58 \end{array}$	$\begin{array}{r} 19.03 \\ 28.19 \end{array}$	$\begin{array}{r} 11.44 \\ 42.25 \end{array}$	$\begin{array}{r} 7.10 \\ 4.29 \end{array}$	$\begin{array}{r} +1.40 \\ \dots \end{array}$	$\begin{array}{r} 6.39 \\ \dots \end{array}$	$\begin{array}{r} 7.79 \\ 4.99 \end{array}$	$\begin{array}{r} 2.10650 \\ 2.09687 \end{array}$	$\begin{array}{r} 4 \\ 4 \end{array}$
22	$\begin{array}{r} 34.08 \\ 43.40 \end{array}$	$\begin{array}{r} 9.32 \\ 36.65 \end{array}$	$\begin{array}{r} 15.77 \\ 39.12 \end{array}$	$\begin{array}{r} 8.19 \\ 5.63 \end{array}$	$\begin{array}{r} +1.28 \\ \dots \end{array}$	$\begin{array}{r} 7.01 \\ \dots \end{array}$	$\begin{array}{r} 8.29 \\ 5.73 \end{array}$	$\begin{array}{r} 2.10819 \\ 2.09944 \end{array}$	$\begin{array}{r} 3\frac{1}{2} \\ 3\frac{1}{2} \end{array}$
23	$\begin{array}{r} 34.29 \\ 43.00 \end{array}$	$\begin{array}{r} 8.71 \\ 38.46 \end{array}$	$\begin{array}{r} 14.80 \\ 36.34 \end{array}$	$\begin{array}{r} 7.57 \\ 4.75 \end{array}$	$\begin{array}{r} +1.41 \\ \dots \end{array}$	$\begin{array}{r} 6.41 \\ \dots \end{array}$	$\begin{array}{r} 7.82 \\ 5.00 \end{array}$	$\begin{array}{r} 2.10660 \\ 2.09691 \end{array}$	$\begin{array}{r} 1\frac{1}{2} \\ 1\frac{1}{2} \end{array}$
Dec. 1	$\begin{array}{r} 36.19 \\ 40.22 \end{array}$	$\begin{array}{r} 4.03 \\ 46.57 \end{array}$	$\begin{array}{r} 17.83 \\ 31.26 \end{array}$	$\begin{array}{r} 5.06 \\ 2.37 \end{array}$	$\begin{array}{r} +1.34 \\ \dots \end{array}$	$\begin{array}{r} 4.70 \\ \dots \end{array}$	$\begin{array}{r} 6.04 \\ 3.36 \end{array}$	$\begin{array}{r} 2.10051 \\ 2.09117 \end{array}$	$\begin{array}{r} 3\frac{1}{2} \\ 3\frac{1}{2} \end{array}$
Dec. 6	$\begin{array}{r} -31^\circ 11' 37.05 \\ +74 33 38.28 \end{array}$	$\begin{array}{r} 43^\circ 22' 1.23 \\ 136 33 49.44 \end{array}$	$\begin{array}{r} 383^\circ 27' 18.63 \\ 246 53 29.19 \end{array}$	$\begin{array}{r} 2' 5.94 \\ 2 3.26 \end{array}$	$\begin{array}{r} +1.34 \\ \dots \end{array}$	$\begin{array}{r} 2' 4.66 \\ \dots \end{array}$	$\begin{array}{r} 2' 6.00 \\ 2 3.32 \end{array}$	$\begin{array}{r} 2.10037 \\ 2.09103 \end{array}$	$\begin{array}{r} 3 \\ 3 \end{array}$

 Stars No. $\left\{ \begin{array}{l} 189 \\ 976 \text{ l. c.} \end{array} \right.$

Date.	δ	$\begin{array}{c} A' \\ B \end{array}$	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	$\log r$	p
1899.									
Oct. 17	$\begin{array}{r} -31^\circ 11' 26.07 \\ +72 11 31.78 \end{array}$	$\begin{array}{r} 41^\circ 0' 5.71 \\ 138 55 30.81 \end{array}$	$\begin{array}{r} 383^\circ 27' 6.64 \\ 244 31 35.83 \end{array}$	$\begin{array}{r} 2' 5.25 \\ 2 18.24 \end{array}$	$\begin{array}{r} -6.49 \\ \dots \end{array}$	$\begin{array}{r} 2' 11.74 \\ \dots \end{array}$	$\begin{array}{r} 2' 5.25 \\ 2 18.23 \end{array}$	$\begin{array}{r} 2.09778 \\ 2.14060 \end{array}$	$\begin{array}{r} 3\frac{1}{2} \\ 3\frac{1}{2} \end{array}$
24	$\begin{array}{r} 27.57 \\ 29.52 \end{array}$	$\begin{array}{r} 1.95 \\ 29.58 \end{array}$	$\begin{array}{r} 6.69 \\ 37.11 \end{array}$	$\begin{array}{r} 7.67 \\ 21.26 \end{array}$	$\begin{array}{r} -6.79 \\ \dots \end{array}$	$\begin{array}{r} 14.23 \\ \dots \end{array}$	$\begin{array}{r} 7.44 \\ 21.02 \end{array}$	$\begin{array}{r} 2.10531 \\ 2.14928 \end{array}$	$\begin{array}{r} 2\frac{1}{2} \\ 2\frac{1}{2} \end{array}$
26	$\begin{array}{r} 27.96 \\ 28.75 \end{array}$	$\begin{array}{r} 41^\circ 0' 0.79 \\ 138 55 31.48 \end{array}$	$\begin{array}{r} 7.51 \\ 36.03 \end{array}$	$\begin{array}{r} 6.50 \\ 20.15 \end{array}$	$\begin{array}{r} -6.82 \\ \dots \end{array}$	$\begin{array}{r} 13.86 \\ \dots \end{array}$	$\begin{array}{r} 7.04 \\ 20.68 \end{array}$	$\begin{array}{r} 2.10394 \\ 2.14823 \end{array}$	$\begin{array}{r} 2\frac{1}{2} \\ 2\frac{1}{2} \end{array}$
31	$\begin{array}{r} 29.16 \\ 27.03 \end{array}$	$\begin{array}{r} 40^\circ 59' 57.87 \\ 138 55 34.96 \end{array}$	$\begin{array}{r} 10.20 \\ 35.24 \end{array}$	$\begin{array}{r} 7.52 \\ 19.26 \end{array}$	$\begin{array}{r} -5.87 \\ \dots \end{array}$	$\begin{array}{r} 13.58 \\ \dots \end{array}$	$\begin{array}{r} 7.71 \\ 19.45 \end{array}$	$\begin{array}{r} 2.10622 \\ 2.14442 \end{array}$	$\begin{array}{r} 1 \\ 1 \end{array}$
Nov. 1	$\begin{array}{r} 29.43 \\ 26.72 \end{array}$	$\begin{array}{r} 57.29 \\ 38.30 \end{array}$	$\begin{array}{r} 10.83 \\ 32.53 \end{array}$	$\begin{array}{r} 5.96 \\ 19.45 \end{array}$	$\begin{array}{r} -6.74 \\ \dots \end{array}$	$\begin{array}{r} 12.20 \\ \dots \end{array}$	$\begin{array}{r} 5.46 \\ 18.94 \end{array}$	$\begin{array}{r} 2.09850 \\ 2.14283 \end{array}$	$\begin{array}{r} 3\frac{1}{2} \\ 3 \end{array}$
6	$\begin{array}{r} 30.55 \\ 24.94 \end{array}$	$\begin{array}{r} 54.39 \\ 37.20 \end{array}$	$\begin{array}{r} 11.44 \\ 34.24 \end{array}$	$\begin{array}{r} 7.10 \\ 20.37 \end{array}$	$\begin{array}{r} -6.63 \\ \dots \end{array}$	$\begin{array}{r} 14.20 \\ \dots \end{array}$	$\begin{array}{r} 7.57 \\ 20.83 \end{array}$	$\begin{array}{r} 2.10575 \\ 2.14869 \end{array}$	$\begin{array}{r} 3\frac{1}{2} \\ 3\frac{1}{2} \end{array}$
22	$\begin{array}{r} 34.08 \\ 18.83 \end{array}$	$\begin{array}{r} 44.75 \\ 44.88 \end{array}$	$\begin{array}{r} 15.77 \\ 30.89 \end{array}$	$\begin{array}{r} 8.19 \\ 22.21 \end{array}$	$\begin{array}{r} -7.01 \\ \dots \end{array}$	$\begin{array}{r} 15.18 \\ \dots \end{array}$	$\begin{array}{r} 8.17 \\ 22.19 \end{array}$	$\begin{array}{r} 2.10779 \\ 2.15287 \end{array}$	$\begin{array}{r} 4 \\ 4 \end{array}$
23	$\begin{array}{r} 34.29 \\ 18.44 \end{array}$	$\begin{array}{r} 44.15 \\ 46.22 \end{array}$	$\begin{array}{r} 14.80 \\ 28.58 \end{array}$	$\begin{array}{r} 7.57 \\ 20.87 \end{array}$	$\begin{array}{r} -6.65 \\ \dots \end{array}$	$\begin{array}{r} 14.81 \\ \dots \end{array}$	$\begin{array}{r} 8.16 \\ 21.46 \end{array}$	$\begin{array}{r} 2.10775 \\ 2.15063 \end{array}$	$\begin{array}{r} 1\frac{1}{2} \\ 1\frac{1}{2} \end{array}$
Dec. 1	$\begin{array}{r} 36.19 \\ 15.59 \end{array}$	$\begin{array}{r} 39.40 \\ 55.72 \end{array}$	$\begin{array}{r} 17.83 \\ 22.11 \end{array}$	$\begin{array}{r} 5.06 \\ 18.48 \end{array}$	$\begin{array}{r} -6.71 \\ \dots \end{array}$	$\begin{array}{r} 12.44 \\ \dots \end{array}$	$\begin{array}{r} 5.73 \\ 19.15 \end{array}$	$\begin{array}{r} 2.09944 \\ 2.14348 \end{array}$	$\begin{array}{r} 3\frac{1}{2} \\ 3\frac{1}{2} \end{array}$
Dec. 6	$\begin{array}{r} -31^\circ 11' 37.05 \\ +72 11 13.62 \end{array}$	$\begin{array}{r} 40^\circ 59' 36.57 \\ 138 55 58.83 \end{array}$	$\begin{array}{r} 383^\circ 27' 18.63 \\ 244 31 19.80 \end{array}$	$\begin{array}{r} 2' 5.94 \\ 2 19.47 \end{array}$	$\begin{array}{r} -6.76 \\ \dots \end{array}$	$\begin{array}{r} 2' 12.30 \\ \dots \end{array}$	$\begin{array}{r} 2' 5.54 \\ 2 19.06 \end{array}$	$\begin{array}{r} 2.09878 \\ 2.14320 \end{array}$	$\begin{array}{r} 3 \\ 3 \end{array}$

Stars No. { 202
 { 911 l. e.

Date.	δ	A' B	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	log r	p
1899.									
Oct. 17	- 29° 22' 41".82 + 76 8 29.76	46° 45' 47".94 133 10 24.60	381° 38' 33".52 248 28 8.92	1' 54".42 1' 53.59	+ 0.41 ..	1' 53".73 ..	1' 54".14 1' 53.32	2.05744 2.05431	4 4
24	43.29 27.20	43.91 24.51	34.17 9.66	56.62 55.79	+ 0.41 ..	55.74 ..	56.15 55.33	2.06502 2.06194	1½ 1½
26	43.65 26.39	42.74 26.61	34.48 7.87	55.96 54.73	+ 0.61 ..	55.32 ..	55.93 54.71	2.06420 2.05960	3 3
31	44.81 24.59	39.78 30.18	38.77 7.59	55.28 55.23	+ 0.02 ..	55.02 ..	55.04 55.00	2.06085 2.06070	2 2
Nov. 1	45.08 24.24	39.16 32.48	37.27 4.79	55.39 54.24	+ 0.57 ..	54.18 ..	54.75 53.61	2.05975 2.05542	3½ 3½
6	46.27 22.34	36.07 32.37	38.12 5.75	56.18 55.25	+ 0.46 ..	55.73 ..	56.19 55.27	2.06517 2.06172	4½ 4½
22	50.04 16.17	26.13 39.42	38 41.61 28 2.19	57.60 56.29	+ 0.65 ..	57.22 ..	57.87 56.57	2.07140 2.06659	4 4
23	50.26 15.78	25.52 42.59	38 42.19 27 59.60	56.75 55.59	+ 0.58 ..	55.94 ..	56.52 55.36	2.06640 2.06205	2 2
Dec. 1	52.36 13.07	20.71 50.99	45.22 54.23	54.52 53.43	+ 0.54 ..	54.15 ..	54.69 53.61	2.05953 2.05542	3 3
Dec. 6	- 29° 22' 53.40 + 76 8 11.20	46 45 17.80 133 10 52.71	381 38 46.31 248 27 53.60	1' 55.33 1' 54.21	+ 0.56 ..	1' 54.74 ..	1' 55.30 1' 54.18	2.06183 2.05759	3 3

Stars No. { 179
 { 944 l. e.

Date.	δ	A' B	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	log r	p
1899.									
Oct. 17	- 32° 49' 23".05 + 74 33 56.81	41° 44' 33".76 138 11 6.62	385° 4' 52".53 246 53 45.91	2' 16".28 2' 2.43	+ 6.92 ..	2' 9".81 ..	2' 16".73 2' 2.89	2.13586 2.08952	4 4
24	24.64 54.36	29.72 7.29	53.88 46.59	19.03 4.97	+ 7.03 ..	11.99 ..	19.02 4.96	2.14308 2.09677	2 2
26	25.03 53.56	28.53 10.04	53.92 43.88	17.83 3.92	+ 6.95 ..	10.71 ..	17.66 3.76	2.13881 2.09258	3 3
31	26.29 51.79	25.50 14.57	58.65 44.08	17.40 3.43	+ 6.98 ..	9.96 ..	16.94 2.98	2.13653 2.08983	1 1
Nov. 1	26.57 51.45	24.88 16.61	57.55 40.94	17.19 3.32	+ 6.93 ..	9.25 ..	16.18 2.32	2.13411 2.08750	4 4
6	27.80 49.58	21.78 15.40	4 57.65 53 42.25	18.29 4.29	+ 7.00 ..	11.41 ..	18.41 4.41	2.14117 2.09485	4 4
22	31.67 43.40	11.73 22.88	5 2.00 53 39.12	19.68 5.63	+ 7.02 ..	12.69 ..	19.71 5.67	2.14523 2.09923	4 4
23	31.90 43.00	11.10 25.18	1.52 36.34	18.69 4.75	+ 6.97 ..	11.81 ..	18.78 4.84	2.14233 2.09635	2 2
Dec. 1	34.01 40.22	6.21 33.67	4.93 31.26	16.12 2.37	+ 6.87 ..	10.06 ..	16.93 3.19	2.13650 2.09057	3 3
Dec. 6	- 32° 49' 35.02 + 74 33 38.28	41 44 3.26 138 11 37.05	385 5 6.26 246 53 29.19	2' 17.11 2' 3.26	+ 6.92 ..	2' 9.84 ..	2' 16.76 2' 2.92	2.13596 2.08962	3 3

Stars No. $\left\{ \begin{array}{l} 179 \\ 976 \text{ l. c.} \end{array} \right.$

Date.	δ	$\frac{A'}{B}$	C'	r'	$\frac{1}{2}(r'_a - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	$\log r$	p
1899.									
Oct. 17	$-32^\circ 49' 23.05$ $+72^\circ 11' 31.78$	$39^\circ 22' 8.73$ $140^\circ 33' 16.70$	$385^\circ 4' 52.53$ $244^\circ 31' 35.83$	$2' 16.28$ $2' 18.24$	-0.98 ..	$2' 17.28$..	$2' 16.30$ $2' 18.26$	2.13450 2.14070	$3\frac{1}{2}$ $3\frac{1}{2}$
24	24.64 29.52	4.88 16.77	53.88 37.11	19.03 21.26	-1.11 ..	19.17 ..	18.06 20.28	2.14007 2.14700	$2\frac{1}{2}$ $2\frac{1}{2}$
26	25.03 28.75	3.72 17.89	53.92 36.03	17.83 20.15	-1.16 ..	19.19 ..	18.03 20.35	2.13997 2.14721	$3\frac{1}{2}$ $3\frac{1}{2}$
31	26.29 27.03	0.74 23.41	58.65 35.24	17.40 19.26	-0.93 ..	17.92 ..	16.99 18.85	2.13669 2.14255	$\frac{1}{2}$ $\frac{1}{2}$
Nov. 1	26.57 26.72	$22^\circ 0.15$ $33^\circ 25.02$	57.55 32.53	17.19 19.45	-1.13 ..	17.41 ..	16.28 18.54	2.13443 2.14157	4 4
6	27.80 24.94	$21^\circ 57.14$ $33^\circ 23.41$	$4^\circ 57.65$ $31^\circ 34.24$	18.29 20.37	-1.04 ..	19.72 ..	18.68 20.76	2.14201 2.14848	$3\frac{1}{2}$ $3\frac{1}{2}$
22	31.67 18.83	47.16 31.11	$5^\circ 2.00$ $31^\circ 30.89$	19.68 22.21	-1.26 ..	20.86 ..	19.60 22.12	2.14488 2.15265	$4\frac{1}{2}$ $4\frac{1}{2}$
23	31.90 18.44	46.54 32.94	1.52 28.58	18.69 20.87	-1.09 ..	20.26 ..	19.17 21.35	2.14355 2.15030	2 2
Dec. 1	34.01 15.59	41.58 42.82	4.93 22.11	16.12 18.48	-1.18 ..	17.75 ..	16.57 18.93	2.13535 2.14280	3 3
Dec. 6	$-32^\circ 49' 35.02$ $+72^\circ 11' 13.62$	$39^\circ 21' 38.60$ $140^\circ 33' 46.36$	$385^\circ 5' 6.26$ $244^\circ 31' 19.90$	$2' 17.11$ $2' 19.47$	-1.18 ..	$2' 17.52$..	$2' 16.34$ $2' 18.70$	2.13462 2.14208	3 3

Stars No. $\left\{ \begin{array}{l} 202 \\ 944 \text{ l. c.} \end{array} \right.$

Date.	δ	$\frac{A'}{B}$	C'	r'	$\frac{1}{2}(r'_a - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	$\log r$	p
1899.									
Oct. 17	$-29^\circ 22' 41.82$ $+74^\circ 33' 56.81$	$45^\circ 11' 14.99$ $134^\circ 44' 47.61$	$381^\circ 38' 33.52$ $246^\circ 53' 45.91$	$1' 54.42$ $2' 2.43$	-4.00 ..	$1' 58.70$..	$1' 54.70$ $2' 2.70$	2.05956 2.08884	4 4
24	43.29 54.36	11.07 47.58	34.17 46.59	56.62 4.97	-4.17 ..	$2^\circ 0.67$..	56.50 4.84	2.06633 2.09635	$1\frac{1}{2}$ $1\frac{1}{2}$
26	43.65 53.56	9.91 50.60	34.48 43.88	55.96 3.92	-3.98 ..	$1^\circ 59.74$..	55.76 3.72	2.06356 2.09244	3 3
31	44.81 51.79	6.98 54.69	38.77 44.08	55.28 3.43	-4.02 ..	59.16 ..	55.14 3.18	2.06123 2.09054	$1\frac{1}{2}$ $1\frac{1}{2}$
Nov. 1	45.08 51.45	6.37 56.33	37.27 40.94	55.39 3.32	-3.96 ..	$1^\circ 58.60$..	54.64 2.56	2.05934 2.08835	4 4
6	46.27 49.58	$11^\circ 3.31$ $44^\circ 55.87$	38.12 42.25	56.18 4.29	-4.05 ..	$2^\circ 0.41$..	56.36 4.46	2.06580 2.09503	4 4
22	50.04 43.40	$10^\circ 53.36$ $45^\circ 2.49$	41.61 39.12	57.60 5.63	-4.01 ..	2.07 ..	58.06 6.08	2.07210 2.10065	$4\frac{1}{2}$ $4\frac{1}{2}$
23	50.26 43.00	52.74 5.85	42.19 36.34	56.75 4.75	-4.00 ..	$2^\circ 0.70$..	56.70 4.70	2.06707 2.09587	2 2
Dec. 1	52.36 40.22	47.86 13.96	45.22 31.26	54.52 2.37	-3.92 ..	$1^\circ 59.09$..	55.17 3.01	2.06134 2.08994	3 3
Dec. 6	$-29^\circ 22' 53.40$ $+74^\circ 33' 38.28$	$45^\circ 10' 44.88$ $134^\circ 45' 17.12$	$381^\circ 38' 46.31$ $246^\circ 53' 29.19$	$1' 55.33$ $2' 3.26$	-3.96 ..	$1^\circ 59.00$..	$1' 55.04$ $2' 2.96$	2.06085 2.08976	3 3

Stars No. { 290
944 l. c.

Date.	δ	A' B	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	log r	p
1899.									
Oct. 17	-30° 45' 48.44 +74 33 56.81	43° 48' 8.37 136 7 45.83	383° 1' 31.74 246 53 45.91	2' 2.27 2 2.43	-0.08 ..	2' 2.90 ..	2' 2.82 2 2.98	2.08927 2.08983	3½ 3½
24	49.93 54.36	4.43 45.43	32.02 46.59	5.21 4.97	+0.12 ..	5.07 ..	5.19 4.95	2.09757 2.09674	2½ 2½
26	50.28 53.56	3.28 49.02	32.90 43.88	3.96 3.92	+0.02 ..	3.85 ..	3.87 3.83	2.09297 2.09283	3½ 3½
31									
Nov. 1	51.67 51.45	48 0.78 7 53.54	34.48 40.94	3.73 3.32	+0.20 ..	2.84 ..	3.04 2.64	2.09005 2.08863	3½ 3½
6	52.95 49.58	47 56.63 7 54.14	36.39 42.25	4.28 4.29	4.61 ..	4.61 4.61	2.09555 2.09555	4 4
22	57.11 43.40	47 46.29 8 2.71	41.83 39.12	5.36 5.63	-0.13 ..	5.50 ..	5.37 5.63	2.09819 2.09909	4 4
23									
Dec. 1	45 59.73 33 40.22	40.49 13.75	45.01 31.26	2.70 2.37	+0.16 ..	2.88 ..	3.04 2.72	2.09005 2.08891	4 4
Dec. 6	-30 46 1.03 +74 33 38.28	43 47 37.25 136 8 16.53	383 1 45.72 246 53 29.19	2 3.51 2 3.26	+0.12 ..	2 3.16 ..	2 3.28 2 3.04	2.09089 2.09005	3 3

Stars No. { 202
1034 l. c.

Date.	δ	A' B	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	log r	p
1899.									
Oct. 17	-29° 22' 41.82 +76 7 58.10	46° 45' 16.28 133 10 56.59	381° 38' 33.52 248 27 36.93	1' 54.42 1 53.33	+0.54 ..	1' 53.56 ..	1' 54.10 1 53.02	2.05729 2.05315	3½ 3½
24	43.29 56.21	12.92 54.86	34.17 39.31	56.62 55.93	+0.34 ..	56.11 ..	56.45 55.77	2.06614 2.06360	2 2
26	43.65 55.54	45 11.89 10 58.04	34.48 36.44	55.96 54.99	+0.48 ..	55.03 ..	55.51 54.55	2.06262 2.05899	3½ 3½
31									
Nov. 1	45.08 53.72	45 8.64 11 1.88	37.27 35.39	55.39 54.57	+0.40 ..	54.74 ..	55.14 54.34	2.06123 2.05820	3½ 3½
6	46.27 52.14	45 5.87 11 3.34	38.12 34.78	56.18 55.12	+0.53 ..	55.39 ..	55.92 54.86	2.06416 2.06017	3½ 3½
22	50.04 46.44	44 56.40 11 9.24	41.61 32.37	57.60 56.26	+0.67 ..	57.18 ..	57.85 56.51	2.07133 2.06636	4½ 4½
23	50.26 46.05	55.79 11.23	42.19 30.96	56.75 55.38	+0.68 ..	56.49 ..	57.17 55.81	2.06882 2.06375	2 2
Dec. 1	52.36 43.25	50.89 20.26	45.22 24.96	54.52 53.63	+0.44 ..	54.42 ..	54.86 53.98	2.06017 2.05683	2½ 2½
Dec. 6	-29 22 53.40 +76 7 41.31	46 44 47.91 133 11 22.57	381 38 46.31 248 27 23.74	1 55.33 1 54.44	+0.44 ..	1 54.76 ..	1 55.20 1 54.32	2.06145 2.05812	2½ 2½

Stars No. $\left\{ \begin{array}{l} 202 \\ 1045 \text{ l. c.} \end{array} \right.$

Date.	δ	A' B	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	log r	p
1899.									
Oct. 17	$-29^\circ 22' 41.82$ $+75^\circ 59' 21.84$	$46^\circ 36' 40.02$ $133^\circ 19' 32.25$	$381^\circ 38' 33.52$ $248^\circ 19' 1.27$	$1^\circ 54.42$ $1^\circ 54.10$	$+0.16$..	$1^\circ 53.86$..	$1^\circ 54.02$ $1^\circ 53.70$	2.05698 2.05576	$3\frac{1}{2}$ $3\frac{1}{2}$
24	43.29 20.01	36.72 30.41	34.17 3.76	56.62 56.77	-0.07 ..	56.43 ..	56.36 56.50	2.06580 2.06633	$2\frac{1}{2}$ $2\frac{1}{2}$
26	43.65 19.36	35.71 34.12	38 34.48 19 0.36	55.96 55.73	+0.11 ..	55.08 ..	55.19 54.97	2.06141 2.06058	$3\frac{1}{2}$ $3\frac{1}{2}$
31									
Nov. 1	45.08 17.56	32.48 37.43	38 37.27 18 59.84	55.39 55.39	55.04 ..	55.04 55.04	2.06085 2.06085	$3\frac{1}{2}$ $3\frac{1}{2}$
6	46.27 16.03	29.76 38.97	38.12 59.15	56.18 55.91	+0.13 ..	55.63 ..	55.76 55.50	2.06356 2.06258	$3\frac{1}{2}$ $3\frac{1}{2}$
22	50.04 10.42	20.38 45.67	41.61 55.94	57.60 57.00	+0.30 ..	56.97 ..	57.27 56.67	2.06919 2.06696	$4\frac{1}{2}$ $4\frac{1}{2}$
23	50.26 10.02	19.76 48.28	42.19 53.91	56.75 56.20	+0.27 ..	55.93 ..	56.20 55.66	2.06521 2.06318	2 2
Dec. 1	52.36 7.24	14.88 56.60	45.22 48.62	54.52 54.43	+0.04 ..	54.26 ..	54.30 54.22	2.05805 2.05774	$2\frac{1}{2}$ $2\frac{1}{2}$
Dec. 6	$-29^\circ 22' 53.40$ $+75^\circ 59' 5.31$	$46^\circ 36' 11.91$ $133^\circ 19' 58.23$	$381^\circ 38' 46.31$ $248^\circ 18' 48.08$	$1^\circ 55.33$ $1^\circ 55.22$	$+0.05$..	$1^\circ 54.93$..	$1^\circ 54.98$ $1^\circ 54.88$	2.06062 2.06024	$2\frac{1}{2}$ $2\frac{1}{2}$

Stars No. $\left\{ \begin{array}{l} 210 \\ 1002 \text{ l. c.} \end{array} \right.$

Date.	δ	A' B	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	log r	p
1899.									
Oct. 17	$-43^\circ 26' 56.56$ $+62^\circ 54' 41.78$	$19^\circ 27' 45.22$ $160^\circ 23' 2.77$	$395^\circ 39' 52.65$ $235^\circ 16' 49.88$	$4^\circ 51.09$ $4^\circ 23.33$	$+13.88$..	$4^\circ 36.00$..	$4^\circ 49.88$ $4^\circ 22.12$	2.46222 2.41850	3 3
24	58.41 39.74	27 41.33 22 55.92	50.61 54.69	57.61 29.26	+14.17 ..	41.37 ..	55.54 27.20	2.47062 2.42684	$1\frac{1}{2}$ $1\frac{1}{2}$
26	26 58.88 54 39.03	27 40.15 23 1.27	52.17 50.90	55.20 27.39	+13.90 ..	39.29 ..	53.19 25.39	2.46715 2.42389	3 3
31									
Nov. 1	27 0.62 54 37.11	36.49 9.36	57.76 48.40	53.71 25.93	+13.89 ..	37.07 ..	50.96 23.18	2.46383 2.42025	3 3
6	2.09 35.44	33.35 9.01	57.01 48.00	55.69 27.44	+14.12 ..	38.82 ..	52.94 24.70	2.46678 2.42275	$3\frac{1}{2}$ $3\frac{1}{2}$
22	6.70 29.54	22.84 9.26	39 58.04 16 48.78	59.67 30.77	+14.45 ..	43.95 ..	58.40 29.50	2.47480 2.43056	4 4
23									
Dec. 1	9.45 26.30	16.85 30.18	40 8.09 16 37.91	51.61 23.94	+13.83 ..	36.48 ..	50.31 22.65	2.46286 2.41938	$2\frac{1}{2}$ $2\frac{1}{2}$
Dec. 6	$-43^\circ 27' 10.70$ $+62^\circ 54' 24.32$	$19^\circ 27' 13.62$ $160^\circ 23' 30.41$	$395^\circ 40' 6.93$ $235^\circ 16' 36.52$	$4^\circ 53.73$ $4^\circ 25.88$	$+13.92$..	$4^\circ 37.98$..	$4^\circ 51.90$ $4^\circ 24.06$	2.46523 2.42170	3 3

Stars No. $\left\{ \begin{array}{l} 210 \\ 1050 \text{ l. c.} \end{array} \right.$

Date.	δ	A' B	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	$\log r$	p
1899.									
Oct. 17	$-43^\circ 26' 56''.56$ $+61 44 39.40$	$18^\circ 17' 42''.84$ $161 32 34.80$	$395^\circ 39' 52''.65$ $234 7 17.85$	$4' 51''.09$ $4 54.79$	-1.85 ..	$4' 51''.18$..	$4' 49''.33$ $4 53.03$	2.46139 2.46691	3 3
24	58.41 37.65	39.24 24.46	50.61 26.15	$4 57.61$ $5 1.84$	-2.11 ..	58.15 ..	$4 56.04$ $5 0.26$	2.47135 2.47750	2 2
26	$26 58.88$ $44 37.01$	38.13 29.90	52.17 22.27	$4 55.20$ $4 59.05$	-1.92 ..	55.98 ..	$4 54.06$ $4 57.90$	2.46844 2.47407	3 3
31									
Nov. 1	$27 0.62$ $44 35.27$	34.65 38.50	57.76 19.26	53.71 58.22	-2.25 ..	53.42 ..	51.17 55.67	2.46415 2.47081	3 3
6	2.09 33.77	31.68 36.67	57.01 20.34	$4 55.69$ $4 59.64$	-1.97 ..	$4 55.82$..	$4 53.85$ $4 57.79$	2.46813 2.47391	$3\frac{1}{2}$ $3\frac{1}{2}$
22	6.70 28.22	$17 21.52$ $32 37.98$	$39 58.04$ $7 20.06$	$4 59.67$ $5 2.56$	-1.44 ..	$5 0.25$..	$4 58.81$ $5 1.69$	2.47539 2.47956	4 4
23									
Dec. 1	9.45 25.05	$17 15.60$ $33 0.44$	$40 8.09$ $7 7.65$	$4 51.61$ $4 55.73$	-2.06 ..	$4 51.98$..	$4 49.92$ $4 54.04$	2.46228 2.46841	$2\frac{1}{2}$ $2\frac{1}{2}$
Dec. 6	$-43 27 10.70$ $+61 44 23.11$	$18 17 12.41$ $161 33 0.17$	$395 40 6.93$ $234 7 6.76$	$4 53.73$ $4 58.12$	-2.19 ..	$4 53.71$..	$4 51.52$ $4 55.90$	2.46467 2.47115	2 2

Stars No. $\left\{ \begin{array}{l} 216 \\ 1002 \text{ l. c.} \end{array} \right.$

Date.	δ	A' B	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	$\log r$	p
1889.									
Oct. 17	$-41^\circ 59' 3''.57$ $+62 54 41.78$	$20^\circ 55' 38''.21$ $158 55 47.40$	$394^\circ 12' 37''.28$ $235 16 49.88$	$4' 13''.38$ $4 23.33$	-4.97 ..	$4' 17''.19$..	$4' 12''.22$ $4 22.16$	2.40178 2.41857	3 3
24	5.41 39.74	34.33 40.60	35.29 54.69	19.01 29.26	-5.12 ..	22.53 ..	17.41 27.65	2.41062 2.42757	2 2
26	5.86 39.03	33.17 45.19	36.09 50.90	16.87 27.39	-5.26 ..	20.82 ..	15.56 26.08	2.40749 2.42501	$3\frac{1}{2}$ $3\frac{1}{2}$
31									
Nov. 1	7.57 37.11	29.54 52.23	40.63 48.40	15.63 25.93	-5.15 ..	19.11 ..	13.96 24.26	2.40476 2.42203	3 3
6	9.05 35.44	26.39 52.03	40.03 48.00	17.28 27.44	-5.08 ..	20.79 ..	15.71 25.87	2.40775 2.42467	$3\frac{1}{2}$ $3\frac{1}{2}$
22	13.65 29.54	15.89 53.82	42.60 48.78	20.81 30.77	-4.98 ..	25.64 ..	20.66 30.62	2.41607 2.43236	4 4
23	13.92 29.15	$55 15.23$ $55 57.52$	43.89 46.37	18.28 28.02	-4.87 ..	23.62 ..	18.75 28.49	2.41288 2.42893	2 2
Dec. 1	16.42 26.30	$55 9.88$ $56 12.96$	50.87 37.91	13.82 23.94	-5.06 ..	18.53 ..	13.47 23.59	2.40393 2.42093	$3\frac{1}{2}$ $3\frac{1}{2}$
Dec. 6	$-41 59 17.69$ $+62 54 24.32$	$20 55 6.63$ $158 56 15.43$	$394 12 51.95$ $235 16 36.52$	$4 15.68$ $4 25.88$	-5.10 ..	$4 18.97$..	$4 13.87$ $4 24.07$	2.40461 2.42172	$3\frac{1}{2}$ $3\frac{1}{2}$

Stars No. $\left\{ \begin{array}{l} 216 \\ 1050 \text{ l. e.} \end{array} \right.$

Date.	δ	A' B	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	log r	p
1899.									
Oct. 17	$-41^\circ 59' 3.57''$ $+61 44 39.40$	$19^\circ 45' 35.83''$ $160 5 19.43$	$394^\circ 12' 37.28''$ $234 7 17.85$	$4' 13.38$ $4 54.79$	-20.70 ..	$4' 32.37$..	$4' 11.67$ $4 53.07$	2.40083 2.46697	3 3
24	5.41 37.65	32.24 9.14	35.29 26.15	$4 19.01$ $5 1.84$	-21.41 ..	39.31 ..	$4 17.90$ $5 0.72$	2.41145 2.47816	$2\frac{1}{2}$ $2\frac{1}{2}$
26	5.86 37.01	32.15 13.82	36.09 22.27	$4 16.87$ $4 59.05$	-21.09 ..	37.01 ..	$4 15.92$ $4 58.10$	2.40810 2.47436	$3\frac{1}{2}$ $3\frac{1}{2}$
31									
Nov. 1	7.57 35.27	27.70 21.37	40.63 19.26	15.63 58.22	-21.29 ..	35.46 ..	14.17 56.75	2.40512 2.47239	3 3
6	9.05 33.77	24.72 19.69	40.03 20.34	$4 17.28$ $4 59.64$	-21.18 ..	37.79 ..	$4 16.61$ $4 58.97$	2.40927 2.47563	$3\frac{1}{2}$ $3\frac{1}{2}$
22	13.65 28.22	14.57 22.54	42.60 20.06	$4 20.81$ $5 2.56$	-20.87 ..	41.44 ..	$4 20.57$ $5 2.31$	2.41592 2.48045	4 4
23	13.92 27.83	13.91 28.87	43.89 15.02	$4 18.28$ $5 0.41$	-21.06 ..	38.61 ..	$4 17.55$ $4 59.67$	2.41086 2.47664	2 2
Dec. 1	16.42 25.05	8.63 43.22	50.87 7.65	$4 13.82$ $4 55.73$	-20.95 ..	34.07 ..	13.12 55.02	2.40333 2.46985	$3\frac{1}{2}$ $3\frac{1}{2}$
Dec. 6	$-41 59 17.69$ $+61 44 23.11$	$19 45 5.42$ $160 5 45.19$	$394 12 51.95$ $234 7 6.76$	$4 15.68$ $4 58.12$	-21.22 ..	$4 34.69$..	$4 13.47$ $4 55.91$	2.40393 2.47116	2 2

Stars No. $\left\{ \begin{array}{l} 228 \\ 1014 \text{ l.e.} \end{array} \right.$

Date.	δ	A' B	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	log r	p
1899.									
Oct. 17	$-50^\circ 42' 52.77''$ $+55 2 8.05$	$4^\circ 19' 15.28''$ $175 13 1.61$	$402^\circ 46' 6.46''$ $227 33 4.85$	$14' 45.74$ $13 22.47$	$+41.63$..	$13' 51.55$..	$14' 33.18$ $13 9.92$	2.94110 2.89758	3 3
24	$42 54.78$ $2 6.18$	$19 11.40$ $12 13.94$	$45 50.85$ $33 36.91$	$15 8.82$ $13 42.17$	$+43.32$..	$14 17.33$..	$15 0.65$ $13 34.01$	2.95456 2.91063	2 2
26									
31									
Nov. 1									
6									
22	$43 3.75$ $1 56.49$	$18 52.74$ $12 16.72$	$45 42.27$ $33 25.55$	$15 18.14$ $13 49.62$	$+44.26$..	$14 25.27$..	$15 9.53$ $13 41.01$	2.95882 2.91435	$4\frac{1}{2}$ $4\frac{1}{2}$
23									
Dec. 1	6.73 53.35	$18 46.62$ $13 16.23$	$46 16.66$ $33 0.43$	$14 49.50$ $13 25.50$	$+42.00$..	$13 58.57$..	$14 40.57$ $13 16.57$	2.94476 2.90122	3 3
Dec. 6	$-50 43 8.11$ $+55 1 51.41$	$4 18 43.30$ $175 13 11.75$	$402 46 15.88$ $227 33 4.13$	$14 58.42$ $13 33.42$	$+42.50$..	$14 2.47$..	$14 44.97$ $13 19.97$	2.94693 2.90307	3 3

Stars No. $\left\{ \begin{array}{l} 270 \\ 1002 \text{ l. c.} \end{array} \right.$

Date.	δ	A' B	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	log r	p
1899.									
Oct. 17	$-42^\circ 32' 14.26''$ $+62 54 41.78$	$20^\circ 22' 27.52''$ $159 28 44.20$	$394^\circ 45' 34.08''$ $235 16 49.88$	$4' 26.46''$ $4 23.33$	$+ 1.56$..	$4' 24.14''$..	$4' 25.70''$ $4 22.58$	2.42439 2.41926	3 3
24	16.06 39.74	23.68 37.91	32.60 54.69	32.68 29.26	$+ 1.71$..	29 20 ..	30.91 27.49	2.43282 2.42731	3 3
26	16.50 39.03	22.53 42.78	33.68 50.90	30.40 27.39	$+ 1.50$..	27.34 ..	28.84 25.84	2.42949 2.42462	$1\frac{1}{2}$ $1\frac{1}{2}$
31									
Nov. 1	18.16 37.11	18.95 49.06	37.46 48.40	29.40 25.93	$+ 1.73$..	25.99 ..	27.72 24.26	2.42768 2.42203	3 3
6	19.66 35.44	15.78 50.83	38.83 48.00	30.69 27.44	$+ 1.62$..	26.69 ..	28.31 25.07	2.42864 2.42336	3 3
22	24.43 29.54	22 5.11 28 52.08	40.86 48.78	33.52 30.77	$+ 1.37$..	31.40 ..	32.77 30.03	2.43580 2.43141	$3\frac{1}{2}$ $3\frac{1}{2}$
23									
Dec. 1	27.37 26.30	21 58.93 29 11.09	49.00 37.91	27 17 23.94	$+ 1.61$..	24.99 ..	26.60 23.38	2.42586 2.42058	3 3
Dec. 6	$-42^\circ 32' 28.80''$ $+62 54 24.32$	$20^\circ 21' 55.52''$ $159 29 12.75$	$394^\circ 45' 49.27''$ $235 16 36.52$	$4' 29.22''$ $4 25.88$	$+ 1.67$..	$4' 25.86''$..	$4' 27.53''$ $4 24.19$	2.42737 2.42192	3 3

Stars No. $\left\{ \begin{array}{l} 297 \\ 1002 \text{ l. c.} \end{array} \right.$

Date.	δ	A' B	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	log r	p
1899.									
Oct. 17	$-42^\circ 3' 2.95''$ $+62 54 41.78$	$20^\circ 51' 38.83''$ $158 59 44.07$	$394^\circ 16' 33.95''$ $235 16 49.88$	$4' 14.82''$ $4 23.33$	$- 4.25$..	$4' 18.55''$..	$4' 14.30''$ $4 22.80$	2.40535 2.41962	3 3
24	4.68 39.74	35.06 37.48	32.17 54.69	21.09 29.26	$- 4.08$..	23.73 ..	19.65 27.81	2.41439 2.42783	$2\frac{1}{2}$ $2\frac{1}{2}$
26	5.09 39.03	34.94 43.48	34.38 50.90	18.34 27.39	$- 4.52$..	20.79 ..	16.27 25.31	2.40870 2.42375	3 3
31									
Nov. 1	6.67 37.11	31.44 49.23	37.63 48.40	18.02 25.93	$- 3.95$..	19.61 ..	15.66 23.56	2.40766 2.42088	3 3
6									
22	12.92 29.54	20 51 17.62 158 59 55.12	43.90 48.78	21.29 30.77	$- 4.74$..	23.63 ..	18.89 28.37	2.41311 2.42873	$4\frac{1}{2}$ $4\frac{1}{2}$
23									
Dec. 1	15.88 26.30	20 51 10.52 159 0 11.03	48.94 37.91	15.76 23.94	$- 4.09$..	19.22 ..	15.13 23.31	2.40676 2.42047	$3\frac{1}{2}$ $3\frac{1}{2}$
Dec. 6	$-42^\circ 3' 17.38''$ $+62 54 24.32$	$20^\circ 51' 6.94''$ $159 0 12.20$	$394^\circ 16' 48.72''$ $233 16 36.52$	$4' 17.54''$ $4 25.88$	$- 4.17$..	$4' 20.43''$..	$4' 16.26''$ $4 24.60$	2.40868 2.42259	3 3

Stars No. $\left\{ \begin{array}{l} 270 \\ 1050 \text{ l. c.} \end{array} \right.$

Date.	δ	$\begin{array}{c} A' \\ B \end{array}$	C'	r'	$\frac{1}{2}(r'_a - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	$\log r$	p
1899.									
Oct. 17	$-42^\circ 32' 14.26''$ $+61^\circ 44' 39.40''$	$19^\circ 12' 25.14''$ $160^\circ 38' 16.23''$	$394^\circ 45' 34.08''$ $234^\circ 7' 17.85''$	$4' 26.46''$ $4' 54.79''$	$-14.16''$..	$4' 39.31''$..	$4' 25.15''$ $4' 53.47''$	2.42349 2.46756	3 3
24	16.06 37.65	21.59 6.45	32.60 26.15	32.68 61.84	-14.58 ..	45.98 ..	31.40 60.56	2.43361 2.47793	3½ 3½
26	16.50 37.01	20.51 11.41	33.68 22.27	30.40 59.05	-14.32 ..	44.04 ..	29.72 58.36	2.43091 2.47474	1½ 1½
31									
Nov. 1	18.16 35.27	17.11 18.20	37.46 19.26	29.40 58.22	-14.41 ..	42.34 ..	27.93 56.75	2.42802 2.47239	3 3
6	19.66 33.77	14.11 18.49	38.83 20.34	30.69 59.64	-14.47 ..	43.70 ..	29.23 58.17	2.43012 2.47446	3 3
22	24.43 28.22	12 3.79 38 20.80	40.86 20.06	33.52 62.56	-14.52 ..	47.70 ..	33.18 62.22	2.43645 2.48042	3½ 3½
23									
Dec. 1	27.37 25.05	11 57.68 38 41.35	49.00 7.65	27.17 55.73	-14.28 ..	40.48 ..	26.20 54.76	2.42521 2.46947	3 3
Dec. 6	$-42^\circ 32' 28.80''$ $+61^\circ 44' 23.11''$	$19^\circ 11' 54.31''$ $160^\circ 38' 42.51''$	$394^\circ 45' 49.27''$ $234^\circ 7' 6.76''$	$4' 29.22''$ $4' 58.12''$	$-14.45''$..	$4' 41.59''$..	$4' 27.14''$ $4' 56.04''$	2.42674 2.47135	2 2

Stars No. $\left\{ \begin{array}{l} 290 \\ 1084 \text{ l. c.} \end{array} \right.$

Date.	δ	$\begin{array}{c} A' \\ B \end{array}$	C'	r'	$\frac{1}{2}(r'_a - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	$\log r$	p
1899.									
Oct. 17	$-30^\circ 45' 48.44''$ $+76^\circ 7' 58.10''$	$45^\circ 22' 9.66''$ $134^\circ 33' 54.81''$	$383^\circ 1' 31.74''$ $248^\circ 27' 36.93''$	$2' 2.27''$ $1' 53.33''$	$+4.47''$..	$1' 57.76''$..	$2' 2.23''$ $1' 53.29''$	2.08718 2.05419	3 3
24	49.93 56.21	6.28 52.71	32.02 39.31	5.21 55.93	+4.64 ..	2 0.50 ..	5.14 55.86	2.09740 2.06393	3 3
26	50.28 55.54	5.26 56.46	32.90 36.44	3.96 54.99	+4.48 ..	1 59.14 ..	3.62 54.66	2.09209 2.05941	4 4
31									
Nov. 1	51.67 53.72	22 2.05 33 59.09	34.48 35.39	3.73 54.57	+4.58 ..	59.43 ..	4.01 54.85	2.09346 2.06013	2 3
6	52.95 52.14	21 59.19 34 1.61	36.39 34.78	4.28 55.12	+4.58 ..	1 59.60 ..	4.18 55.02	2.09405 2.06077	3½ 3½
22	57.11 46.44	49.33 9.46	41.83 32.37	5.36 56.26	+4.55 ..	2 0.60 ..	5.15 56.05	2.09743 2.06464	4 4
23									
Dec. 1	45 59.73 7 43.25	43.52 20.05	45.01 24.96	2.70 53.63	+4.53 ..	1 58.21 ..	2.74 53.68	2.08899 2.05568	3½ 3½
Dec. 6	$-30^\circ 46' 1.03''$ $+76^\circ 7' 41.31''$	$45^\circ 21' 40.28''$ $134^\circ 34' 21.98''$	$383^\circ 1' 45.72''$ $248^\circ 27' 23.74''$	$2' 3.51''$ $1' 54.44''$	$+4.53''$..	$1' 58.87''$..	$2' 3.40''$ $1' 54.34''$	2.09131 2.05820	2½ 2½

Stars No. $\left\{ \begin{array}{l} 290 \\ 1045 \text{ l. e.} \end{array} \right.$

Date.	δ	$\begin{array}{c} A' \\ B \end{array}$	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	$\log r$	p
1899.									
Oct. 17	$\begin{array}{r} -30^\circ 45' 48''.44 \\ +75 59 21.84 \end{array}$	$\begin{array}{r} 45^\circ 13' 33''.40 \\ 134 42 30.47 \end{array}$	$\begin{array}{r} 383^\circ 1' 31''.74 \\ 248 19 1.27 \end{array}$	$\begin{array}{r} 2' 2''.27 \\ 1 54.10 \end{array}$	$\begin{array}{r} + 4''.08 \\ \dots \end{array}$	$\begin{array}{r} 1' 58''.06 \\ \dots \end{array}$	$\begin{array}{r} 2' 2''.14 \\ 1 53.98 \end{array}$	$\begin{array}{r} 2.08686 \\ 2.05683 \end{array}$	$\begin{array}{r} 3 \\ 3 \end{array}$
24	$\begin{array}{r} 49.93 \\ 20.01 \end{array}$	$\begin{array}{r} 30.08 \\ 28.26 \end{array}$	$\begin{array}{r} 32.02 \\ 3.76 \end{array}$	$\begin{array}{r} 5.21 \\ 56.77 \end{array}$	$\begin{array}{r} + 4.22 \\ \dots \end{array}$	$\begin{array}{r} 2 \ 0.83 \\ \dots \end{array}$	$\begin{array}{r} 5.05 \\ 56.61 \end{array}$	$\begin{array}{r} 2.09708 \\ 2.06674 \end{array}$	$\begin{array}{r} 3\frac{1}{2} \\ 3\frac{1}{2} \end{array}$
26	$\begin{array}{r} 50.28 \\ 19.36 \end{array}$	$\begin{array}{r} 29.08 \\ 32.54 \end{array}$	$\begin{array}{r} 1 \ 32.90 \\ 19 \ 0.36 \end{array}$	$\begin{array}{r} 3.96 \\ 55.73 \end{array}$	$\begin{array}{r} + 4.11 \\ \dots \end{array}$	$\begin{array}{r} 1 \ 59.19 \\ \dots \end{array}$	$\begin{array}{r} 3.30 \\ 55.08 \end{array}$	$\begin{array}{r} 2.09096 \\ 2.06100 \end{array}$	$\begin{array}{r} 4 \\ 4 \end{array}$
31									
Nov. 1	$\begin{array}{r} 51.67 \\ 17.56 \end{array}$	$\begin{array}{r} 25.89 \\ 34.64 \end{array}$	$\begin{array}{r} 1 \ 34.48 \\ 18 \ 59.84 \end{array}$	$\begin{array}{r} 3.73 \\ 55.39 \end{array}$	$\begin{array}{r} + 4.17 \\ \dots \end{array}$	$\begin{array}{r} 59.73 \\ \dots \end{array}$	$\begin{array}{r} 3.90 \\ 55.56 \end{array}$	$\begin{array}{r} 2.09307 \\ 2.06281 \end{array}$	$\begin{array}{r} 3 \\ 3 \end{array}$
6	$\begin{array}{r} 52.95 \\ 16.03 \end{array}$	$\begin{array}{r} 24.08 \\ 37.24 \end{array}$	$\begin{array}{r} 36.39 \\ 59.15 \end{array}$	$\begin{array}{r} 4.28 \\ 55.91 \end{array}$	$\begin{array}{r} + 4.18 \\ \dots \end{array}$	$\begin{array}{r} 1 \ 59.34 \\ \dots \end{array}$	$\begin{array}{r} 3.52 \\ 55.16 \end{array}$	$\begin{array}{r} 2.09174 \\ 2.06130 \end{array}$	$\begin{array}{r} 3\frac{1}{2} \\ 3\frac{1}{2} \end{array}$
22	$\begin{array}{r} 57.11 \\ 10.42 \end{array}$	$\begin{array}{r} 13.31 \\ 45.89 \end{array}$	$\begin{array}{r} 41.83 \\ 55.94 \end{array}$	$\begin{array}{r} 5.36 \\ 57.00 \end{array}$	$\begin{array}{r} + 4.18 \\ \dots \end{array}$	$\begin{array}{r} 2 \ 0.40 \\ \dots \end{array}$	$\begin{array}{r} 4.58 \\ 56.22 \end{array}$	$\begin{array}{r} 2.09545 \\ 2.06528 \end{array}$	$\begin{array}{r} 4 \\ 4 \end{array}$
23									
Dec. 1	$\begin{array}{r} 45 \ 59.73 \\ 59 \ 7.24 \end{array}$	$\begin{array}{r} 7.51 \\ 56.39 \end{array}$	$\begin{array}{r} 45.01 \\ 48.62 \end{array}$	$\begin{array}{r} 2.70 \\ 54.43 \end{array}$	$\begin{array}{r} + 4.13 \\ \dots \end{array}$	$\begin{array}{r} 1 \ 57.55 \\ \dots \end{array}$	$\begin{array}{r} 1.68 \\ 53.42 \end{array}$	$\begin{array}{r} 2.08522 \\ 2.05469 \end{array}$	$\begin{array}{r} 3\frac{1}{2} \\ 3\frac{1}{2} \end{array}$
Dec. 6	$\begin{array}{r} -30 \ 46 \ 1.03 \\ +75 \ 59 \ 5.31 \end{array}$	$\begin{array}{r} 45 \ 13 \ 4.28 \\ 134 \ 42 \ 57.64 \end{array}$	$\begin{array}{r} 383 \ 1 \ 45.72 \\ 248 \ 18 \ 48.08 \end{array}$	$\begin{array}{r} 2 \ 3.51 \\ 1 \ 55.22 \end{array}$	$\begin{array}{r} + 4.14 \\ \dots \end{array}$	$\begin{array}{r} 1 \ 59.04 \\ \dots \end{array}$	$\begin{array}{r} 2 \ 3.18 \\ 1 \ 54.90 \end{array}$	$\begin{array}{r} 2.09054 \\ 2.06032 \end{array}$	$\begin{array}{r} 2\frac{1}{2} \\ 2\frac{1}{2} \end{array}$

Stars No. $\left\{ \begin{array}{l} 279 \\ 1050 \text{ l. e.} \end{array} \right.$

Date.	δ	$\begin{array}{c} A' \\ B \end{array}$	C'	r'	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A' + B)$	r	$\log r$	p
1899.									
Oct. 17	$\begin{array}{r} -42^\circ 3' 2''.95 \\ +61 44 39.40 \end{array}$	$\begin{array}{r} 19^\circ 41' 36''.45 \\ 160 \ 9 \ 16.10 \end{array}$	$\begin{array}{r} 394^\circ 16' 33''.95 \\ 234 \ 7 \ 17.85 \end{array}$	$\begin{array}{r} 4' 14''.82 \\ 4 \ 54.79 \end{array}$	$\begin{array}{r} -19''.98 \\ \dots \end{array}$	$\begin{array}{r} 4' 33''.72 \\ \dots \end{array}$	$\begin{array}{r} 4' 13''.74 \\ 4 \ 53.70 \end{array}$	$\begin{array}{r} 2.40439 \\ 2.46790 \end{array}$	$\begin{array}{r} 3 \\ 3 \end{array}$
24	$\begin{array}{r} 4.68 \\ 37.65 \end{array}$	$\begin{array}{r} 32.97 \\ 6.02 \end{array}$	$\begin{array}{r} 32.17 \\ 26.15 \end{array}$	$\begin{array}{r} 21.09 \\ 61.84 \end{array}$	$\begin{array}{r} -20.37 \\ \dots \end{array}$	$\begin{array}{r} 40.50 \\ \dots \end{array}$	$\begin{array}{r} 4 \ 20.13 \\ 5 \ 0.87 \end{array}$	$\begin{array}{r} 2.41519 \\ 2.47838 \end{array}$	$\begin{array}{r} 3 \\ 3 \end{array}$
26	$\begin{array}{r} 5.09 \\ 37.01 \end{array}$	$\begin{array}{r} 31.92 \\ 12.11 \end{array}$	$\begin{array}{r} 34.38 \\ 22.27 \end{array}$	$\begin{array}{r} 18.34 \\ 59.05 \end{array}$	$\begin{array}{r} -20.35 \\ \dots \end{array}$	$\begin{array}{r} 37.98 \\ \dots \end{array}$	$\begin{array}{r} 4 \ 17.63 \\ 4 \ 58.33 \end{array}$	$\begin{array}{r} 2.41100 \\ 2.47470 \end{array}$	$\begin{array}{r} 3 \\ 3 \end{array}$
31									
Nov. 1	$\begin{array}{r} 6.67 \\ 35.27 \end{array}$	$\begin{array}{r} 28.60 \\ 18.37 \end{array}$	$\begin{array}{r} 37.63 \\ 19.26 \end{array}$	$\begin{array}{r} 18.02 \\ 58.22 \end{array}$	$\begin{array}{r} -20.10 \\ \dots \end{array}$	$\begin{array}{r} 36.51 \\ \dots \end{array}$	$\begin{array}{r} 4 \ 16.41 \\ 4 \ 56.61 \end{array}$	$\begin{array}{r} 2.40893 \\ 2.47219 \end{array}$	$\begin{array}{r} 3 \\ 3 \end{array}$
6									
22	$\begin{array}{r} 12.92 \\ 28.22 \end{array}$	$\begin{array}{r} 15.30 \\ 23.84 \end{array}$	$\begin{array}{r} 43.90 \\ 20.06 \end{array}$	$\begin{array}{r} 21.29 \\ 62.56 \end{array}$	$\begin{array}{r} -20.63 \\ \dots \end{array}$	$\begin{array}{r} 40.43 \\ \dots \end{array}$	$\begin{array}{r} 4 \ 19.80 \\ 5 \ 1.06 \end{array}$	$\begin{array}{r} 2.41464 \\ 2.47865 \end{array}$	$\begin{array}{r} 4\frac{1}{2} \\ 4\frac{1}{2} \end{array}$
23									
Dec. 1	$\begin{array}{r} 15.88 \\ 25.05 \end{array}$	$\begin{array}{r} 9.17 \\ 41.29 \end{array}$	$\begin{array}{r} 48.94 \\ 7.65 \end{array}$	$\begin{array}{r} 15.76 \\ 55.73 \end{array}$	$\begin{array}{r} -19.98 \\ \dots \end{array}$	$\begin{array}{r} 34.77 \\ \dots \end{array}$	$\begin{array}{r} 4 \ 14.79 \\ 4 \ 54.75 \end{array}$	$\begin{array}{r} 2.40618 \\ 2.46945 \end{array}$	$\begin{array}{r} 3\frac{1}{2} \\ 3\frac{1}{2} \end{array}$
Dec. 6	$\begin{array}{r} -42 \ 3 \ 17.38 \\ +61 \ 44 \ 23.11 \end{array}$	$\begin{array}{r} 19 \ 41 \ 5.73 \\ 160 \ 9 \ 41.96 \end{array}$	$\begin{array}{r} 394 \ 16 \ 48.72 \\ 234 \ 7 \ 6.76 \end{array}$	$\begin{array}{r} 4 \ 17.54 \\ 4 \ 58.12 \end{array}$	$\begin{array}{r} -20.29 \\ \dots \end{array}$	$\begin{array}{r} 4 \ 36.15 \\ \dots \end{array}$	$\begin{array}{r} 4 \ 15.86 \\ 4 \ 56.44 \end{array}$	$\begin{array}{r} 2.40800 \\ 2.47194 \end{array}$	$\begin{array}{r} 2 \\ 2 \end{array}$

The next table is given as a sample of the reduction for $\Delta = d \log r = d \log a$. The second column contains the logarithms of the computed refractions; the next contains the logarithms of the observed refractions; and the last, the difference between the two preceding in the sense O—C; the column p contains the weights.

In the computation of Δ the $\log r$ of the stars that have been grouped with more than one other star is taken to be the weighted mean of the $\log r$ from all the determinations. The weight assigned to it is the mean of the weights of the individual determinations to the nearest half unit, in all cases where the number to be combined is the same for every date. In cases where different dates have different numbers combined the mean of the weights is multiplied by the square root of the number of determinations taken into account, and this product, taken to the nearest half unit, is the adopted weight.

Star No. 5.

Date.	$\log r'$	$\log r$	Δ	p
Oct. 17	2.60700	2.60307	-0.00393	5
24	1694	1317	377	3
26	1264	0981	283	4½
31	1731	1556	175	2
Nov. 1	1091	0712	379	4½
6	1541	1367	174	5
22	1964	1699	265	5½
23	1784	1369	415	1½
Dec. 1	0762	0467	295	4
Dec. 6	2.61216	2.60811	-0.00405	3

$$[p] = 38$$

$$\Delta = -0.00310 \pm 0.00018$$

$$\log [p \sqrt{v}] = 5.4101$$

The values of Δ are collected in the following table. The weight for each Δ has been derived from the probable error r of the Δ . The approximate zenith distance of each star is also given.

Star No.	Δ	r	p	Z. D.	Star No.	Δ	r	p	Z. D.
5	-0.00310	± 0.00018	12	84°	174	-	28	39	52
9	-		23	73	179	-	81	26	70
788	-		10	84	944	+	40	31	68
793	-		20	72	187	-	198	23	78
38	-		5	84	190	+	1275	147	16
49	+		2	30	202	-	58	20	67
58	+		5	67	210	-	263	19	81
66	-		8	84	976	-	13	33	70
76	+		1	30	216	-	177	24	79
79	+		1	34	223	-	501	37	88
83	+		3	30	235	+	238	29	47
89	-		3	51	1002	-	232	25	80
96	+		2	35	1014	-	554	40	87
103	+		1	30	1019	-	329	21	84
108	+		2	29	264	+	240	32	48
885	-		13	78	270	-	208	20	80
136	+		1	29	1034	-	27	20	66
139	+		5	68	1045	-	167	18	67
149	+		1	30	1050	-	182	14	81
153	+		1	29	290	-	64	13	68
911	-		8	66	297	-0.00235	± 0.00021	9	79
161	+		1:7	16					

It will be noticed that for this series also, Δ is clearly a function of the zenith distance, thus confirming the result found from the reduction of the summer series that the Constant of Refraction is a function of the zenith distance. Upon plotting these values, using the zenith distance z for abscissa, and Δ for ordinate, it is seen that a straight line, inclined about 145° to the z axis, and cutting it at $z =$ about 60° appears to represent Δ fairly well. Accordingly, assuming Z to be the zenith distance for $\Delta = 0$, we can set up an observation equation of the following type for each star,

$$\log a = \log a_0 + (Z - z)x$$

or

$$\log a - \log a_0 = \Delta = Zx - zx = D - zx$$

where

$$D = Zx$$

and where a_0 is the Constant of Refraction of the tables used (Pulkowa).

Equations of this kind were, therefore, formed and solved for Z and x by the method of Least Squares. The details of the solution for the autumn series alone will not be given here. The result of the solution is

$$x = +0.000117 \pm 0.000011$$

$$Z = 59^\circ.3 \pm 8^\circ.7$$

The solution of the summer series alone gave

$$x = +0.000101 \pm 0.000013$$

$$Z = 56^\circ.6 \pm 12^\circ.2$$

The details of the solution for the two series combined are here given. To reduce the number of equations those nearly alike were combined into one equation, and given a weight equal to the sum of the weights of the equations combined.

This leads to the following equations of condition:

No.	a	b	n	p
1	D	$-79.63 x$	$= -0.00191$	138
2		-29.92	$= + 368$	16
3		-87.10	$= - 352$	32
4		-67.29	$= - 56$	94
5		-83.57	$= - 296$	167
6		-88.78	$= - 485$	10
7		-21.47	$= + 385$	7
8		-57.31	$= - 117$	29
9		-27.15	$= - 103$	5
10		-62.75	$= - 7$	25
11		-65.34	$= + 41$	18
12		-72.65	$= - 241$	43
13		-34.86	$= + 187$	3
14		-51.51	$= - 27$	6
15		-77.83	$= - 200$	20
16		-15.78	$= + 1266$	0.31
17		-70.25	$= - 54$	10
18	D	$-47.65 x$	$= +0.00239$	9

WEIGHTED OBSERVATION EQUATIONS.

No.	a	b	n
1	11.75 <i>D</i>	- 935.7 <i>x</i>	= -0.02244
2	4.00	- 119.7	= + 1472
3	5.66	- 493.0	= - 1992
4	9.70	- 652.7	= - 543
5	12.92	- 1079.7	= - 3824
6	3.16	- 280.5	= - 1533
7	2.65	- 56.9	= + 1020
8	5.39	- 308.9	= - 631
9	2.24	- 60.8	= - 231
10	5.00	- 313.7	= - 35
11	4.24	- 277.0	= + 174
12	6.56	- 476.6	= - 1581
13	1.73	- 60.3	= + 324
14	2.45	- 126.2	= - 66
15	4.47	- 347.9	= - 894
16	0.56	- 8.8	= + 709
17	3.16	- 222.0	= - 171
18	3.00 <i>D</i>	- 142.9 <i>x</i>	= +0.00717

To render these more nearly homogeneous, let $10D = d$; $1000x = y$; and multiply the absolute term by 100.

WEIGHTED HOMOGENEOUS OBSERVATION EQUATIONS.

No.	a	b	n
1	1.175 <i>d</i>	- 0.936 <i>y</i>	= - 2.244
2	0.400	- 0.120	= + 1.472
3	0.566	- 0.493	= - 1.992
4	0.970	- 0.653	= - 0.543
5	1.292	- 1.080	= - 3.824
6	0.316	- 0.280	= - 1.533
7	0.265	- 0.057	= + 1.020
8	0.539	- 0.309	= - 0.631
9	0.224	- 0.061	= - 0.231
10	0.500	- 0.314	= - 0.035
11	0.424	- 0.277	= + 0.174
12	0.656	- 0.477	= - 1.581
13	0.173	- 0.060	= + 0.324
14	0.245	- 0.126	= - 0.066
15	0.447	- 0.348	= - 0.894
16	0.056	- 0.009	= + 0.709
17	0.316	- 0.222	= - 0.171
18	0.300 <i>d</i>	- 0.143	= + 0.717

Combining these by the method of Least Squares we obtain the following

NORMAL EQUATIONS.

$$+ 6.325d - 4.637y = - 10.389$$

$$- 4.637 + 3.520 = + 8.916$$

The solution of these gives

$$x = + 0.000108 \pm 0.000009$$

$$Z = 58^{\circ}.1 \pm 7^{\circ}.8$$

It is evident, therefore, that the Constant of Refraction used in the Pulkowa tables needs not only a correction, but a correction for every zenith distance. In other words, the formulæ from which the refractions are computed, and hence the theory also, need to be modified. Or, the formulæ may be retained unchanged and the desired agreement between observation and computation may be obtained by correcting the tables used (Pulkowa) not by a constant amount, but by an amount represented in magnitude by

$$\Delta \mu = + 0.000108 (58^{\circ} - z)$$

where z is the zenith distance in degrees. The table for $\mu + \log \operatorname{tg} z$ given hereafter is the Pulkowa table corrected by this amount.

The result of this investigation leads to an interesting question, viz., Could not the different values of the Constant of Refraction determined by various observers be brought into harmony by reducing them to the same zenith distance

by this formula? The average zenith distance of the stars used in these determinations are undoubtedly different in every case, and this may account for the varying results. This is an investigation which the writer hopes to undertake as soon as the data and the time are available.

The reductions for latitude and a readjustment of the declinations of the stars used in these two series, based upon the observations made, will form the subject matter of a later paper.

In his preliminary reductions of his observations of the PIAZZI southern stars TUCKER (*Publications of the Lick Observatory*, Vol. VI) has used the Pulkowa refractions. In his final reductions, he has corrected the declinations for errors in refraction by the amounts exhibited below

δ	Z. D.	$\Delta \delta$
-20°	57	+ 0.07
-25	62	+ 0.14
-30	67	+ 0.24
-35	72	+ 0.39
-40	77	+ 0.65
-45	82	+ 2.44

Applying to these AUWERS' corrections (*Abhandlungen zu den A. N.* No. 7) we would have, for the same declinations, the amounts exhibited in the second column below.

Z. D.	$\Delta \delta$	$\Delta \delta$	A - C	p. e.
57°	+ 0.07	- 0.02	+ 0.09	± 0.25
62	+ 0.12	+ 0.10	+ 0.02	0.26
67	+ 0.19	+ 0.29	- 0.10	0.37
72	+ 0.41	+ 0.60	- 0.19	0.49
77	+ 0.90	+ 1.15	- 0.25	0.77
82	+ 2.84	+ 2.29	+ 0.55	± 1.00

In the third column are the corrections which would have resulted by applying the correction, found on page 198, to the Pulkowa tables. The fourth column is the difference between the second and third (in the sense AUWERS-CRAWFORD). The fifth column contains the probable errors of a single observation given by TUCKER (Vol. VI *Publication of the Lick Observatory*) for stars observed at zenith distances indicated in the first column. This table shows conclusively that had this series of PIAZZI stars been reduced originally with the refractions found from this investigation, any error in the observed positions due to errors of refraction would have been much less than the probable error of the observation itself.



REFRACTION TABLES.

REFRACTION TABLES.

In order that all the refraction tables may be conveniently together the three tables giving the terms B , γ and T published in Volume I of these *Publications* are given again here as Tables I, II, III respectively. Table IV gives A , λ and σ . Table V gives i . [Tables IV and V are reprinted from the Pulkowa Tables.] Table VI is the Pulkowa table for $\mu + \log \operatorname{tg} z$ corrected by

$$\Delta \mu = +0.000108 (58^\circ - z)$$

where z is in degrees. The refractions are then to be computed from these tables by the formula

$$\log \operatorname{Refr.} = \mu + \log \operatorname{tg} z + A (B + T) + \lambda \gamma - \sigma i.$$

R. T. C.

TABLE I.
VALUES OF THE REFRACTION TERM B.
(Units of the fifth decimal place.)

Bar. Inches.	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09	With left hand col.
25.0	-7330	-7313	-7296	-7278	-7261	-7243	-7226	-7209	-7191	-7174	20 × Bar. 500
.1	7157	7140	7122	7105	7088	7070	7053	7036	7019	7001	502
.2	6984	6967	6950	6933	6915	6898	6881	6864	6847	6829	504
.3	6812	6795	6778	6761	6744	6726	6709	6692	6675	6658	506
.4	6641	6624	6607	6590	6573	6555	6538	6521	6504	6487	508
.5	-6470	-6453	-6436	-6419	-6402	-6385	-6368	-6351	-6334	-6317	510
.6	6300	6283	6266	6249	6232	6215	6199	6182	6165	6148	512
.7	6131	6114	6097	6080	6063	6046	6030	6013	5996	5979	514
.8	5962	5945	5929	5912	5895	5878	5861	5845	5828	5811	516
.9	5794	5777	5761	5744	5727	5711	5694	5677	5660	5644	518
26.0	-5627	-5610	-5594	-5577	-5560	-5543	-5527	-5510	-5493	-5477	520
.1	5460	5444	5427	5410	5394	5377	5360	5344	5327	5311	522
.2	5294	5278	5261	5244	5228	5211	5195	5178	5162	5145	524
.3	5129	5112	5096	5079	5063	5046	5030	5013	4997	4980	526
.4	4964	4947	4931	4915	4898	4882	4865	4849	4832	4816	528
.5	-4800	-4783	-4767	-4751	-4734	-4718	-4701	-4685	-4669	-4652	530
.6	4636	4620	4603	4587	4571	4555	4538	4522	4506	4489	532
.7	4473	4457	4441	4424	4408	4392	4376	4359	4343	4327	534
.8	4311	4295	4278	4262	4246	4230	4214	4197	4181	4165	536
.9	4149	4133	4117	4101	4084	4068	4052	4036	4020	4004	538
27.0	-3988	-3972	-3956	-3940	-3923	-3907	-3891	-3875	-3859	-3843	540
.1	3827	3811	3795	3779	3763	3747	3731	3715	3699	3683	542
.2	3667	3651	3635	3619	3603	3587	3572	3556	3540	3524	544
.3	3508	3492	3476	3460	3444	3428	3413	3397	3381	3365	546
.4	3349	3333	3317	3302	3286	3270	3254	3238	3222	3207	548
.5	-3191	-3175	-3159	-3144	-3128	-3112	-3096	-3080	-3065	-3049	550
.6	3033	3018	3002	2986	2970	2955	2939	2923	2908	2892	552
.7	2876	2860	2845	2829	2814	2798	2782	2767	2751	2735	554
.8	2720	2704	2689	2673	2657	2642	2626	2610	2595	2579	556
.9	2564	2548	2533	2517	2502	2486	2470	2455	2439	2424	558
28.0	-2408	-2393	-2377	-2362	-2346	-2331	-2315	-2300	-2285	-2269	560
With right hand col.	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	20 × Bar.

N. B.—The left hand column and top argument give the whole argument for a barometer reading in inches; the right hand column and bottom argument give the entry for 20 times the barometer reading in inches.

TABLE II.

VALUES OF THE REFRACTION TERM γ .

(Units of the fifth decimal place.)

t. Fahr.	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0°	+4412	+4318	+4222	+4128	+4032	+3938	+3846	+3750	+3658	+3562	9	19	28	37	47	56	66	75	84
10	3470	3376	3284	3190	3099	3008	2914	2823	2730	2640	9	18	28	37	46	55	64	74	83
20	2547	2457	2365	2275	2184	2094	2004	1913	1824	1734	9	18	27	36	45	54	63	72	81
30	1645	1554	1466	1378	1288	1201	1111	1023	934	847	9	18	27	35	44	53	62	71	80
40	+ 759	+ 672	+ 586	+ 497	+ 411	+ 324	+ 238	+ 150	+ 65	- 22	9	17	26	35	43	52	61	69	78
50	- 107	- 193	- 279	- 364	- 450	- 534	- 620	- 704	- 789	- 873	9	17	26	34	43	51	60	68	77
60	957	1042	1125	1209	1293	1377	1460	1543	1626	1708	8	17	25	33	42	50	58	67	75
70	1791	1873	1956	2038	2121	2202	2285	2366	2446	2528	8	16	24	33	41	49	57	65	73
80	2609	2691	2771	2852	2932	3014	3093	3172	3253	3334	8	16	24	32	40	48	56	64	72
90	-3414	-3493	-3573	-3652	-3731	-3810	-3889	-3967	-4046	-4124	8	16	23	31	39	47	55	63	70

TABLE III.

VALUES OF REFRACTION TERM T.

(Units of the fifth decimal place.)

t. (Fahr.)	0	1	2	3	4	5	6	7	8	9
0°	+125	+121	+117	+113	+109	+105	+101	+ 98	+ 94	+ 90
10	+ 86	+ 82	+ 78	+ 74	+ 70	+ 66	+ 62	+ 58	+ 55	+ 51
20	+ 47	+ 43	+ 39	+ 35	+ 31	+ 27	+ 23	+ 20	+ 16	+ 12
30	+ 8	+ 4	0	- 4	- 8	- 12	- 16	- 20	- 23	- 27
40	- 31	- 35	- 39	- 43	- 47	- 51	- 55	- 58	- 62	- 66
50	- 70	- 74	- 78	- 82	- 86	- 90	- 93	- 97	-101	-105
60	-109	-113	-117	-121	-124	-128	-132	-136	-140	-144
70	-148	-152	-156	-160	-164	-168	-171	-175	-179	-183
80	-187	-191	-195	-199	-202	-206	-210	-214	-218	-221
90	-225	-229	-233	-237	-241	-244	-248	-252	-256	-260

TABLE IV.

VALUES OF THE REFRACTION TERMS λ , A AND σ .

z	λ	A	σ	z	λ	A	σ
45°	1.0018			85° 0'	1.1235	1.0127	0.00146
50	1.0022			5	1264		
55	1.0029			10	1294		
60	1.0044			15	1325		
61	1.0047			20	1357	141	170
62	1.0051			25	1390		
63	1.0055			30	1424		
64	1.0059			35	1459		
65	1.0064			40	1495	156	202
66	1.0070			45	1532		
67	1.0077			50	1571		
68	1.0085			55	1611		
69	1.0093			86 0	1.1652	1.0172	0.00241
70	1.0103			5	1695		
71	1.0115			10	1740	181	264
72	1.0130			15	1786		
73	1.0147			20	1833	192	290
74	1.0166			25	1882		
75 0'	1.0188			30	1934	203	320
75 30	1.0200			35	1987		
76 0	1.0216			40	2040	214	352
76 30	1.0235			45	2095		
77 0	1.0253	1.0029		50	2153	227	386
77 30	1.0271	1.0030		55	2214		
78 0	1.0293	1.0033		87 0	1.2277	1.0241	0.00421
78 30	1.0318	1.0035		2	2303		
79 0	1.0344	1.0038		4	2329	247	437
79 30	1.0374	1.0041		6	2356		
80 0	1.0409	1.0044	0.00019	8	2383	254	453
10	1.0421			10	2410		
20	1.0433			12	2438	261	471
30	1.0447	1.0048	0.00022	14	2466		
40	1.0461			16	2495	268	489
50	1.0475			18	2524		
81 0	1.0491	1.0052	0.00025	20	2554	275	509
10	1.0508			22	2584		
20	1.0525			24	2615	282	529
30	1.0542	1.0057	0.00029	26	2646		
40	1.0561			28	2677	290	550
50	1.0580			30	2708		
82 0	1.0600	1.0063	0.00035	32	2740	297	572
10	1.0622			34	2772		
20	1.0645			36	2805	305	594
30	1.0669	1.0070	0.00045	38	2839		
40	1.0694			40	2873	313	617
50	1.0720			42	2908		
83 0	1.0747	1.0078	0.00057	44	2942	321	641
10	1.0776			46	2978		
20	1.0807			48	3014	328	666
30	1.0839	0.0087	0.00073	50	3051		
40	1.0874			52	3088	338	693
50	1.0911			54	3125		
84 0	1.0949	1.0098	0.00091	56	3163	347	721
10	1.0990			58	3202		
20	1.1034			87 58			
30	1.1080	1.0112	0.00116	88 0	1.3241	1.0357	0.00749
40	1.1128						
50	1.1180						
85 0	1.1235	1.0127	0.00146				

TABLE VI.

 $\mu + \log \tan z$.

z	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°	z
0'	$-\infty$	0.0083	0.3094	0.4856	0.6108	0.7080	0.7875	0.8550	0.9135	0.9653	0'
1	8.230	155	130	880	126	094	888	560	144	661	1
2	.531	226	166	904	144	109	900	571	153	669	2
3	.707	295	201	928	162	123	912	581	162	678	3
4	8.832	0.0364	0.3236	0.4952	.06180	0.7138	0.7924	0.8591	0.9172	0.9686	4
5	8.929	0.0431	0.3271	0.4975	0.6198	0.7152	0.7936	0.8602	0.9181	0.9694	5
6	9.008	497	306	0.4999	215	166	948	612	190	702	6
7	.075	563	341	0.5022	233	180	960	622	199	710	7
8	.133	627	375	045	250	195	972	632	208	718	8
9	9.184	0.0691	0.3408	0.5068	0.6268	0.7209	0.7983	0.8643	0.9217	0.9726	9
10	9.230	0.0753	0.3442	0.5091	0.6285	0.7223	0.7995	0.8653	0.9226	0.9734	10
11	.271	815	475	114	303	237	0.8007	663	235	742	11
12	.309	875	508	137	320	251	019	673	244	750	12
13	.344	935	541	160	338	265	031	683	253	758	13
14	9.376	0.0994	0.3574	0.5182	0.6355	0.7279	0.8042	0.8694	0.9262	0.9766	14
15	9.406	0.1053	0.3606	0.5205	0.6372	0.7293	0.8054	0.8704	0.9271	0.9774	15
16	.434	110	638	227	389	307	066	714	279	782	16
17	.460	167	670	249	406	320	077	724	288	790	17
18	.485	223	702	271	423	334	089	734	297	798	18
19	9.509	0.1279	0.3733	0.5293	0.6440	0.7348	0.8101	0.8744	0.9306	0.9806	19
20	9.531	0.1333	0.3764	0.5315	0.6457	0.7362	0.8112	0.8754	0.9315	0.9814	20
21	.552	387	795	336	473	375	124	764	324	822	21
22	.572	440	826	358	490	389	135	774	332	829	22
23	.592	493	856	380	507	402	147	784	341	836	23
24	9.610	0.1545	0.3887	0.5401	0.6523	0.7416	0.8158	0.8794	0.9350	0.9844	24
25	9.628	0.1596	0.3917	0.5422	0.6540	0.7429	0.8169	0.8803	0.9359	0.9852	25
26	.645	647	947	443	556	443	181	813	367	860	26
27	.661	608	0.3976	464	572	456	192	823	376	868	27
28	.677	747	0.4006	485	589	469	203	833	384	876	28
29	9.692	0.1796	0.4035	0.5506	0.6605	0.7483	0.8215	0.8843	0.9392	0.9883	29
30	9.707	0.1845	0.4064	0.5527	0.6621	0.7496	0.8226	0.8852	0.9401	0.9891	30
31	.721	893	093	548	637	509	237	862	410	899	31
32	.735	940	122	568	653	522	248	872	418	907	32
33	.749	0.1987	150	589	669	535	259	881	427	914	33
34	9.762	0.2034	0.4179	0.5609	0.6685	0.7549	0.8271	0.8890	0.9436	0.9922	34
35	9.774	0.2080	0.4207	0.5630	0.6701	0.7562	0.8282	0.8900	0.9444	0.9930	35
36	.786	125	235	650	717	575	292	910	453	937	36
37	.798	170	263	670	733	588	303	919	461	945	37
38	.810	215	290	690	748	601	314	929	470	953	38
39	9.821	0.2259	0.4317	0.5710	0.6764	0.7613	0.8325	0.8938	0.9478	0.9961	39
40	9.832	0.2303	0.4345	0.5730	0.6780	0.7625	0.8336	0.8948	0.9487	0.9968	40
41	.843	346	372	749	795	638	347	957	495	976	41
42	.853	389	399	769	811	651	358	967	504	983	42
43	.864	431	426	789	826	664	369	976	512	991	43
44	9.874	0.2473	0.4452	0.5808	0.6841	0.7677	0.8379	0.8986	0.9520	0.9999	44
45	9.883	0.2515	0.4479	0.5828	0.6856	0.7689	0.8390	0.8995	0.9529	1.0006	45
46	.893	556	505	847	871	702	401	0.9005	537	014	46
47	.902	597	531	866	886	714	412	014	546	021	47
48	.911	637	557	885	902	727	422	024	554	029	48
49	9.920	0.2677	0.4583	0.5904	0.6917	0.7740	0.8433	0.9033	0.9562	1.0036	49
50	9.929	0.2717	0.4608	0.5922	0.6932	0.7752	0.8444	0.9043	0.9571	1.0044	50
51	.938	756	634	941	947	765	455	052	579	051	51
52	.946	795	659	960	962	777	465	061	587	059	52
53	.954	834	685	979	977	790	476	071	595	066	53
54	9.962	0.2872	0.4709	0.5997	0.6991	0.7802	0.8486	0.9080	0.9604	1.0074	54
55	9.970	0.2909	0.4734	0.6016	0.7006	0.7814	0.8497	0.9089	0.9612	1.0081	55
56	.978	947	758	034	021	827	508	098	620	089	56
57	.986	0.2984	783	053	036	839	518	107	629	096	57
58	9.994	0.3021	808	071	050	851	529	117	637	104	58
59	0.0011	0.3058	0.4832	0.6089	0.7065	0.7863	0.8539	0.9126	0.9645	1.0111	59
60	0.0083	0.3094	0.4856	0.6108	0.7080	0.7875	0.8550	0.9135	0.9653	1.0118	60

TABLE VI.

 $\mu + \log \tan z$.

z	10°	11°	12°	13°	14°	15°	16°	17°	18°	19°	z
0'	1.0118	1.0540	1.0928	1.1286	1.1618	1.1929	1.2223	1.2500	1.2764	1.3014	0'
1	126	547	934	291	623	934	227	505	768	018	1
2	133	554	940	297	628	940	232	509	772	023	2
3	140	561	946	303	634	945	237	514	776	027	3
4	148	567	952	309	639	950	242	518	781	031	4
5	155	574	959	314	644	955	246	523	785	035	5
6	162	581	965	319	650	960	251	527	789	039	6
7	170	587	971	325	655	965	256	532	794	043	7
8	177	594	977	330	660	970	261	536	798	047	8
9	1.0184	1.0601	1.0983	1.1336	1.1666	1.1975	1.2266	1.2541	1.2802	1.3051	9
10	192	1.0607	1.0989	1.1342	1.1671	1.1980	1.2270	1.2545	1.2806	055	10
11	199	614	1.0995	348	676	985	275	550	811	059	11
12	206	620	1.1001	353	682	990	280	554	815	063	12
13	213	626	007	359	687	1.1995	284	559	819	067	13
14	221	633	013	365	692	1.2000	289	563	823	072	14
15	228	640	019	370	698	005	294	567	828	076	15
16	235	646	025	376	703	010	299	572	832	080	16
17	242	653	031	382	708	015	303	576	836	084	17
18	249	659	037	387	714	020	308	581	840	088	18
19	1.0256	1.0666	1.1043	1.1393	1.1719	1.2025	1.2313	1.2585	1.2845	1.3092	19
20	1.0263	1.0673	1.1049	1.1399	1.1724	1.2030	1.2317	1.2590	1.2849	1.3096	20
21	270	679	055	404	729	034	322	594	853	100	21
22	277	686	061	410	735	039	327	599	857	104	22
23	284	692	067	415	740	044	331	603	861	108	23
24	291	699	073	421	745	049	336	607	866	112	24
25	298	705	079	427	750	054	341	612	870	116	25
26	305	712	086	432	756	059	345	616	874	120	26
27	312	718	092	438	761	064	350	621	878	124	27
28	319	725	098	443	766	069	355	625	883	128	28
29	1.0327	1.0731	1.1104	1.1449	1.1771	1.2074	1.2359	1.2630	1.2887	1.3132	29
30	1.0334	1.0738	1.1109	1.1454	1.1776	1.2079	1.2364	1.2634	1.2891	1.3136	30
31	341	744	115	460	782	084	369	638	895	140	31
32	348	751	121	466	787	089	373	643	899	144	32
33	355	757	127	471	792	093	378	647	904	147	33
34	362	763	133	477	797	098	382	652	908	151	34
35	369	770	139	482	803	103	387	656	912	155	35
36	376	776	145	488	808	108	392	660	916	159	36
37	383	783	151	493	813	113	396	665	920	163	37
38	390	789	157	499	818	118	401	669	924	167	38
39	1.0397	1.0795	1.1163	1.1504	1.1823	1.2123	1.2405	1.2673	1.2928	1.3171	39
40	1.0404	1.0802	1.1169	1.1510	1.1828	1.2128	1.2410	1.2678	1.2932	1.3175	40
41	411	808	175	515	833	132	415	682	936	179	41
42	418	815	181	521	839	137	419	687	940	183	42
43	424	821	187	526	844	142	424	691	944	187	43
44	431	827	192	532	849	147	428	695	948	191	44
45	438	834	198	537	854	152	433	699	952	195	45
46	445	840	204	543	859	157	437	703	957	199	46
47	452	846	210	548	864	161	442	707	961	203	47
48	459	853	216	554	869	166	447	712	965	207	48
49	1.0466	1.0859	1.1222	1.1559	1.1874	1.2171	1.2451	1.2716	1.2969	1.3211	49
50	1.0473	1.0865	1.1228	1.1565	1.1879	1.2176	1.2455	1.2720	1.2973	1.3215	50
51	479	871	233	570	885	181	459	725	977	219	51
52	486	878	239	575	890	186	464	729	981	223	52
53	493	884	245	581	895	190	468	733	986	227	53
54	500	890	251	586	900	194	473	738	990	230	54
55	507	897	257	592	904	199	477	742	994	234	55
56	514	903	262	597	909	204	482	746	1.2998	238	56
57	520	909	268	602	914	208	486	750	1.3002	242	57
58	527	915	274	608	919	213	491	755	006	246	58
59	1.0534	1.0921	1.1280	1.1613	1.1924	1.2218	1.2496	1.2759	1.3010	1.3250	59
60	1.0540	1.0928	1.1286	1.1618	1.1929	1.2223	1.2500	1.2764	1.3014	1.3254	60

TABLE VI.

 $\mu + \log \tan \alpha.$

α	20°	21°	22°	23°	24°	25°	26°	27°	28°	29°	α
0'	1.3254	1.3484	1.3705	1.3919	1.4125	1.4325	1.4519	1.4707	1.4891	1.5070	0'
1	258	488	709	922	128	328	522	710	894	073	1
2	262	492	713	926	132	331	525	713	897	076	2
3	266	495	716	929	135	335	528	716	900	079	3
4	270	499	720	933	139	338	531	719	903	082	4
5	274	503	724	936	142	341	534	722	906	085	5
6	278	507	727	940	145	345	537	725	909	088	6
7	282	511	731	943	149	348	540	728	912	091	7
8	286	514	734	947	152	351	543	732	915	094	8
9	1.3289	1.3518	1.3738	1.3950	1.4156	1.4354	1.4547	1.4735	1.4918	1.5097	9
10	1.3293	1.3522	1.3742	1.3954	1.4159	1.4357	1.4550	1.4738	1.4921	1.5100	10
11	297	526	745	957	162	360	553	741	924	103	11
12	301	529	749	961	165	363	556	744	927	106	12
13	305	533	753	964	168	367	559	747	930	109	13
14	309	537	756	968	172	370	563	750	933	112	14
15	313	540	760	971	175	373	566	753	936	115	15
16	317	544	763	974	178	376	569	756	939	118	16
17	321	548	767	977	182	380	572	760	942	121	17
18	325	552	770	981	185	383	575	763	945	124	18
19	1.3328	1.3555	1.3774	1.3984	1.4188	1.4386	1.4579	1.4766	1.4948	1.5127	19
20	1.3332	1.3559	1.3778	1.3988	1.4192	1.4390	1.4582	1.4769	1.4951	1.5129	20
21	336	563	781	991	195	393	585	772	954	132	21
22	340	567	784	995	198	396	588	775	957	135	22
23	344	570	788	1.3998	202	399	591	778	960	138	23
24	348	574	791	1.4002	205	403	594	781	963	141	24
25	352	578	795	005	209	406	597	784	966	144	25
26	356	582	798	009	212	409	601	787	969	147	26
27	359	585	802	012	215	412	604	790	972	150	27
28	363	588	805	015	219	416	607	794	975	153	28
29	1.3367	1.3592	1.3809	1.4019	1.4222	1.4419	1.4610	1.4797	1.4978	1.5156	29
30	1.3370	1.3595	1.3812	1.4022	1.4225	1.4422	1.4613	1.4800	1.4981	1.5159	30
31	374	599	816	026	229	425	617	803	984	162	31
32	378	603	820	029	232	429	620	806	987	165	32
33	381	606	823	032	235	432	623	809	990	168	33
34	385	610	827	036	239	435	626	812	993	171	34
35	389	614	830	040	242	438	629	815	996	174	35
36	393	618	834	043	245	442	632	818	1.4999	177	36
37	397	621	838	046	249	445	636	821	1.5002	180	37
38	401	625	841	050	252	448	639	824	005	183	38
39	1.3404	1.3629	1.3845	1.4053	1.4255	1.4451	1.4642	1.4827	1.5008	1.5186	39
40	1.3408	1.3632	1.3848	1.4057	1.4259	1.4455	1.4645	1.4830	1.5011	1.5188	40
41	412	636	852	060	262	458	648	834	014	191	41
42	416	640	855	064	265	461	651	837	017	194	42
43	420	643	859	067	269	464	654	840	020	197	43
44	423	647	862	070	272	467	658	843	023	200	44
45	427	651	866	074	275	471	661	846	026	203	45
46	431	654	870	077	279	474	664	849	029	206	46
47	435	658	873	081	282	477	667	852	032	209	47
48	439	662	877	084	285	480	670	855	035	211	48
49	1.3443	1.3665	1.3880	1.4088	1.4288	1.4484	1.4673	1.4858	1.5038	1.5214	49
50	1.3446	1.3669	1.3884	1.4091	1.4292	1.4487	1.4676	1.4861	1.5040	1.5217	50
51	450	673	887	094	295	490	680	864	043	220	51
52	454	676	891	098	298	493	683	867	046	223	52
53	458	680	894	101	302	496	686	870	049	226	53
54	462	684	898	105	305	500	689	873	052	228	54
55	465	687	901	108	308	503	692	875	055	231	55
56	469	691	905	111	312	506	695	878	058	234	56
57	473	695	908	115	315	509	698	881	061	237	57
58	477	698	912	118	318	513	701	885	064	240	58
59	1.3481	1.3702	1.3915	1.4122	1.4322	1.4516	1.4704	1.4888	1.5067	1.5243	59
60	1.3484	1.3705	1.3919	1.4125	1.4325	1.4519	1.4707	1.4891	1.5070	1.5246	60

TABLE VI.

 $\mu + \log \text{tang } z.$

z	30°	31°	32°	33°	34°	35°	36°	37°	38°	39°	z
0'	1.5246	1.5418	1.5587	1.5753	1.5917	1.6078	1.6237	1.6394	1.6550	1.6705	0'
1	49	21	90	56	19	81	40	397	53	07	1
2	52	24	93	59	22	83	42	400	55	10	2
3	55	27	96	62	25	86	45	02	58	12	3
4	58	30	598	64	28	89	48	05	61	15	4
5	61	32	601	67	30	91	50	08	63	17	5
6	63	35	04	70	33	94	53	10	66	19	6
7	66	38	07	73	36	97	56	13	68	22	7
8	69	41	10	75	38	099	58	15	71	24	8
9	1.5272	1.5444	1.5612	1.5778	1.5941	1.6102	1.6261	1.6418	1.6574	1.6727	9
10	1.5275	1.5447	1.5615	1.5781	1.5944	1.6105	1.6264	1.6421	1.6576	1.6729	10
11	78	50	18	84	47	07	66	23	78	32	11
12	81	52	21	86	49	10	69	25	80	34	12
13	84	55	24	89	52	13	72	28	83	37	13
14	87	58	26	92	55	16	74	30	86	40	14
15	90	61	29	95	58	18	77	33	88	42	15
16	92	64	32	797	60	21	79	35	91	45	16
17	95	67	35	800	63	23	81	38	93	47	17
18	298	70	38	03	66	26	84	41	96	50	18
19	1.5301	1.5472	1.5640	1.5806	1.5968	1.6129	1.6287	1.6443	1.6598	1.6752	19
20	1.5304	1.5475	1.5643	1.5808	1.5971	1.6131	1.6289	1.6446	1.6601	1.6755	20
21	07	78	46	11	74	34	92	48	04	58	21
22	10	81	49	14	76	36	94	51	06	60	22
23	13	84	51	17	79	39	297	54	09	63	23
24	16	87	54	19	81	41	300	56	11	65	24
25	19	89	57	22	84	44	02	59	14	68	25
26	21	92	60	25	87	47	05	62	17	70	26
27	24	95	63	28	90	49	08	64	19	73	27
28	27	498	65	30	92	52	10	67	22	76	28
29	1.5330	1.5501	1.5668	1.5833	1.5994	1.6155	1.6313	1.6469	1.6624	1.6778	29
30	1.5333	1.5504	1.5671	1.5835	1.5997	1.6157	1.6315	1.6472	1.6627	1.6781	30
31	36	06	74	38	1.6000	60	18	75	30	83	31
32	39	09	76	40	02	63	21	77	32	86	32
33	42	12	78	43	05	65	23	80	35	88	33
34	45	15	81	46	08	68	26	82	37	91	34
35	47	18	84	48	11	71	29	85	40	94	35
36	50	21	87	51	13	73	31	88	43	96	36
37	53	23	90	54	16	76	34	90	45	799	37
38	56	26	92	57	19	79	37	93	48	801	38
39	1.5359	1.5528	1.5695	1.5859	1.6021	1.6181	1.6339	1.6495	1.6650	1.6804	39
40	1.5362	1.5531	1.5698	1.5862	1.6024	1.6184	1.6342	1.6498	1.6653	1.6806	40
41	65	34	701	65	27	87	44	501	56	09	41
42	68	37	03	68	30	89	47	03	58	12	42
43	70	39	06	70	32	92	50	06	61	14	43
44	72	42	09	73	35	94	52	09	63	1.6817	44
45	75	45	12	76	38	197	55	11	66	19	45
46	78	48	14	79	40	200	58	14	68	22	46
47	81	51	17	81	43	02	60	16	71	24	47
48	84	53	20	84	46	05	63	19	74	27	48
49	1.5387	1.5556	1.5723	1.5887	1.6048	1.6208	1.6365	1.6521	1.6676	1.6830	49
50	1.5390	1.5559	1.5726	1.5889	1.6051	1.6210	1.6368	1.6524	1.6679	1.6832	50
51	92	62	28	92	54	13	71	27	81	35	51
52	95	65	31	95	56	16	73	30	84	37	52
53	398	67	34	898	59	18	76	32	86	40	53
54	401	70	37	900	62	21	79	35	89	42	54
55	04	73	39	03	64	24	81	37	92	45	55
56	07	76	42	06	67	26	84	40	94	47	56
57	10	79	45	09	70	29	87	42	97	50	57
58	12	82	48	11	73	32	89	45	699	53	58
59	1.5415	1.5584	1.5750	1.5914	1.6075	1.6234	1.6392	1.6548	1.6702	1.6855	59
60	1.5418	1.5587	1.5753	1.5917	1.6078	1.6237	1.6394	1.6550	1.6705	1.6857	60



REFRACTION TABLES.

In order that all the refraction tables may be conveniently together the three tables giving the terms B , γ and T published in Volume I of these *Publications* are given again here as Tables I, II, III respectively. Table IV gives A , λ and σ . Table V gives i . [Tables IV and V are reprinted from the Pulkowa Tables.] Table VI is the Pulkowa table for $\mu + \log \operatorname{tg} z$ corrected by

$$\Delta \mu = +0.000108 (58^\circ - z)$$

where z is in degrees. The refractions are then to be computed from these tables by the formula

$$\log \text{Refr.} = \mu + \log \operatorname{tg} z + A (B + T) + \lambda \gamma - \sigma i.$$

R. T. C.

TABLE I.
VALUES OF THE REFRACTION TERM B.
(Units of the fifth decimal place.)

Bar. Inches.	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09	With left hand col.
											20 × Bar.
25.0	-7330	-7313	-7296	-7278	-7261	-7243	-7226	-7209	-7191	-7174	500
.1	7157	7140	7122	7105	7088	7070	7053	7036	7019	7001	502
.2	6984	6967	6950	6933	6915	6898	6881	6864	6847	6829	504
.3	6812	6795	6778	6761	6744	6726	6709	6692	6675	6658	506
.4	6641	6624	6607	6590	6573	6555	6538	6521	6504	6487	508
.5	-6470	-6453	-6436	-6419	-6402	-6385	-6368	-6351	-6334	-6317	510
.6	6300	6283	6266	6249	6232	6215	6199	6182	6165	6148	512
.7	6131	6114	6097	6080	6063	6046	6030	6013	5996	5979	514
.8	5962	5945	5929	5912	5895	5878	5861	5845	5828	5811	516
.9	5794	5777	5761	5744	5727	5711	5694	5677	5660	5644	518
26.0	-5627	-5610	-5594	-5577	-5560	-5543	-5527	-5510	-5493	-5477	520
.1	5460	5444	5427	5410	5394	5377	5360	5344	5327	5311	522
.2	5294	5278	5261	5244	5228	5211	5195	5178	5162	5145	524
.3	5129	5112	5096	5079	5063	5046	5030	5013	4997	4980	526
.4	4964	4947	4931	4915	4898	4882	4865	4849	4832	4816	528
.5	-4800	-4783	-4767	-4751	-4734	-4718	-4701	-4685	-4669	-4652	530
.6	4636	4620	4603	4587	4571	4555	4538	4522	4506	4489	532
.7	4473	4457	4441	4424	4408	4392	4376	4359	4343	4327	534
.8	4311	4295	4278	4262	4246	4230	4214	4197	4181	4165	536
.9	4149	4133	4117	4101	4084	4068	4052	4036	4020	4004	538
27.0	-3988	-3972	-3956	-3940	-3923	-3907	-3891	-3875	-3859	-3843	540
.1	3827	3811	3795	3779	3763	3747	3731	3715	3699	3683	542
.2	3667	3651	3635	3619	3603	3587	3572	3556	3540	3524	544
.3	3508	3492	3476	3460	3444	3428	3413	3397	3381	3365	546
.4	3349	3333	3317	3302	3286	3270	3254	3238	3222	3207	548
.5	-3191	-3175	-3159	-3144	-3128	-3112	-3096	-3080	-3065	-3049	550
.6	3033	3018	3002	2986	2970	2955	2939	2923	2908	2892	552
.7	2876	2860	2845	2829	2814	2798	2782	2767	2751	2735	554
.8	2720	2704	2689	2673	2657	2642	2626	2610	2595	2579	556
.9	2564	2548	2533	2517	2502	2486	2470	2455	2439	2424	558
28.0	-2408	-2393	-2377	-2362	-2346	-2331	-2315	-2300	-2285	-2269	560
With right hand col.	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	20 × Bar.

N. B.—The left hand column and top argument give the whole argument for a barometer reading in inches; the right hand column and bottom argument give the entry for 20 times the barometer reading in inches.

TABLE II.

VALUES OF THE REFRACTION TERM γ .

(Units of the fifth decimal place.)

t. Fahr.	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0°	+4412	+4318	+4222	+4128	+4032	+3938	+3846	+3750	+3658	+3562	9	19	28	37	47	56	66	75	84
10	3470	3376	3284	3190	3099	3008	2914	2823	2730	2640	9	18	28	37	46	55	64	74	83
20	2547	2457	2365	2275	2184	2094	2004	1913	1824	1734	9	18	27	36	45	54	63	72	81
30	1645	1554	1466	1378	1288	1201	1111	1023	934	847	9	18	27	35	44	53	62	71	80
40	+ 759	+ 672	+ 586	+ 497	+ 411	+ 324	+ 238	+ 150	+ 65	- 22	9	17	26	35	43	52	61	69	78
50	- 107	- 193	- 279	- 364	- 450	- 534	- 620	- 704	- 789	- 873	9	17	26	34	43	51	60	68	77
60	957	1042	1125	1209	1293	1377	1460	1543	1626	1708	8	17	25	33	42	50	58	67	75
70	1791	1873	1956	2038	2121	2202	2285	2366	2446	2528	8	16	24	33	41	49	57	65	73
80	2609	2691	2771	2852	2932	3014	3093	3172	3253	3334	8	16	24	32	40	48	56	64	72
90	-3414	-3493	-3573	-3652	-3731	-3810	-3889	-3967	-4046	-4124	8	16	23	31	39	47	55	63	70

TABLE III.

VALUES OF REFRACTION TERM T.

(Units of the fifth decimal place.)

t. (Fahr.)	0	1	2	3	4	5	6	7	8	9
0°	+125	+121	+117	+113	+109	+105	+101	+98	+94	+90
10	+86	+82	+78	+74	+70	+66	+62	+58	+55	+51
20	+47	+43	+39	+35	+31	+27	+23	+20	+16	+12
30	+8	+4	0	-4	-8	-12	-16	-20	-23	-27
40	-31	-35	-39	-43	-47	-51	-55	-58	-62	-66
50	-70	-74	-78	-82	-86	-90	-93	-97	-101	-105
60	-109	-113	-117	-121	-124	-128	-132	-136	-140	-144
70	-148	-152	-156	-160	-164	-168	-171	-175	-179	-183
80	-187	-191	-195	-199	-202	-206	-210	-214	-218	-221
90	-225	-229	-233	-237	-241	-244	-248	-252	-256	-260

TABLE IV.

VALUES OF THE REFRACTION TERMS λ , A AND σ .

z	λ	A	σ	z	λ	A	σ
45°	1.0018			85° 0'	1.1235	1.0127	0.00146
50	1.0022			5	1264		
55	1.0029			10	1294		
60	1.0044			15	1325		
61	1.0047			20	1357	141	170
62	1.0051			25	1390		
63	1.0055			30	1424		
64	1.0059			35	1459		
65	1.0064			40	1495	156	202
66	1.0070			45	1532		
67	1.0077			50	1571		
68	1.0085			55	1611		
69	1.0093						
70	1.0103			86 0	1.1652	1.0172	0.00241
71	1.0115			5	1695		
72	1.0130			10	1740	181	264
73	1.0147			15	1786		
74	1.0166			20	1833	192	290
				25	1882		
75 0'	1.0188			30	1934	203	320
75 30	1.0200			35	1987		
76 0	1.0216			40	2040	214	352
76 30	1.0235			45	2095		
77 0	1.0253	1.0029		50	2153	227	386
77 30	1.0271	1.0030		55	2214		
78 0	1.0293	1.0033		87 0	1.2277	1.0241	0.00421
78 30	1.0318	1.0035		2	2303		
79 0	1.0344	1.0038		4	2329	247	437
79 30	1.0374	1.0041		6	2356		
				8	2383	254	453
80 0	1.0409	1.0044	0.00019				
10	1.0421			10	2410		
20	1.0433			12	2438	261	471
30	1.0447	1.0048	0.00022	14	2466		
40	1.0461			16	2495	268	489
50	1.0475			18	2524		
81 0	1.0491	1.0052	0.00025	20	2554	275	509
10	1.0508			22	2584		
20	1.0525			24	2615	282	529
30	1.0542	1.0057	0.00029	26	2646		
40	1.0561			28	2677	290	550
50	1.0580						
82 0	1.0600	1.0063	0.00035	30	2708		
10	1.0622			32	2740	297	572
20	1.0645			34	2772		
30	1.0669	1.0070	0.00045	36	2805	305	594
40	1.0694			38	2839		
50	1.0720			40	2873	313	617
83 0	1.0747	1.0078	0.00057	42	2908		
10	1.0776			44	2942	321	641
20	1.0807			46	2978		
30	1.0839	0.0087	0.00073	48	3014	328	666
40	1.0874						
50	1.0911			50	3051		
84 0	1.0949	1.0098	0.00091	52	3088	338	693
10	1.0990			54	3125		
20	1.1034			56	3163	347	721
30	1.1080	1.0112	0.00116	58	3202		
40	1.1128			88 0	1.3241	1.0357	0.00749
50	1.1180						
85 0	1.1235	1.0127	0.00146				

TABLE IV.

VALUES OF THE REFRACTION TERMS λ , A AND σ .

z	λ	A	σ	z	λ	A	σ
88° 0'	1.3241	1.0357	0.00749	89° 15'	1.5218		
2	3280			16	5252	1.0609	0.01582
4	3321	367	778	17	5287		
6	3362			18	5322		1613
8	3403	376	810	19	5357		
10	3445			20	5392	628	1644
12	3487	387	842	21	5427		
14	3531			22	5463		1675
16	3575	397	876	23	5500		
18	3620			24	5537	648	1709
20	3665	409	912	25	5574		
22	3710			26	5611		1744
24	3757	420	948	27	5648		
26	3805			28	5685	669	1779
28	3853	431	986	29	5723		
30	3902			30	5762		1813
32	3951	443	1026	31	5801		
34	4002			32	5840	690	1849
36	4053	455	1067	33	5879		
38	4107			34	5918		1886
40	4161	468	1111	35	5957		
42	4214			36	5997	712	1923
44	4268	482	1155	37	6037		
46	4323			38	6077		1961
48	4378	496	1201	39	6117		
50	4433			40	6158	734	1999
52	4490	511	1249	41	6199		
54	4549			42	6241		2038
56	4608	526	1300	43	6283		
88 58	4669			44	6325	758	2079
89 0	1.4729	1.0541	0.01352	45	6367		
1	4760			46	6410		2119
2	4791		1379	47	6453		
3	4823			48	6497	782	2161
4	4855	558	1407	49	6541		
5	4887			50	6585		2203
6	4919		1435	51	6630		
7	4951			52	6675	807	2246
8	4984	575	1464	53	6720		
9	5017			54	6765		2288
10	5050		1493	55	6811		
11	5083			56	6857	833	2332
12	5116	1.0592	1522	57	6904		
13	5150			58	6951		2377
14	5184		0.01552	89 59	6998		
89 15	1.5218			90 0	1.7046	1.0859	0.02424

TABLE V.

VALUES OF THE REFRACTION TERM i .

Jan. 15	-0.34	July 15	+0.33
Feb. 15	-0.27	Aug. 15	+0.30
Mar. 15	-0.05	Sept. 15	+0.19
Apr. 15	+0.08	Oct. 15	-0.16
May 15	+0.20	Nov. 15	-0.33
June 15	+0.26	Dec. 15	-0.37

TABLE VI.

 $\mu + \log \tan z$.

z	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°	z
0'	— ∞	0.0083	0.3094	0.4856	0.6108	0.7080	0.7875	0.8550	0.9135	0.9653	0'
1	8.230	155	130	880	126	094	888	560	144	661	1
2	.531	226	166	904	144	109	900	571	153	669	2
3	.707	295	201	928	162	123	912	581	162	678	3
4	8.832	0.0364	0.3236	0.4952	.06180	0.7138	0.7924	0.8591	0.9172	0.9686	4
5	8.929	0.0431	0.3271	0.4975	0.6198	0.7152	0.7936	0.8602	0.9181	0.9694	5
6	9.008	497	306	0.4999	215	166	948	612	190	702	6
7	.075	563	341	0.5022	233	180	960	622	199	710	7
8	.133	627	375	045	250	195	972	632	208	718	8
9	9.184	0.0691	0.3408	0.5068	0.6268	0.7209	0.7983	0.8643	0.9217	0.9726	9
10	9.230	0.0753	0.3442	0.5091	0.6285	0.7223	0.7995	0.8653	0.9226	0.9734	10
11	.271	815	475	114	303	237	0.8007	663	235	742	11
12	.309	875	.508	137	320	251	019	673	244	750	12
13	.344	935	541	160	338	265	031	683	253	758	13
14	9.376	0.0994	0.3574	0.5182	0.6355	0.7279	0.8042	0.8694	0.9262	0.9766	14
15	9.406	0.1053	0.3606	0.5205	0.6372	0.7293	0.8054	0.8704	0.9271	0.9774	15
16	.434	110	638	227	389	307	066	714	279	782	16
17	.460	167	670	249	406	320	077	724	288	790	17
18	.485	223	702	271	423	334	089	734	297	798	18
19	9.509	0.1279	0.3733	0.5293	0.6440	0.7348	0.8101	0.8744	0.9306	0.9806	19
20	9.531	0.1333	0.3764	0.5315	0.6457	0.7362	0.8112	0.8754	0.9315	0.9814	20
21	.552	387	795	336	473	375	124	764	324	822	21
22	.572	440	826	358	490	389	135	774	332	829	22
23	.592	493	856	380	507	402	147	784	341	836	23
24	9.610	0.1545	0.3887	0.5401	0.6523	0.7416	0.8158	0.8794	0.9350	0.9844	24
25	9.628	0.1596	0.3917	0.5422	0.6540	0.7429	0.8169	0.8803	0.9359	0.9852	25
26	.645	647	947	443	556	443	181	813	367	860	26
27	.661	698	0.3976	464	572	456	192	823	376	868	27
28	.677	747	0.4006	485	589	469	203	833	384	876	28
29	9.692	0.1796	0.4035	0.5506	0.6605	0.7483	0.8215	0.8843	0.9392	0.9883	29
30	9.707	0.1845	0.4064	0.5527	0.6621	0.7496	0.8226	0.8852	0.9401	0.9891	30
31	.721	893	093	548	637	509	237	862	410	899	31
32	.735	940	122	568	653	522	248	872	418	907	32
33	.749	0.1987	150	589	669	535	259	881	427	914	33
34	9.762	0.2034	0.4179	0.5609	0.6685	0.7549	0.8271	0.8890	0.9436	0.9922	34
35	9.774	0.2080	0.4207	0.5630	0.6701	0.7562	0.8282	0.8900	0.9444	0.9930	35
36	.786	125	235	650	717	575	292	910	453	937	36
37	.798	170	263	670	733	588	303	919	461	945	37
38	.810	215	290	690	748	601	314	929	470	953	38
39	9.821	0.2259	0.4317	0.5710	0.6764	0.7613	0.8325	0.8938	0.9478	0.9961	39
40	9.832	0.2303	0.4345	0.5730	0.6780	0.7625	0.8336	0.8948	0.9487	0.9968	40
41	.843	346	372	749	795	638	347	957	495	976	41
42	.853	389	399	769	811	651	358	967	504	983	42
43	.864	431	426	789	826	664	369	976	512	991	43
44	9.874	0.2473	0.4452	0.5808	0.6841	0.7677	0.8379	0.8986	0.9520	0.9999	44
45	9.883	0.2515	0.4479	0.5828	0.6856	0.7689	0.8390	0.8995	0.9529	1.0006	45
46	.893	556	505	847	871	702	401	0.9005	537	014	46
47	.902	597	531	866	886	714	412	014	546	021	47
48	.911	637	557	885	902	727	422	024	554	029	48
49	9.920	0.2677	0.4583	0.5904	0.6917	0.7740	0.8433	0.9033	0.9562	1.0036	49
50	9.929	0.2717	0.4608	0.5922	0.6932	0.7752	0.8444	0.9043	0.9571	1.0044	50
51	.938	756	634	941	947	765	455	052	579	051	51
52	.946	795	659	960	962	777	465	061	587	059	52
53	.954	834	685	979	977	790	476	071	595	066	53
54	9.962	0.2872	0.4709	0.5997	0.6991	0.7802	0.8486	0.9080	0.9604	1.0074	54
55	9.970	0.2909	0.4734	0.6016	0.7006	0.7814	0.8497	0.9089	0.9612	1.0081	55
56	.978	947	758	034	021	827	508	098	620	089	56
57	.986	0.2984	783	053	036	839	518	107	629	096	57
58	9.994	0.3021	808	071	050	851	529	117	637	104	58
59	0.0011	0.3058	0.4832	0.6089	0.7065	0.7863	0.8539	0.9126	0.9645	1.0111	59
60	0.0083	0.3094	0.4856	0.6108	0.7080	0.7875	0.8550	0.9135	0.9653	1.0118	60

TABLE VI.

 $\mu + \log \tan z$.

z	10°	11°	12°	13°	14°	15°	16°	17°	18°	19°	z
0'	1.0118	1.0540	1.0928	1.1286	1.1618	1.1929	1.2223	1.2500	1.2764	1.3014	0'
1	126	547	934	291	623	934	227	505	768	018	1
2	133	554	940	297	628	940	232	509	772	023	2
3	140	561	946	303	634	945	237	514	776	027	3
4	148	567	952	309	639	950	242	518	781	031	4
5	155	574	959	314	644	955	246	523	785	035	5
6	162	581	965	319	650	960	251	527	789	039	6
7	170	587	971	325	655	965	256	532	794	043	7
8	177	594	977	330	660	970	261	536	798	047	8
9	1.0184	1.0601	1.0983	1.1336	1.1666	1.1975	1.2266	1.2541	1.2802	1.3051	9
10	192	1.0607	1.0989	1.1342	1.1671	1.1980	1.2270	1.2545	1.2806	055	10
11	199	614	1.0995	348	676	985	275	550	811	059	11
12	206	620	1.1001	353	682	990	280	554	815	063	12
13	213	626	007	359	687	1.1995	284	559	819	067	13
14	221	633	013	365	692	1.2000	289	563	823	072	14
15	228	640	019	370	698	005	294	567	828	076	15
16	235	646	025	376	703	010	299	572	832	080	16
17	242	653	031	382	708	015	303	576	836	084	17
18	249	659	037	387	714	020	308	581	840	088	18
19	1.0256	1.0666	1.1043	1.1393	1.1719	1.2025	1.2313	1.2585	1.2845	1.3092	19
20	1.0263	1.0673	1.1049	1.1399	1.1724	1.2030	1.2317	1.2590	1.2849	1.3096	20
21	270	679	055	404	729	034	322	594	853	100	21
22	277	686	061	410	735	039	327	599	857	104	22
23	284	692	067	415	740	044	331	603	861	108	23
24	291	699	073	421	745	049	336	607	866	112	24
25	298	705	079	427	750	054	341	612	870	116	25
26	305	712	086	432	756	059	345	616	874	120	26
27	312	718	092	438	761	064	350	621	878	124	27
28	319	725	098	443	766	069	355	625	883	128	28
29	1.0327	1.0731	1.1104	1.1449	1.1771	1.2074	1.2359	1.2630	1.2887	1.3132	29
30	1.0334	1.0738	1.1109	1.1454	1.1776	1.2079	1.2364	1.2634	1.2891	1.3136	30
31	341	744	115	460	782	084	369	638	895	140	31
32	348	751	121	466	787	089	373	643	899	144	32
33	355	757	127	471	792	093	378	647	904	147	33
34	362	763	133	477	797	098	382	652	908	151	34
35	369	770	139	482	803	103	387	656	912	155	35
36	376	776	145	488	808	108	392	660	916	159	36
37	383	783	151	493	813	113	396	665	920	163	37
38	390	789	157	499	818	118	401	669	924	167	38
39	1.0397	1.0795	1.1163	1.1504	1.1823	1.2123	1.2405	1.2673	1.2928	1.3171	39
40	1.0404	1.0802	1.1169	1.1510	1.1828	1.2128	1.2410	1.2678	1.2932	1.3175	40
41	411	808	175	515	833	132	415	682	936	179	41
42	418	815	181	521	839	137	419	687	940	183	42
43	424	821	187	526	844	142	424	691	944	187	43
44	431	827	192	532	849	147	428	695	948	191	44
45	438	834	198	537	854	152	433	699	952	195	45
46	445	840	204	543	859	157	437	703	957	199	46
47	452	846	210	548	864	161	442	707	961	203	47
48	459	853	216	554	869	166	447	712	965	207	48
49	1.0466	1.0859	1.1222	1.1559	1.1874	1.2171	1.2451	1.2716	1.2969	1.3211	49
50	1.0473	1.0865	1.1228	1.1565	1.1879	1.2176	1.2455	1.2720	1.2973	1.3215	50
51	479	871	233	570	885	181	459	725	977	219	51
52	486	878	239	575	890	186	464	729	981	223	52
53	493	884	245	581	895	190	468	733	986	227	53
54	500	890	251	586	900	194	473	738	990	230	54
55	507	897	257	592	904	199	477	742	994	234	55
56	514	903	262	597	909	204	482	746	1.2998	238	56
57	520	909	268	602	914	208	486	750	1.3002	242	57
58	527	915	274	608	919	213	491	755	006	246	58
59	1.0534	1.0921	1.1280	1.1613	1.1924	1.2218	1.2496	1.2759	1.3010	1.3250	59
60	1.0540	1.0928	1.1286	1.1618	1.1929	1.2223	1.2500	1.2764	1.3014	1.3254	60

TABLE VI.

 $\mu + \log \tan z$.

z	20°	21°	22°	23°	24°	25°	26°	27°	28°	29°	z
0'	1.3254	1.3484	1.3705	1.3919	1.4125	1.4325	1.4519	1.4707	1.4891	1.5070	0'
1	258	488	709	922	128	328	522	710	894	073	1
2	262	492	713	926	132	331	525	713	897	076	2
3	266	495	716	929	135	335	528	716	900	079	3
4	270	499	720	933	139	338	531	719	903	082	4
5	274	503	724	936	142	341	534	722	906	085	5
6	278	507	727	940	145	345	537	725	909	088	6
7	282	511	731	943	149	348	540	728	912	091	7
8	286	514	734	947	152	351	543	732	915	094	8
9	1.3289	1.3518	1.3738	1.3950	1.4156	1.4354	1.4547	1.4735	1.4918	1.5097	9
10	1.3293	1.3522	1.3742	1.3954	1.4159	1.4357	1.4550	1.4738	1.4921	1.5100	10
11	297	526	745	957	162	360	553	741	924	103	11
12	301	529	749	961	165	363	556	744	927	106	12
13	305	533	753	964	168	367	559	747	930	109	13
14	309	537	756	968	172	370	563	750	933	112	14
15	313	540	760	971	175	373	566	753	936	115	15
16	317	544	763	974	178	376	569	756	939	118	16
17	321	548	767	977	182	380	572	760	942	121	17
18	325	552	770	981	185	383	575	763	945	124	18
19	1.3328	1.3555	1.3774	1.3984	1.4188	1.4386	1.4579	1.4766	1.4948	1.5127	19
20	1.3332	1.3559	1.3778	1.3988	1.4192	1.4390	1.4582	1.4769	1.4951	1.5129	20
21	336	563	781	991	195	393	585	772	954	132	21
22	340	567	784	995	198	396	588	775	957	135	22
23	344	570	788	1.3998	202	399	591	778	960	138	23
24	348	574	791	1.4002	205	403	594	781	963	141	24
25	352	578	795	005	209	406	597	784	966	144	25
26	356	582	798	009	212	409	601	787	969	147	26
27	359	585	802	012	215	412	604	790	972	150	27
28	363	588	805	015	219	416	607	794	975	153	28
29	1.3367	1.3592	1.3809	1.4019	1.4222	1.4419	1.4610	1.4797	1.4978	1.5156	29
30	1.3370	1.3595	1.3812	1.4022	1.4225	1.4422	1.4613	1.4800	1.4981	1.5159	30
31	374	599	816	026	229	425	617	803	984	162	31
32	378	603	820	029	232	429	620	806	987	165	32
33	381	606	823	032	235	432	623	809	990	168	33
34	385	610	827	036	239	435	626	812	993	171	34
35	389	614	830	040	242	438	629	815	996	174	35
36	393	618	834	043	245	442	632	818	1.4999	177	36
37	397	621	838	046	249	445	636	821	1.5002	180	37
38	401	625	841	050	252	448	639	824	005	183	38
39	1.3404	1.3629	1.3845	1.4053	1.4255	1.4451	1.4642	1.4827	1.5008	1.5186	39
40	1.3408	1.3632	1.3848	1.4057	1.4259	1.4455	1.4645	1.4830	1.5011	1.5188	40
41	412	636	852	060	262	458	648	834	014	191	41
42	416	640	855	064	265	461	651	837	017	194	42
43	420	643	859	067	269	464	654	840	020	197	43
44	423	647	862	070	272	467	658	843	023	200	44
45	427	651	866	074	275	471	661	846	026	203	45
46	431	654	870	077	279	474	664	849	029	206	46
47	435	658	873	081	282	477	667	852	032	209	47
48	439	662	877	084	285	480	670	855	035	211	48
49	1.3443	1.3665	1.3880	1.4088	1.4288	1.4484	1.4673	1.4858	1.5038	1.5214	49
50	1.3446	1.3669	1.3884	1.4091	1.4292	1.4487	1.4676	1.4861	1.5040	1.5217	50
51	450	673	887	094	295	490	680	864	043	220	51
52	454	676	891	098	298	493	683	867	046	223	52
53	458	680	894	101	302	496	686	870	049	226	53
54	462	684	898	105	305	500	689	873	052	228	54
55	465	687	901	108	308	503	692	875	055	231	55
56	469	691	905	111	312	506	695	878	058	234	56
57	473	695	908	115	315	509	698	881	061	237	57
58	477	698	912	118	318	513	701	885	064	240	58
59	1.3481	1.3702	1.3915	1.4122	1.4322	1.4516	1.4704	1.4888	1.5067	1.5243	59
60	1.3484	1.3705	1.3919	1.4125	1.4325	1.4519	1.4707	1.4891	1.5070	1.5246	60

TABLE VI.

 $\mu + \log \tan z$.

z	30°	31°	32°	33°	34°	35°	36°	37°	38°	39°	z
0'	1.5246	1.5418	1.5587	1.5753	1.5917	1.6078	1.6237	1.6394	1.6550	1.6705	0'
1	49	21	90	56	19	81	40	397	53	07	1
2	52	24	93	59	22	83	42	400	55	10	2
3	55	27	96	62	25	86	45	02	58	12	3
4	58	30	598	64	28	89	48	05	61	15	4
5	61	32	601	67	30	91	50	08	63	17	5
6	63	35	04	70	33	94	53	10	66	19	6
7	66	38	07	73	36	97	56	13	68	22	7
8	69	41	10	75	38	099	58	15	71	24	8
9	1.5272	1.5444	1.5612	1.5778	1.5941	1.6102	1.6261	1.6418	1.6574	1.6727	9
10	1.5275	1.5447	1.5615	1.5781	1.5944	1.6105	1.6264	1.6421	1.6576	1.6729	10
11	78	50	18	84	47	07	66	23	78	32	11
12	81	52	21	86	49	10	69	25	80	34	12
13	84	55	24	89	52	13	72	28	83	37	13
14	87	58	26	92	55	16	74	30	86	40	14
15	90	61	29	95	58	18	77	33	88	42	15
16	92	64	32	797	60	21	79	35	91	45	16
17	95	67	35	800	63	23	81	38	93	47	17
18	298	70	38	03	66	26	84	41	96	50	18
19	1.5301	1.5472	1.5640	1.5806	1.5968	1.6129	1.6287	1.6443	1.6598	1.6752	19
20	1.5304	1.5475	1.5643	1.5808	1.5971	1.6131	1.6289	1.6446	1.6601	1.6755	20
21	07	78	46	11	74	34	92	48	04	58	21
22	10	81	49	14	76	36	94	51	06	60	22
23	13	84	51	17	79	39	297	54	09	63	23
24	16	87	54	19	81	41	300	56	11	65	24
25	19	89	57	22	84	44	02	59	14	68	25
26	21	92	60	25	87	47	05	62	17	70	26
27	24	95	63	28	90	49	08	64	19	73	27
28	27	498	65	30	92	52	10	67	22	76	28
29	1.5330	1.5501	1.5668	1.5833	1.5994	1.6155	1.6313	1.6469	1.6624	1.6778	29
30	1.5333	1.5504	1.5671	1.5835	1.5997	1.6157	1.6315	1.6472	1.6627	1.6781	30
31	36	06	74	38	1.6000	60	18	75	30	83	31
32	39	09	76	40	02	63	21	77	32	86	32
33	42	12	78	43	05	65	23	80	35	88	33
34	45	15	81	46	08	68	26	82	37	91	34
35	47	18	84	48	11	71	29	85	40	94	35
36	50	21	87	51	13	73	31	88	43	96	36
37	53	23	90	54	16	76	34	90	45	799	37
38	56	26	92	57	19	79	37	93	48	801	38
39	1.5359	1.5528	1.5695	1.5859	1.6021	1.6181	1.6339	1.6495	1.6650	1.6804	39
40	1.5362	1.5531	1.5698	1.5862	1.6024	1.6184	1.6342	1.6498	1.6653	1.6806	40
41	65	34	701	65	27	87	44	501	56	09	41
42	68	37	03	68	30	89	47	03	58	12	42
43	70	39	06	70	32	92	50	06	61	14	43
44	72	42	09	73	35	94	52	09	63	1.6817	44
45	75	45	12	76	38	197	55	11	66	19	45
46	78	48	14	79	40	200	58	14	68	22	46
47	81	51	17	81	43	02	60	16	71	24	47
48	84	53	20	84	46	05	63	19	74	27	48
49	1.5387	1.5556	1.5723	1.5887	1.6048	1.6208	1.6365	1.6521	1.6676	1.6830	49
50	1.5390	1.5559	1.5726	1.5889	1.6051	1.6210	1.6368	1.6524	1.6679	1.6832	50
51	92	62	28	92	54	13	71	27	81	35	51
52	95	65	31	95	56	16	73	30	84	37	52
53	398	67	34	898	59	18	76	32	86	40	53
54	401	70	37	900	62	21	79	35	89	42	54
55	04	73	39	03	64	24	81	37	92	45	55
56	07	76	42	06	67	26	84	40	94	47	56
57	10	79	45	09	70	29	87	42	97	50	57
58	12	82	48	11	73	32	89	45	999	53	58
59	1.5415	1.5584	1.5750	1.5914	1.6075	1.6234	1.6392	1.6548	1.6702	1.6855	59
60	1.5418	1.5587	1.5753	1.5917	1.6078	1.6237	1.6394	1.6550	1.6705	1.6857	60

TABLE VI.

 $\mu + \log \tan z.$

z	40°	41°	42°	43°	44°	45°	46°	47°	48°	49°	z
0'	1.6857	1.7009	1.7160	1.7311	1.7462	1.7612	1.7762	1.7913	1.8063	1.8215	0'
1	59	12	63	14	64	15	65	15	66	17	1
2	62	14	66	16	67	17	67	18	69	20	2
3	64	17	68	19	69	20	70	20	71	22	3
4	67	19	71	21	72	22	72	23	74	25	4
5	70	22	73	24	74	25	75	25	76	28	5
6	72	24	76	26	77	27	77	28	79	30	6
7	75	27	78	29	79	30	80	30	81	33	7
8	77	29	81	31	82	32	82	33	84	35	8
9	1.6880	1.7032	1.7183	1.7334	1.7484	1.7635	1.7785	1.7935	1.8086	1.8238	9
10	1.6882	1.7034	1.7186	1.7337	1.7487	1.7637	1.7787	1.7938	1.8089	1.8240	10
11	85	37	88	39	89	40	90	40	91	43	11
12	87	39	91	42	92	42	93	43	94	45	12
13	90	42	93	44	94	45	95	46	96	48	13
14	92	45	96	47	497	47	798	48	099	50	14
15	95	47	198	49	500	50	800	51	102	53	15
16	898	50	201	52	02	52	03	53	04	56	16
17	900	52	03	54	05	55	05	56	06	58	17
18	03	55	06	57	07	57	08	58	09	60	18
19	1.6905	1.7057	1.7209	1.7359	1.7510	1.7660	1.7810	1.7961	1.8112	1.8262	19
20	1.6908	1.7060	1.7211	1.7362	1.7512	1.7662	1.7813	1.7963	1.8114	1.8265	20
21	10	62	14	64	15	65	15	66	17	67	21
22	13	65	16	67	17	68	18	68	19	70	22
23	16	68	19	69	20	70	20	71	21	72	23
24	18	70	21	72	22	73	23	73	23	75	24
25	21	73	24	74	25	75	25	76	26	78	25
26	23	75	26	77	27	78	28	78	28	80	26
27	26	78	29	79	30	80	30	81	31	83	27
28	28	80	31	82	32	83	33	83	33	85	28
29	1.6931	1.7083	1.7234	1.7385	1.7535	1.7685	1.7835	1.7985	1.8136	1.8288	29
30	1.6933	1.7085	1.7236	1.7387	1.7537	1.7688	1.7837	1.7988	1.8139	1.8290	30
31	36	88	39	90	40	90	39	90	41	93	31
32	38	90	41	92	42	92	42	93	44	95	32
33	41	93	44	95	45	94	45	95	46	98	33
34	44	95	47	397	47	97	47	1.7998	49	300	34
35	46	098	49	400	50	699	50	1.8000	51	03	35
36	49	101	52	02	53	702	52	03	54	06	36
37	51	03	54	05	55	04	55	05	56	08	37
38	54	06	57	07	58	07	57	08	59	11	38
39	1.6956	1.7108	1.7259	1.7410	1.7560	1.7709	1.7860	1.8010	1.8161	13	39
40	1.6959	1.7111	1.7262	1.7412	1.7562	1.7712	1.7862	1.8013	1.8164	1.8316	40
41	61	13	64	15	64	14	65	15	66	18	41
42	64	16	67	17	67	17	67	18	69	21	42
43	67	18	69	19	69	19	70	20	72	23	43
44	69	21	72	21	72	22	72	23	74	26	44
45	72	23	74	24	74	24	75	25	77	29	45
46	74	26	77	26	77	27	77	28	79	31	46
47	77	28	79	29	79	29	80	31	82	34	47
48	79	31	81	31	82	32	82	33	84	36	48
49	1.6982	1.7133	1.7284	1.7434	1.7584	1.7735	1.7885	1.8036	1.8187	1.8339	49
50	1.6984	1.7135	1.7286	1.7437	1.7587	1.7737	1.7887	1.8038	1.8189	1.8341	50
51	87	38	89	39	89	40	90	41	92	44	51
52	90	40	91	42	92	42	92	43	95	46	52
53	92	43	94	44	94	45	95	46	197	49	53
54	95	45	96	47	97	47	897	48	200	52	54
55	97	48	299	49	599	50	900	51	02	54	55
56	6999	50	301	52	602	52	03	53	05	57	56
57	1.7001	53	04	54	04	55	05	56	07	59	57
58	04	55	06	57	07	57	08	58	10	62	58
59	1.7006	1.7158	1.7309	1.7459	1.7610	1.7760	1.7910	1.8061	1.8212	1.8364	59
60	1.7009	1.7160	1.7311	1.7462	1.7612	1.7762	1.7913	1.8063	1.8215	1.8367	60

TABLE VI.

 $\mu + \log \tan z$.

z	50°	51°	52°	53°	54°	55°	56°	57°	58°	59°	z
0'	1.8367	1.8520	1.8674	1.8829	1.8985	1.9144	1.93044	1.94672	1.96325	1.98005	0'
1	69	22	76	31	88	46	072	700	353	033	1
2	72	25	79	33	90	49	099	728	381	062	2
3	75	28	82	36	93	52	126	755	408	090	3
4	77	30	84	39	96	54	153	782	436	118	4
5	80	33	87	41	1.8998	57	180	809	464	146	5
6	82	35	89	44	1.9001	60	207	837	492	174	6
7	85	37	91	47	04	62	234	865	520	203	7
8	87	39	94	49	06	65	261	892	548	232	8
9	1.8390	1.8542	1.8696	1.8852	1.9009	1.9168	1.93288	1.94919	1.96575	1.98260	9
10	1.8392	1.8545	1.8699	1.8854	1.9012	1.9170	1.93315	1.94946	1.96603	1.98288	10
11	94	47	701	57	14	73	342	1.94973	631	316	11
12	97	50	04	60	17	76	369	1.95001	659	345	12
13	399	52	07	62	19	78	396	028	687	373	13
14	402	55	09	65	22	81	423	055	714	402	14
15	04	58	12	68	25	84	450	083	742	430	15
16	07	60	14	70	27	86	477	111	770	458	16
17	09	63	17	73	30	89	504	138	798	487	17
18	12	65	20	75	33	92	531	166	826	515	18
19	1.8415	1.8568	1.8722	1.8878	1.9035	1.9195	1.93558	1.95193	1.96854	1.98544	19
20	1.8417	1.8570	1.8725	1.8881	1.9038	1.9197	1.93585	1.95220	1.96881	1.98573	20
21	20	73	27	83	41	200	612	248	909	601	21
22	22	76	30	86	43	03	639	275	937	629	22
23	25	78	33	88	46	05	666	303	965	658	23
24	27	81	35	91	49	08	693	330	1.96993	687	24
25	30	83	38	94	51	11	720	357	1.97021	715	25
26	33	86	40	96	54	13	747	385	049	744	26
27	35	88	43	99	57	16	774	413	077	772	27
28	38	91	46	002	59	19	802	441	105	800	28
29	1.8440	1.8594	1.8748	1.8904	1.9062	1.9221	1.93829	1.95468	1.97133	1.98829	29
30	1.8443	1.8596	1.8751	1.8907	1.9065	1.9224	1.93856	1.95495	1.97161	1.98857	30
31	45	599	53	10	67	27	883	522	189	886	31
32	48	601	56	12	70	30	910	550	217	915	32
33	50	04	59	15	73	32	937	578	245	944	33
34	53	07	61	17	75	35	965	606	273	1.98972	34
35	56	09	64	20	78	38	1.93992	634	301	1.99000	35
36	58	12	67	23	81	40	1.94019	661	329	029	36
37	61	14	69	25	83	43	046	688	357	058	37
38	63	17	72	28	86	46	073	716	385	086	38
39	1.8466	1.8620	1.8774	1.8931	1.9089	1.9248	1.94100	1.95743	1.97413	1.99115	39
40	1.8468	1.8622	1.8777	1.8933	1.9091	1.9251	1.94127	1.95771	1.97441	1.99143	40
41	71	25	80	36	94	54	154	798	469	171	41
42	74	27	82	39	96	57	182	825	497	200	42
43	76	30	85	41	099	59	209	853	525	229	43
44	79	32	87	44	102	62	236	881	554	258	44
45	81	35	90	46	05	65	263	909	582	287	45
46	84	38	93	49	07	67	290	937	610	315	46
47	87	40	95	52	10	70	317	964	638	344	47
48	89	43	798	54	12	72	345	1.95991	666	372	48
49	1.8492	1.8645	1.8800	1.8957	1.9115	1.9274	1.94372	1.96019	1.97694	1.99401	49
50	1.8494	1.8648	1.8803	1.8960	1.9117	1.9277	1.94400	1.96047	1.97723	1.99430	50
51	97	51	06	62	20	80	427	075	751	459	51
52	499	53	08	65	22	82	454	102	779	487	52
53	502	56	11	68	25	85	481	130	807	516	53
54	04	58	14	70	28	88	508	158	835	545	54
55	07	61	16	72	30	91	536	186	864	574	55
56	10	63	19	74	33	93	564	214	892	603	56
57	12	66	22	77	36	96	591	242	920	632	57
58	15	69	24	80	38	299	618	269	948	660	58
59	1.8517	1.8671	1.8827	1.8982	1.9141	1.9301	1.94645	1.96297	1.97977	1.99689	59
60	1.8520	1.8674	1.8829	1.8985	1.9144	1.9304	1.94672	1.96325	1.98005	1.99718	60

TABLE VI.

 $\mu + \log \tan z.$

z	60°	61°	62°	63°	64°	65°	66°	67°	68°	69°	z
0'	1.99718	2.01462	2.03244	2.05068	2.06938	2.08860	2.10837	2.12874	2.14979	2.17160	0'
1	747	492	274	099	2.06970	893	871	909	2.15015	197	1
2	776	521	304	130	2.07002	925	904	943	051	234	2
3	804	550	334	160	033	957	937	2.12977	087	270	3
4	833	580	364	191	064	2.08990	2.10971	1.13012	122	307	4
5	862	609	394	222	096	2.09023	2.11005	047	158	344	5
6	891	639	425	253	128	055	038	082	194	381	6
7	920	669	455	284	160	088	072	116	230	419	7
8	949	698	485	315	192	121	105	151	266	456	8
9	1.99977	2.01727	2.03515	2.05345	2.07224	2.09153	2.11138	2.13185	2.15302	2.17493	9
10	2.00006	2.01757	2.03545	2.05376	2.07255	2.09186	2.11172	2.13220	2.15337	2.17530	10
11	035	786	575	407	286	219	206	254	373	568	11
12	064	816	605	438	318	251	239	289	409	605	12
13	093	845	635	469	350	284	273	324	445	642	13
14	121	875	665	500	382	316	306	358	481	679	14
15	150	904	695	531	414	349	340	393	517	716	15
16	179	934	726	562	445	382	374	428	553	754	16
17	208	963	756	593	477	415	408	463	589	792	17
18	237	2.01993	787	624	509	448	442	498	625	829	18
19	2.00266	2.02023	2.03817	2.05655	2.07541	2.09481	2.11476	2.13533	2.15661	2.17866	19
20	2.00295	2.02052	2.03847	2.05686	2.07573	2.09513	2.11509	2.13568	2.15697	2.17903	20
21	324	082	877	717	605	546	543	603	733	941	21
22	353	111	908	749	637	579	577	638	769	2.17979	22
23	382	141	938	780	669	612	611	673	805	2.18016	23
24	411	171	968	811	701	645	645	708	842	054	24
25	440	200	2.03998	842	733	677	678	742	878	091	25
26	469	230	2.04029	873	765	710	712	777	915	128	26
27	498	259	059	904	797	743	746	812	951	166	27
28	528	289	090	935	829	776	780	847	2.15987	204	28
29	2.00557	2.02319	2.04120	2.05966	2.07861	2.09809	2.11814	2.13883	2.16023	2.18242	29
30	2.00586	2.02348	2.04150	2.05997	2.07893	2.09842	2.11848	2.13918	2.16059	2.18280	30
31	615	378	181	2.06028	925	874	881	953	096	317	31
32	644	407	211	059	957	907	916	2.13988	133	355	32
33	673	437	242	090	2.07989	940	950	2.14023	169	393	33
34	702	467	272	122	2.08021	2.09973	2.11984	058	205	431	34
35	731	496	302	153	053	2.10006	2.12018	093	242	469	35
36	760	526	332	184	085	038	052	128	278	506	36
37	789	555	363	215	117	071	086	163	315	544	37
38	819	585	393	247	150	105	120	199	351	582	38
39	2.00848	2.02615	2.04424	2.06278	2.08182	2.10138	2.12154	2.14235	2.16387	2.18620	39
40	2.00877	2.02645	2.04454	2.06309	2.08214	2.10171	2.12188	2.14270	2.16423	2.18658	40
41	906	675	484	340	246	204	222	305	420	696	41
42	935	705	515	372	278	237	256	340	497	734	42
43	964	735	546	403	310	270	291	376	534	772	43
44	2.00993	765	577	435	343	304	325	411	571	811	44
45	2.01023	795	607	466	375	337	359	447	608	849	45
46	052	825	638	497	407	370	393	482	644	887	46
47	081	855	668	528	439	403	427	517	680	925	47
48	110	885	699	560	471	436	462	552	717	2.18963	48
49	2.01139	2.02915	2.04730	2.06591	2.08504	2.10470	2.12496	2.14588	2.16754	2.19001	49
50	2.01169	2.02945	2.04761	2.06623	2.08536	2.10504	2.12530	2.14623	2.16791	2.19039	50
51	198	2.02975	792	655	568	537	565	659	828	078	51
52	227	2.03004	822	686	600	570	599	694	864	115	52
53	256	034	852	718	633	603	633	729	901	153	53
54	286	064	883	749	665	637	668	765	938	192	54
55	315	094	914	781	698	670	702	801	2.16975	230	55
56	345	124	945	813	730	704	737	836	2.17012	269	56
57	374	154	2.04976	844	763	737	771	872	049	307	57
58	403	184	2.05006	875	795	770	805	907	086	345	58
59	2.01433	2.03214	2.05037	2.06907	2.08828	2.10804	2.12840	2.14943	2.17123	2.19384	59
60	2.01462	2.03244	2.05068	2.06938	2.08860	2.10837	2.12874	2.14979	2.17160	2.19422	60

TABLE VI.

 $\mu + \log \tan z$.

z	70°	71°	72°	73°	74°	75°	76°	77°	78°	79°	z
0'	2.19422	2.21778	2.24236	2.26811	2.29517	2.32367	2.35383	2.38591	2.42015	2.45693	0'
1	461	818	278	855	564	415	435	646	074	757	1
2	500	858	320	900	610	464	487	701	133	821	2
3	538	898	362	944	656	512	539	756	192	884	3
4	2.19576	2.21938	2.24404	2.26988	2.29702	2.32561	2.35591	2.38811	2.42251	2.45948	4
5	2.19615	2.21978	2.24446	2.27032	2.29749	2.32610	2.35643	2.38867	2.42310	2.46012	5
6	654	2.22019	489	076	795	660	695	922	370	076	6
7	693	059	531	120	842	709	747	2.38978	430	141	7
8	731	099	573	165	889	758	799	2.39034	490	205	8
9	2.19769	2.22139	2.24615	2.27209	2.29935	2.32807	2.35851	2.39089	2.42550	2.46269	9
10	2.19808	2.22180	2.24657	2.27253	2.29981	2.32857	2.35904	2.39144	2.42610	2.46333	10
11	847	220	699	297	2.30028	906	2.35956	200	670	398	11
12	886	261	742	342	075	2.32956	2.36009	256	730	462	12
13	925	301	784	386	122	2.33005	061	312	790	527	13
14	2.19963	2.22341	2.24826	2.27431	2.30168	2.33054	2.36113	2.39368	2.42850	2.46591	14
15	2.20002	2.22382	2.24869	2.27475	2.30215	2.33104	2.36166	2.39424	2.42910	2.46656	15
16	041	423	912	519	262	154	218	480	2.42970	721	16
17	080	464	954	564	309	203	271	536	2.43030	786	17
18	119	505	2.24997	609	356	253	324	593	090	851	18
19	2.20158	2.22545	2.25039	2.27654	2.30403	2.33303	2.36377	2.39649	2.43150	2.46917	19
20	2.20197	2.22585	2.25081	2.27699	2.30450	2.33352	2.36430	2.39706	2.43210	2.46982	20
21	236	626	124	744	497	402	483	762	271	2.47047	21
22	275	667	167	788	545	452	536	818	332	112	22
23	314	708	209	833	592	502	590	875	393	178	23
24	2.20353	2.22749	2.25252	2.27878	2.30639	2.33552	2.36643	2.39932	2.43454	2.47243	24
25	2.20392	2.22789	2.25294	2.27923	2.30685	2.33602	2.36696	2.39989	2.43515	2.47308	25
26	431	830	337	2.27968	733	652	749	2.40046	576	374	26
27	470	871	380	2.28013	780	702	802	103	637	440	27
28	510	912	423	057	827	752	855	159	698	506	28
29	2.20549	2.22953	2.25466	2.28102	2.30875	2.33803	2.36909	2.40216	2.43759	2.47572	29
30	2.20588	2.22994	2.25509	2.28148	2.30922	2.33853	2.36963	2.40273	2.43820	2.47638	30
31	627	2.23035	551	193	2.30967	903	2.37016	330	881	704	31
32	666	076	594	238	2.31017	2.33953	069	387	2.43943	770	32
33	706	117	637	283	065	2.34004	123	444	2.44005	836	33
34	2.20745	2.23158	2.25680	2.28328	2.31113	2.34054	2.37176	2.40501	2.44066	2.47903	34
35	2.20785	2.23200	2.25723	2.28374	2.31161	2.34105	2.37230	2.40559	2.44128	2.47970	35
36	824	241	766	419	208	155	283	616	189	2.48036	36
37	863	282	809	464	256	205	337	674	251	103	37
38	903	323	852	510	304	256	391	732	313	170	38
39	2.20943	2.23364	2.25896	2.28555	2.31352	2.34307	2.37445	2.40790	2.44375	2.48237	39
40	2.20983	2.23405	2.25939	2.28600	2.31400	2.34358	2.37499	2.40847	2.44437	2.48304	40
41	2.21022	446	2.25982	645	447	408	553	904	499	371	41
42	061	487	2.26025	691	495	459	607	2.40962	561	439	42
43	101	529	069	736	543	510	661	2.41020	624	506	43
44	2.21141	2.23570	2.26112	2.28782	2.31592	2.34561	2.37715	2.41078	2.44686	2.48574	44
45	2.21181	2.23612	2.26156	2.28828	2.31640	2.34612	2.37770	2.41136	2.44749	2.48641	45
46	220	653	199	873	688	663	824	194	811	709	46
47	259	694	242	919	736	714	878	252	873	776	47
48	299	736	286	2.28965	784	765	933	311	936	844	48
49	2.21339	2.23778	2.26329	2.29011	2.31833	2.34817	2.37988	2.41370	2.44999	2.48912	49
50	2.21379	2.23819	2.26373	2.29057	2.31881	2.34868	2.38042	2.41428	2.45061	2.48980	50
51	419	861	417	103	930	919	096	487	124	2.49048	51
52	458	902	460	148	2.31978	2.34970	150	545	187	116	52
53	498	944	504	195	2.32026	2.35021	205	603	250	184	53
54	2.21538	2.23986	2.26548	2.29241	2.32075	2.35073	2.38260	2.41662	2.45313	2.49253	54
55	2.21578	2.24028	2.26592	2.29287	2.32124	2.35125	2.38315	2.41721	2.45376	2.49322	55
56	618	070	636	333	173	177	370	780	440	390	56
57	658	112	680	379	221	228	425	838	503	459	57
58	698	153	723	425	269	279	480	896	566	57	58
59	2.21738	2.24194	2.26767	2.29471	2.32318	2.35331	2.38535	2.41955	2.45629	2.49596	59
60	2.21778	2.24236	2.26811	2.29517	2.32367	2.35383	2.38591	2.42015	2.45693	2.49665	60

TABLE VI.

 $\mu + \log \tan z.$

z	80°	81°	82°	83°	84°	85°	86°	87°	88°	89°	z
0'	2.49665	2.53986	2.58722	2.63958	2.69809	2.76427	2.84003	2.92802	3.03197	3.15661	0'
1	734	2.54061	805	2.64050	2.69913	545	139	2.92961	3.03388	3.15890	1
2	803	136	888	142	2.70017	663	275	2.93121	3.03579	3.16120	2
3	872	211	2.58971	234	120	781	412	2.93280	3.03769	3.16350	3
4	2.49942	2.54286	2.59054	2.64327	2.70224	2.76900	2.84548	2.93440	3.03960	3.16581	4
5	2.50011	2.54362	2.59137	2.64420	2.70328	2.77018	2.84685	2.93600	3.04151	3.16813	5
6	081	438	221	513	432	137	823	2.93761	3.04343	3.17046	6
7	150	514	305	607	537	257	2.84961	2.93923	3.04536	3.17279	7
8	220	591	389	701	642	377	2.85099	2.94086	3.04729	3.17513	8
9	2.50289	2.54667	2.59473	2.64794	2.70747	2.77497	2.85238	2.94249	3.04921	3.17747	9
10	2.50359	2.54743	2.59557	2.64888	2.70852	2.77617	2.85376	2.94412	3.05114	3.17982	10
11	429	820	642	2.64982	2.70958	737	515	2.94576	3.05309	3.18218	11
12	499	897	726	2.65077	2.71064	858	655	2.94740	3.05506	3.18455	12
13	569	2.54974	811	171	170	2.77979	795	2.94905	3.05703	3.18693	13
14	2.50639	2.55051	2.59895	2.65265	2.71276	2.78099	2.85935	2.95070	3.05900	3.18932	14
15	2.50709	2.55128	2.59980	2.65360	2.71382	2.78221	2.86076	2.95235	3.06097	3.19172	15
16	779	205	2.60066	455	489	344	217	2.95401	3.06295	3.19413	16
17	850	282	151	551	597	466	358	2.95567	3.06494	3.19655	17
18	921	359	236	646	704	589	500	2.95734	3.06693	3.19898	18
19	2.50992	2.55437	2.60321	2.65742	2.71812	2.78712	2.86642	2.95902	3.06893	3.20142	19
20	2.51063	2.55514	2.60406	2.65837	2.71919	2.78834	2.86785	2.96070	3.07093	3.20385	20
21	134	592	492	2.65933	2.72027	2.78957	2.86928	2.96238	3.07294	3.20629	21
22	205	669	578	2.66029	135	2.79081	2.87071	2.96406	3.07495	3.20874	22
23	276	747	664	125	244	205	214	2.96574	3.07697	3.21120	23
24	2.51347	2.55826	2.60750	2.66221	2.72353	2.79330	2.87358	2.96743	3.07899	3.21368	24
25	2.51418	2.55904	2.60836	2.66317	2.72461	2.79454	2.87502	2.96913	3.08101	3.21615	25
26	489	2.55982	2.60923	414	570	579	647	2.97083	3.08305	3.21864	26
27	561	2.56060	2.61010	511	680	704	792	2.97254	3.08510	3.22113	27
28	633	138	097	609	790	830	2.87937	2.97426	3.08716	3.22364	28
29	2.51705	2.56217	2.61185	2.66706	2.72900	2.79955	2.88083	2.97598	3.08923	3.22616	29
30	2.51777	2.56296	2.61272	2.66803	2.73010	2.80081	2.88229	2.97771	3.09131	3.22868	30
31	849	376	359	900	120	207	376	2.97944	3.09338	3.23120	31
32	922	455	446	2.66997	231	333	523	2.98117	3.09546	3.23373	32
33	2.51994	534	534	2.67095	342	460	671	2.98291	3.09755	3.23627	33
34	2.52067	2.56613	2.61622	2.67193	2.73453	2.80587	2.88818	2.98466	3.09965	3.23882	34
35	2.52140	2.56693	2.61710	2.67291	2.73565	2.80714	2.88966	2.98642	3.10175	3.24139	35
36	212	772	798	389	676	841	2.89115	2.98817	3.10386	3.24396	36
37	284	852	886	488	788	2.80969	264	2.98993	3.10597	3.24654	37
38	357	2.56932	2.61974	587	2.73900	2.81097	414	2.99170	3.10809	3.24913	38
39	2.52430	2.57012	2.62063	2.67687	2.74012	2.81226	2.89554	2.99348	3.11023	3.25173	39
40	2.52503	2.57092	2.62152	2.67786	2.74125	2.81355	2.89714	2.99526	3.11238	3.25434	40
41	576	172	240	885	237	484	2.89864	2.99705	3.11452	3.25695	41
42	649	252	329	2.67984	350	614	2.90015	2.99885	3.11667	3.25957	42
43	723	333	418	2.68084	463	744	167	3.00066	3.11882	3.26220	43
44	2.52797	2.57413	2.62508	2.68184	2.74577	2.81874	2.90319	3.00246	3.12098	3.26484	44
45	2.52870	2.57494	2.62598	2.68284	2.74691	2.82005	2.90471	3.00427	3.12315	3.26749	45
46	2.52944	575	687	384	805	136	623	3.00608	3.12532	3.27016	46
47	2.53017	656	776	484	2.74918	267	776	3.00789	3.12750	3.27284	47
48	091	737	866	585	2.75032	398	2.90929	3.00970	3.12969	3.27552	48
49	2.53165	2.57819	2.62956	2.68686	2.75147	2.82530	2.91083	3.01152	3.13189	3.27821	49
50	2.53239	2.57900	2.63046	2.68787	2.75262	2.82663	2.91237	3.01334	3.13411	3.28091	50
51	313	2.57982	137	888	377	795	392	3.01517	3.13633	3.28362	51
52	387	2.58063	227	2.68989	492	2.82927	547	3.01700	3.13855	3.28634	52
53	462	146	317	2.69091	608	2.83060	702	3.01884	3.14078	3.28907	53
54	2.53536	2.58228	2.63408	2.69193	2.75724	2.83194	2.91858	3.02070	3.14302	3.29181	54
55	2.53611	2.58310	2.63499	2.69295	2.75841	2.83328	2.92014	3.02256	3.14527	3.29456	55
56	686	393	591	398	2.75958	463	171	3.02443	3.14753	3.29732	56
57	761	475	682	501	2.76075	598	328	3.02630	3.14979	3.30009	57
58	835	557	773	603	192	732	485	3.02817	3.15206	3.30286	58
59	2.53910	2.58639	2.63865	2.69706	2.76309	2.83867	2.92643	3.03006	3.15433	3.30564	59
60	2.53986	2.58722	2.63958	2.69809	2.76427	2.84003	2.92802	3.03197	3.15661	3.30842	60

SHORT METHODS OF DETERMINING ORBITS

SECOND PAPER.

With Plates.

By A. O. LEUSCHNER,

DIRECTOR OF THE STUDENTS' OBSERVATORY,

BERKELEY ASTRONOMICAL DEPARTMENT

Section of the American Mathematical Society. In this case the solution of the orbit without assumption regarding the eccentricity was indeterminate by any known method. But in this connection it was shown that the *Short Method* readily permits of the determination of the limits of the period and eccentricity within which, in more or less indeterminate cases, the observations may be satisfied, and hints were given for the computation of a parabola when the upper limit of the period or semimajor axis is found to be very large. With the application of these principles the *Short Method* gave parabolic orbits of comet *a* 1904 for the different sets of observations under consideration in accordance with the parabolic orbits deduced by the usual methods. Hence, even in this exceptional case, the *Short Method* in a modified form has yielded results as accurate as may be desired.

Concerning LAPLACE'S original method,¹ BAUSCHINGER quotes LAGRANGE² as having reached the conclusion that analytically it constitutes the simplest solution of the problem, but that in practice it does not afford corresponding advantages, because the differential coefficients could not be determined with the necessary accuracy. POINCARÉ, on the contrary, in a paper entitled *Sur la Détermination des Orbits par la Méthode de Laplace*³ has recently shown that in the first hypothesis LAPLACE'S method is superior to that of GAUSS, inasmuch as for unequal intervals LAPLACE'S error in the geocentric distance ρ is only of the second order as compared to GAUSS'S error of the first order.

If we examine further into the question in dispute, it appears that of late a very important factor has been left out of consideration in judging the efficiency of the methods which are based on LAPLACE. All published improvements of the same are methods of determining orbits without assumption regarding the eccentricity. They ought, therefore, to be compared only with similar methods, as POINCARÉ has done. The preliminary orbits of comets derived, for instance, by the *Short Method* in its original form (Part I) are not always comparable to parabolic orbits derived from the same observations by OLBERS'S method. If the physical orbit of a comet happen to be nearly parabolic, then a preliminary orbit derived from a short arc by OLBERS'S method will approximate the true orbit more or less closely. If the physical orbit differ considerably from a parabola, then a preliminary orbit derived by parabolic methods is based on erroneous assumptions and will fail to represent subsequent observations. The *Short Method* in its original form (Part I) produces a preliminary orbit without assumption regarding the eccentricity. The degree of approximation of this orbit to the physical orbit depends on the accuracy with which the observations, in the nature of things, permit of the determination of a general orbit. If the range of practical solutions within which the observations may be represented be limited, then the *Short*

¹LAPLACE, *Méc. Cél.*, t. I, première partie, livre II, chap. IV.

²LAGRANGE, *Oeuvres*, t. XIV, p. 108.

³*Bulletin Astronomique*, t. XXIII.

the necessity of introducing an additional observation. The method of determining circular orbits outlined in these pages is, therefore, not intended to lessen the numerical work required by the usual methods, but it has a far more extended range of usefulness.

The comet orbits computed at the Students' Observatory on the basis of the principles just outlined, demonstrate that the accuracy of the differential coefficients of the geocentric coördinates is such as to lead to orbits which agree more closely with the true orbits than orbits computed from similar arcs by other methods. Even a mere cursory¹ examination of the orbits, contained in *Lick Observatory Bulletins* 73 and thereafter, reveals that, with but one exception, they approximate the final orbits as closely as those computed from much longer arcs by other methods, in fact in general from arcs twice the length. The material for this comparison may be derived from the *A. N.* The final orbits are probably not as yet available for any of these comets, but the orbit computed from the longest geocentric arc will serve the purpose of comparison equally well. It is safe to conclude on the basis of these results that if, formerly, discrepancies from the true orbits have been observed in orbits derived by LAPLACEAN methods, the explanation therefor must be sought not in the insufficiency of the differential coefficients, but in the range of practical solutions, for it is safe to assume that no orbit has been published that does not actually satisfy the observations on which it is based. This range becomes highly restricted if an assumption be made regarding the eccentricity.

Orbits which are indeterminate are of two classes. They may be indeterminate, on the one hand, because the observations can actually be represented by a variety of orbits; on the other hand, because the adopted methods of solution may lead to an indeterminate result, although the orbit may be well defined by the given observations. Thus, for example, a solution by GAUSS's method becomes indeterminate if the three observations lie in a great circle; but this geometrical condition in itself does not in the least affect the accuracy of an orbit derived by LAPLACEAN methods, as for example by the *Short Method*. In dealing with cases of the first class, recourse must be had to an assumption regarding the eccentricity. For the second class, for which general methods are applicable, and for determinate cases, the solutions based on LAPLACEAN methods give in the first hypothesis results of greater accuracy than solutions by GAUSS's method, as shown above by practical experience as well as analytically by reference to POINCARÉ's investigation and my own.

The foregoing considerations apply to the first hypothesis in preliminary orbits, which is always based on moderate arcs. To discuss the relative merits in the first hypothesis of various methods on the basis of arcs of *any* length would be of purely theoretical interest and of no practical value, as moderate arcs are always available. In the study of the practical problem of determining preliminary orbits

¹ Compare, also, CRAWFORD's conclusions in Part 8.

coördinates in the methods based on LAPLACE. The results of the first hypothesis are used for the determination of more approximate values of these quantities, which then form the basis of the second hypothesis, and so forth. Where ratios of triangles are used, the convergence of the successive hypotheses depends in the main on the intervals, on r , and $\frac{dr}{dt}$ and may become very slow. If the highest accuracy be aimed at, as in v. OPPOLZER's general method, the computation is very laborious. Thus, in v. OPPOLZER's theoretically very elegant and precise method, it is necessary, as I have mentioned in my paper *On the General Applicability*, etc., to derive hypothesis after hypothesis, to perform a series of trials for the distances in each hypothesis, and to perform for each triangular ratio a series of approximations in passing from one hypothesis to the next. GIBBS's vector method, although the most accurate method of all in the first hypothesis, but unfortunately requiring too large an amount of numerical work, does not readily lend itself for application to a second hypothesis, and POINCARÉ's recent presentation of LAPLACE's method has not as yet been cast in a suitable form for practical application.

Instead of forming a second hypothesis in the usual sense by correcting the geocentric velocities and accelerations on the basis of the first solution, it is here proposed, as in Part I, to determine at once from differential relations such corrections to the geocentric distance and to the velocities in the heliocentric rectangular coördinates,—which, together with the geocentric coördinates at the middle date, practically, are the elements of the orbit,—as will remove whatever residuals the first solution may leave in the first and third geocentric places. This mode of approximation is not comparable to the method of forming successive hypotheses, but obviously leads directly to the required result. Hitherto it has been applied to the solution of more or less definitive orbits from normal places, with the aid of the differential coefficients of the geocentric coördinates with respect to the geometrical elements. For this purpose very elegant solutions have been arranged, chief among which is that by BAUSCHINGER.¹ It may safely be asserted that, with proper adaptation to the computation of strictly parabolic and circular preliminary orbits so as to include cases in which a general solution is unwarranted, the methods of differential correction will lead more directly to the final result than the method of successive hypotheses in the usual sense.

The advantages of the method of differential correction, over the method of successive hypotheses in improving an orbit, are exactly the same as the advantages of the method of differential correction over the ordinary trials in the successive approximations for the distances in OLBERS's and GAUSS's methods, etc. In the course of the ordinary trials, the final values of the distances of one trial form the initial values in the next trial, etc. In the method of differential correction, such corrections to the initial values of one trial are derived differentially from

¹*Veröffentlichungen des Kgl. Recheninstituts, Berlin, No. 23.*

the differences between the initial and final values in the same trial, as will produce an agreement of the initial and final values in the next trial. But the number of approximations required by the ordinary trials is, in general, far in excess of that required by the method of differential correction.

Another simplification involved in the direct solution and the differential correction proposed here is that the consideration of various topics essential to the older methods becomes unnecessary, as, for instance, the consideration of the ratio of sector to triangle, etc.

If it had been deemed desirable or essential, formulæ on the plan of successive hypotheses might have been set up without difficulty. For this purpose it is not necessary to determine the corrections to the initial values of the velocities and accelerations from the results of the first solution. Instead, an expression of $\sigma = \rho \cos \delta$ may be derived in which account is taken of the third and higher derivatives expressed in terms of the coördinates and their lower derivatives. The actual derivation of these expressions of the higher derivatives is somewhat complicated, although theoretically quite simple, but the resulting formula for σ admits of a second hypothesis as soon as r and $r' = \frac{1}{k} \frac{dr}{dt}$ become known in the first hypothesis. In this manner an expression for κ has been derived on page 268, formula (4a), for the purpose of discussing the degree of approximation of the only hypothesis made in the *Short Method*. The formula, however, also serves the purpose of indicating the principles on which we would proceed in making a second hypothesis. For the purpose in hand, it was deemed unnecessary to take account of derivatives of the geocentric coördinates higher than the third.

POINCARÉ'S discussion of LAPLACE'S method is the most elegant treatment of that subject yet attained. His choice of the direction cosines, their velocities, and accelerations as fundamental data leads to a symmetry in the developments affording a clear perspective of the method and permitting of easy determination of its accuracy in relation to other methods. His discussion is also full of suggestions from a practical point of view. If none of these have been adopted here it is due to the fact that the formulæ presented below had already been applied to numerous cases prior to the publication of POINCARÉ'S paper and had proved entirely sufficient.

With reference to the corrections for parallax and aberration, POINCARÉ points out that they can be dealt with in a simpler manner than has been done in Parts 1-3. This was recognized soon after the publication of the *Short Method* in 1902 and was referred to in my *Notes on the Short Method of Determining Orbits*, etc., presented at the 1902 meeting of the Astronomical and Astrophysical Society of America.¹ The details of the elimination referred to are given on page 233 *et seq.*

With reference to the question of multiple solutions it is shown, among other things, on page 291, that there never can exist more than one physical parabolic

¹*Publications*, Vol. 1, pp. 194-195.

solution. By applying the criteria on page 288 the computer may detect the existence of additional mathematical solutions, of which there may be two. By following the directions outlined on page 291 he may then discard the two fictitious out of the three mathematical parabolic solutions with the aid of the tables at the end of this paper. It is, indeed, one of the chief advantages of the method outlined here that the possibility of adopting a fictitious for the true solution is eliminated in the course of the computation. Further advantages are that the feasibility of a parabolic solution may be readily tested; that transference may be made to a general solution with little additional labor, and that the degree of accuracy of the adopted solution, which determines the range of practical solutions, is readily ascertained. While, however, there never can exist more than one solution when an assumption is made regarding the eccentricity, two solutions may sometimes occur when no such assumption is made; but at the present stage of the investigation it is found possible to distinguish the mathematical from the physical solution only in certain cases; that is to say, in the case of a general solution two orbits may sometimes result, satisfying the three given observations within the errors of observation and the degree of accuracy of the direct solution.

To assist the computer in the application of the methods derived here and in Part I, the various formulæ for the direct computation of an orbit, with or without assumption regarding the eccentricity, and for the derivation of an orbit on the basis of a previous approximation, are collected in a "Synopsis of Formulæ" at the end of this paper.

Acknowledgment is due to Professor CRAWFORD for many valuable suggestions, particularly as regards the arrangement of the formulæ in a convenient form for computation.

Geneva, Switzerland, April, 1909.

THE INITIAL GEOCENTRIC VELOCITIES AND ACCELERATIONS.

The geocentric velocities and accelerations derived from three places at short intervals as in formulæ II, page 15, are practically always sufficiently accurate to serve as the basis of an orbit computation. But, if the geocentric motion be very irregular, as is sometimes the case for comets, or if it be desired to base the preliminary orbit from the start on a longer arc, then it may become expedient to deduce the velocities and accelerations from more than three places. For long arcs or intervals, however, it is preferable to derive initial values of the fundamental data σ, x', y', z' from the existing approximate orbits.

If more than three places be made the basis of the computation, the initial geocentric velocities and accelerations may be determined graphically or analytically.

The graphical determination of the velocities and accelerations is accomplished in the usual manner by drawing smooth curves for the right ascensions and declinations, then tabulating these coördinates from the curves for equidistant intervals and finally computing their first and second derivatives by the formulæ of numerical differentiation.

The analytical determination of the velocities and accelerations from any number of observations is readily accomplished by successive approximation.

In either case, it is essential that the observed or apparent places are expressed in right ascension and declination, and are reduced to the mean equinox at the beginning of the year in the usual manner by including the aberration of the fixed stars, while the dates remain uncorrected for planetary aberration until the geocentric distances shall have been determined.

We select as epoch the time of an observation lying about the middle of the available range of dates. As in Part I, we shall call this date the normal or zero date t_0 , and designate the corresponding right ascension and declination by α_0 and δ_0 . Ultimately the epoch will be the true date of this observation. Any other of the available right ascensions may then be represented by TAYLOR'S theorem as follows:

$$\alpha_i = \alpha_0 + \frac{(t_i - t_0)}{1} \alpha' + \frac{(t_i - t_0)^2}{2} \alpha'' + \frac{(t_i - t_0)^3}{6} \alpha''' + \dots \quad (1)$$

where

$$\alpha''' = \frac{\partial^3 \alpha}{\partial t^3},$$

with a similar series for the declinations. The numerical values of the differential coefficients of α and δ at the epoch with respect to the time are to be determined from the observed values α_i and δ_i .

The number of coefficients that can be determined is restricted by the number of positions available in a preliminary series of observations. The number of differential coefficients which is necessary and sufficient in a given case depends on the nature of the geocentric motion. The number is sufficient if intermediate places not used in the derivation of the differential coefficients are represented by (1) within the accuracy of the observations.

The number of observations necessary to determine n differential coefficients is $n + 1$. The computation of these differential coefficients by a solution of n simultaneous equations of the form (1) would be very laborious without a convenient method of procedure, even if, as is generally the case, the number of differential coefficients to be determined is limited.

In the first approximation we confine ourselves to the determination of the first two differential coefficients from the first, last, and the normal observation of the series. This is accomplished in the same manner as the velocities and accelerations in α and δ are determined in formulæ II, page 15. The unit of time in these formulæ is $1/k$ mean solar days, where k^2 is the solar constant of attraction. Inasmuch as this unit can be introduced more conveniently later, we shall for the present choose the mean solar day as the unit of time in determining the successive derivatives. Hence, we let $k = 1$ in II, page 15. Let the velocities and accelerations so determined from three observations be denoted by

$$\alpha'_0, \delta'_0, \alpha''_0, \delta''_0 \quad (2a)$$

If determined from five observations by formula (1) the velocities and accelerations may be written

$$\alpha'_0 + \partial_1 \alpha'_0, \delta'_0 + \partial_1 \delta'_0, \alpha''_0 + \partial_1 \alpha''_0, \delta''_0 + \partial_1 \delta''_0 \quad (2b)$$

and if from seven observations

$$\alpha'_0 + \partial_1 \alpha'_0 + \partial_2 \alpha'_0, \delta'_0 + \partial_1 \delta'_0 + \partial_2 \delta'_0, \alpha''_0 + \partial_1 \alpha''_0 + \partial_2 \alpha''_0, \delta''_0 + \partial_1 \delta''_0 + \partial_2 \delta''_0 \quad (2c)$$

and so forth. For the true values α' , δ' , α'' , δ'' of the velocities and accelerations at the normal date we shall have

$$\begin{aligned} \alpha' &= \alpha'_0 + \partial_1 \alpha'_0 + \partial_2 \alpha'_0 + \dots \\ \alpha'' &= \alpha''_0 + \partial_1 \alpha''_0 + \partial_2 \alpha''_0 + \dots \end{aligned} \quad (2d)$$

with similar expressions for δ' and δ'' . In the successive approximations of the velocities and accelerations, the higher differential coefficients are neglected, viz., differential coefficients higher than the second in the first approximation, higher than the fourth in the second, and so forth. In general, we may denote a differential coefficient of any order resulting from the first approximation by α''_0 ; from the second by $\alpha''_0 + \partial_1 \alpha''_0$ from the third by $\alpha''_0 + \partial_1 \alpha''_0 + \partial_2 \alpha''_0$ *a. s. f.*, so that

$$\alpha''' = \alpha''_0 + \partial_1 \alpha''_0 + \partial_2 \alpha''_0 + \partial_3 \alpha''_0 + \dots,$$

and similarly for δ''' .

Hence,

$$\alpha''_0 = \alpha''_0 = \alpha''_0 = \alpha''_0 = \dots = 0$$

and

$$\partial_1 \alpha''_0 = \partial_1 \alpha''_0 = \dots = 0. \quad a. s. f.$$

For the second approximation, based on five observations, we have from (1) and (2)

$$\alpha_i = \alpha_0 + (t_i - t_0)[\alpha'_0 + \partial_1 \alpha'_0] + \frac{(t_i - t_0)^2}{2}[\alpha''_0 + \partial_1 \alpha''_0] + \frac{(t_i - t_0)^3}{6}\partial_1 \alpha'''_0 + \frac{(t_i - t_0)^4}{24}\partial_1 \alpha^{iv}_0, \quad (3)$$

$i = 1, 2, 4, 5$; $\alpha_0 = \alpha_3$, where the i are taken in the order of time.

Let us suppose that the first approximation was based on

$$\alpha_1, \quad \alpha_3, \quad \alpha_5.$$

Then putting

$$\left. \begin{aligned} \frac{\alpha_i - [\alpha_0 + (t_i - t_0)\alpha'_0 + \frac{(t_i - t_0)^2}{2}\alpha''_0]}{t_i - t_0} &= \partial \alpha_i = n_{i,a} \\ w_a &= \partial_1 \alpha^{iv}_0; & a_i &= \frac{(t_i - t_0)^3}{4!} \\ x_a &= \partial_1 \alpha'''_0; & b_i &= \frac{(t_i - t_0)^2}{3!} \\ y_a &= \partial_1 \alpha''_0; & c_i &= \frac{t_i - t_0}{2!} \\ z_a &= \partial_1 \alpha'_0; & d_i &= 1 \end{aligned} \right\} \quad (4)$$

we have the following system of equations

$$\left. \begin{aligned} a_1 w_a + b_1 x_a + c_1 y_a + z_a &= 0 & (a) \\ a_2 w_a + b_2 x_a + c_2 y_a + z_a &= n_{2,a} & (b) \\ a_4 w_a + b_4 x_a + c_4 y_a + z_a &= n_{4,a} & (c) \\ a_5 w_a + b_5 x_a + c_5 y_a + z_a &= 0 & (d) \end{aligned} \right\} \quad (5)$$

In the same way we shall have for the declinations

$$\left. \begin{aligned} a_1 w_\delta + b_1 x_\delta + c_1 y_\delta + z_\delta &= 0 & (a) \\ a_2 w_\delta + b_2 x_\delta + c_2 y_\delta + z_\delta &= n_{2,\delta} & (b) \\ a_4 w_\delta + b_4 x_\delta + c_4 y_\delta + z_\delta &= n_{4,\delta} & (c) \\ a_5 w_\delta + b_5 x_\delta + c_5 y_\delta + z_\delta &= 0 & (d) \end{aligned} \right\} \quad (6)$$

As only the unknowns y and z will ultimately be required, it is not necessary to compute the w and x . The following form of solution of (5) and (6) has been arranged after GAUSS's method of elimination. Neglecting the subscripts α and δ , equations (a) give:

$$w = -\frac{b_1}{a_1} x - \frac{c_1}{a_1} y - \frac{1}{a_1} z. \quad (7)$$

Eliminating w from (b), (c), (d), by means of (7) and letting

$$\left. \begin{aligned} p_i - \frac{a_i}{a_1} p_1 &= [p_i \cdot 1]; \quad p_i = b_i, c_i, 1 \\ i &= 2, 4, 5. \end{aligned} \right\} \quad (8)$$

we obtain

$$\left. \begin{aligned} [b_2 \cdot 1] x + [c_2 \cdot 1] y + [1_2 \cdot 1] z &= n_2 & (e) \\ [b_4 \cdot 1] x + [c_4 \cdot 1] y + [1_4 \cdot 1] z &= n_4 & (f) \\ [b_5 \cdot 1] x + [c_5 \cdot 1] y + [1_5 \cdot 1] z &= 0 & (g) \end{aligned} \right\} \quad (9)$$

Eliminating x from (e) and (f) by means of (g) and letting

$$\left. \begin{aligned} [p_i \cdot 1] - \frac{[b_i \cdot 1]}{[b_s \cdot 1]} [p_s \cdot 1] &= [p_i \cdot 2] \\ p_i &= c_i, \quad 1; \quad i = 2, 4, \end{aligned} \right\} \quad (10)$$

we have the following equation in y and z

$$\left. \begin{aligned} [c_2 \cdot 2] y + [1_2 \cdot 2] z &= n_2 \\ [c_4 \cdot 2] y + [1_4 \cdot 2] z &= n_4. \end{aligned} \right\} \quad (11)$$

From these

$$y = \frac{n_2 [1_4 \cdot 2] - n_4 [1_2 \cdot 2]}{[c_2 \cdot 2] [1_4 \cdot 2] - [c_4 \cdot 2] [1_2 \cdot 2]} ; \quad z = \frac{n_4 [c_2 \cdot 2] - n_2 [c_4 \cdot 2]}{[c_2 \cdot 2] [1_4 \cdot 2] - [c_4 \cdot 2] [1_2 \cdot 2]} \quad (12)$$

In practice this form of solution is extremely simple. Its chief advantage consists in the identity of the coefficients for both systems of equations, (5) and (6), so that after the $[p_i \cdot 2]$ coefficients are computed, equations (12) give:

$$y_a = \partial_1 \alpha'_0, \quad z_a = \partial_1 \alpha'_0, \text{ from } n_{2,a} \text{ and } n_{4,a},$$

and

$$y_s = \partial_1 \delta''_0, \quad z_s = \partial_1 \delta''_0, \text{ from } n_{2,s} \text{ and } n_{4,s}.$$

The combined number of $[p_i \cdot 1]$ and $[p_i \cdot 2]$ coefficients to be computed is thirteen. The numerical values of

$$\partial_1 \alpha'_0, \partial_1 \delta'_0, \partial_1 \alpha''_0, \partial_1 \delta''_0$$

are to be substituted in (2d) together with the first approximation for the velocities and accelerations defined in (2a), in order to obtain

$$\alpha', \delta', \alpha'', \delta''.$$

The procedure outlined above is readily extended to any given odd number of observations. The successive approximations have been based on an odd number of observations merely on account of the symmetry of the resulting formulæ. Similar forms of solution may readily be written out for an even number of observations. The velocities and accelerations may now be expressed in the proper unit of time. If the unit of time is to be $\frac{1}{k}$ mean solar days, α' and δ' must be divided by k , and α'' and δ'' by k^2 .

and still simpler procedure was proposed for the elimination of the parallax, the aberration requiring no further consideration in a preliminary orbit calculation. This procedure is as follows:

In the equations

$$\begin{aligned}\xi &= \rho \cos \delta_g \cos \alpha_g = x + X; & \eta &= \rho \cos \delta_g \sin \alpha_g = y + Y \\ \zeta &= \rho \sin \delta_g = z + Z,\end{aligned}\quad (1)$$

let $\xi, \eta, \zeta, \rho, \alpha_g, \delta_g, X, Y, Z$ be geocentric coördinates. Let $p_a^x \rho$ and $p_b^x \rho$ designate the parallax factors for the reduction of the observed place to the center of the Earth, expressed in seconds of arc, and let α and δ be the observed coördinates. Then we may write

$$\begin{aligned}\rho \cos (\delta + p_b^g) \cos (\alpha + p_a^g) &= x + X; & \rho \cos (\delta + p_b^g) \sin (\alpha + p_a^g) &= y + Y \\ \rho \sin (\delta + p_b^g) &= z + Z,\end{aligned}\quad (2)$$

or, expanding the sines and cosines and neglecting the second and higher powers of p_a^x and p_b^x , as well as their products,

$$\begin{aligned}\rho \cos \delta \cos \alpha - \rho \cos \delta \sin \alpha p_a^g - \rho \sin \delta \cos \alpha p_b^g &= x + X \\ \rho \cos \delta \sin \alpha + \rho \cos \delta \cos \alpha p_a^g - \rho \sin \delta \sin \alpha p_b^g &= y + Y \\ \rho \sin \delta + \rho \cos \delta p_b^g &= z + Z.\end{aligned}\quad (3)$$

Since the parallax factors $p_a^x \rho$ and $p_b^x \rho$ are known, the terms containing them may be applied as corrections to the solar coördinates. Hence, if we let

$$\begin{aligned}\Delta X &= \cos \delta \sin \alpha p_a^g \rho + \sin \delta \cos \alpha p_b^g \rho; & \Delta Y &= -\cos \delta \cos \alpha p_a^g \rho + \sin \delta \sin \alpha p_b^g \rho \\ \Delta Z &= -\cos \delta p_b^g \rho,\end{aligned}\quad (4)$$

we may write

$$\begin{aligned}(\xi) &= \rho \cos \delta \cos \alpha = x + X + \Delta X; & (\eta) &= \rho \cos \delta \sin \alpha = y + Y + \Delta Y \\ (\zeta) &= \rho \sin \delta = z + Z + \Delta Z,\end{aligned}\quad (5)$$

where ρ is the geocentric distance, while α and δ are referred to the observer. In computing the corrections to the solar coördinates by (4), $p_a^x \rho$ and $p_b^x \rho$ must be multiplied by $\sin 1''$ to reduce the corrections to the proper unit.

It will be observed that by correcting the solar coördinates by (4), the parallax is fully taken into account in computing coördinates by (5). Thus, for instance, after the value of $\sigma_0 = \rho_0 \cos \delta_0$ has been derived by V, page 16, the heliocentric coördinates at the middle date, needed in VI, follow at once from (5). Similarly in the comparison between theory and observation for the first and third places no correction for parallax needs to be applied, it being contained in (5). But it remains to investigate what effect this elimination of the parallax has on the velocities and accelerations which are involved.

By successive differentiation of expressions (5) we obtain

$$(\xi)'' = (\rho \cos \delta \cos \alpha)'' = x'' + [X + \Delta X]'' \quad (6)$$

and similar expressions for $(\eta)''$ and $(\zeta)''$. If the α_g and δ_g were given at the start, the same operation might have been performed on (1). We should then have to derive the velocities and accelerations in α and δ from geocentric positions and the expressions would have been correct, so far as *this* parallax is concerned. But as the α_g and δ_g are not available at the start, it is proposed to use (6) and to introduce into these formulæ the velocities and accelerations derived by numerical differ-

with similar equations in Y and Z , where m and m_1 represent the mass of the Earth and of the Moon, respectively, and where the subscript m signifies that the coördinates are referred to the center of mass of the system Earth-Moon. The necessity of reducing the geocentric solar coördinates taken from an astronomical ephemeris to the center of mass of the system Earth-Moon, before they may be introduced for the elimination of the solar accelerations in direct orbit solutions by LAPLACEAN methods has been pointed out by BRUNS in his discussion of LAMBERT'S theorem.¹ The reduced places are referred to by BRUNS as *barycentric* places.

In the equation $\xi = x + X$, it is, of course, immaterial whether ξ and X are referred to the observer, or to the center of the Earth, or to the center of mass of Earth-Moon system, or to any other arbitrary intermediate origin, so long as ξ and X are referred to the *same* intermediate origin. The same statement holds for the equations $\xi' = x' + X'$ and $\xi'' = x'' + X''$, which give the heliocentric velocities and accelerations. But since in the development of the formulæ of the LAPLACEAN methods the solar accelerations are to be eliminated by means of the equations of motion (9a) and since these equations are rigorously true only with reference to the center of mass of the Earth-Moon system, therefore ξ and X must also be referred to the center of mass. In a rigorous solution of the orbit, equations (1) and their derivations must, therefore, be replaced by the equations

$$\xi_m = \rho_m \cos \alpha_m \cos \delta_m = x + X_m \quad (1a)$$

$$\xi'_m = (\rho_m \cos \alpha_m \cos \delta_m)' = x' + X'_m \quad (1b)$$

$$\xi''_m = (\rho_m \cos \alpha_m \cos \delta_m)'' = x'' + X''_m \quad (1c)$$

and the corresponding equations in η_m and \mathcal{Z}_m , where the subscript m again signifies that the coördinates, velocities, and accelerations are referred to the center of mass of the Earth-Moon system. But just as the geocentric places, so also must the barycentric places of the body remain unknown until after the geocentric distances shall have been determined.

The angle subtended at the body by the line joining the observer and the center of the Earth will hereafter be called the *geocentric parallax* and, as above, be denoted by p^g . The angle subtended at the body by the line joining the center of the Earth and the center of mass of the Earth-Moon system will be called the *barycentric parallax* and be denoted by p^m , while the angle subtended at the body by the line joining the observer and the center of mass will simply be called the *parallax* and be denoted by p . Similarly the corresponding parallaxes in α and δ will be denoted by $p^g_\alpha, p^g_\delta; p^m_\alpha, p^m_\delta$; and p_α, p_δ , respectively. The corresponding parallax factors will be denoted by $p^g_\alpha \rho, p^g_\delta \rho; p^m_\alpha \rho_m, p^m_\delta \rho_m$; and $p_\alpha \rho_m, p_\delta \rho_m$, respectively.

Since $p_\alpha = p^g_\alpha + p^m_\alpha$; $p_\delta = p^g_\delta + p^m_\delta$, we have, after multiplying by ρ_m and writing, with an error of the second order only of the parallax, $p^g_\alpha \rho$ for $p^g_\alpha \rho_m$; $p^g_\delta \rho$ for $p^g_\delta \rho_m$; $p_\alpha \rho_m = p^g_\alpha \rho + p^m_\alpha \rho_m$; $p_\delta \rho_m = p^g_\delta \rho + p^m_\delta \rho_m$.

It has been the rule to neglect the geocentric as well as the barycentric parallax in the first direct solution of the geocentric distances. HARZER repeats the direct solution after correcting the observations only for geocentric parallax on

¹H. BRUNS, *der Lambert'sche Satz*, A. N. 2482.

the basis of geocentric distances resulting from the first direct solution. BRUNS, however, corrects also for barycentric parallax, before repeating the direct solution.

As stated above, on page 233, it was pointed out in Part I, page 9, that the second direct solution might be omitted altogether, and that the small changes in the orbit resulting from the geocentric parallax corrections might be determined just as well from the observation equations as part of the final corrections of the heliocentric coördinates and velocities. This remark holds also for the barycentric parallax.

It is evident, however, that the method defined by equations (4) and (5) of eliminating the geocentric parallax in the first direct solution may at once be extended to the simultaneous elimination of the barycentric parallax. For this purpose it is only necessary to introduce in equations (4) the parallax factors $p_a \rho_m$, $p_b \rho_m$ defined above, which correspond to a reduction of the observed α and δ to the center of mass. These *parallax factors* we have seen are equal to the algebraical sum of the corresponding *geocentric* and *barycentric parallax factors*. As stated above, hereafter this sum will simply be designated by *parallax factor*, without modification. Equations (4) will therefore be applicable to the elimination of the *parallax* or of the *geocentric parallax alone*, or of the *barycentric parallax alone*, as we substitute in that equation the corresponding factors for $p_a^g \rho$ and $p_b^g \rho$.

Let the corrections to the solar coördinates resulting from equation (4) on the basis of $p_a \rho_m$ and $p_b \rho_m$ be denoted by $\Delta_2 X$, $\Delta_2 Y$, and $\Delta_2 Z$; and let $\Delta_1 X$, $\Delta_1 Y$, $\Delta_1 Z$ represent the corrections which must be applied to the geocentric solar coördinates in order to reduce them to the center of mass. Then

$$\Delta X = \Delta_1 X + \Delta_2 X; \quad \Delta Y = \Delta_1 Y + \Delta_2 Y; \quad \Delta Z = \Delta_1 Z + \Delta_2 Z$$

are the complete corrections which must be applied to the interpolated solar coördinates for the *total elimination of the parallax* in α and δ (geocentric and barycentric), including the reduction of the solar coördinates to the center of mass.

We shall first derive $\Delta_1 X$, $\Delta_1 Y$, $\Delta_1 Z$. Let d_1 and d be the distances of the center of mass and of the Moon, respectively, from the center of the Earth, α_1 , δ_1 the geocentric coördinates of the Moon, as given in the astronomical ephemerides. Then the corrections to be applied to the geocentric solar coördinates X , Y , Z are

$$\Delta_1 X = -d_1 \cos \delta_1 \cos \alpha_1; \quad \Delta_1 Y = -d_1 \cos \delta_1 \sin \alpha_1; \quad \Delta_1 Z = -d_1 \sin \delta_1.$$

Let π and π_1 be the values of the mean horizontal equatorial parallax of the Sun and Moon respectively, and m and m_1 the masses of the Earth and of the Moon, respectively. Then we have with more than sufficient accuracy

$$d_1 = d \frac{m_1}{m + m_1} = \frac{\pi}{\pi_1} \frac{\mu}{1 + \mu},$$

where μ is the ratio of the Moon's mass to the Earth's mass and where d is given in astronomical units. With HINKS'S¹ value of $\pi = 8''.807$ and NEWCOMB'S² values of $\pi_1 = 57' 2''.68$ and $\mu = \frac{1}{81.43}$ we obtain

$$\frac{d_1}{\sin 1''} = 6''.437.$$

¹M. N. R. A. S. Vol. LXIX, page 567.

²The Elements of the Four Inner Planets and the Fundamental Constants of Astronomy. Supplement to the American Ephemeris and Nautical Almanac for 1897.

The barycentric parallax factors in α and δ are given by the left hand sides of equations (15), page 11, if $\Delta_1 X$, $\Delta_1 Y$, $\Delta_1 Z$ be substituted on the right hand side for ∂x , ∂y , ∂z , respectively. In the first of these equations $\partial \alpha = \cos \delta \partial \alpha$. Hence

$$p_a^m \rho_m = \frac{\sin \alpha}{\cos \delta} d_1 \cos \delta_1 \cos \alpha_1 - \frac{\cos \alpha}{\cos \delta} d_1 \cos \delta_1 \sin \alpha_1$$

$$p_\delta^m \rho_m = \sin \delta \cos \alpha d_1 \cos \delta_1 \cos \alpha_1 + \sin \delta \sin \alpha d_1 \cos \delta_1 \sin \alpha_1 - \cos \delta d_1 \sin \delta_1,$$

or

$$p_a^m \rho_m = d_1 \frac{\cos \delta_1}{\cos \delta} \sin (\alpha - \alpha_1)$$

$$p_\delta^m \rho_m = d_1 [\sin \delta \cos \delta_1 \cos (\alpha - \alpha_1) - \cos \delta \sin \delta_1].$$

By adding these *barycentric* to the *geocentric parallax factors* which are generally given with the observations, the required *parallax factors* $p_a \rho_m$ and $p_\delta \rho_m$ may now be formed.

For equations (1a) we now write with reference to the center of mass

$$\xi_m = \rho_m \cos (\delta + p_\delta) \cos (\alpha + p_a) = x + X_m; \quad \eta_m = \rho_m \cos (\delta + p_\delta) \sin (\alpha + p_a) = y + Y_m$$

$$z_m = \rho_m \sin (\delta + p_\delta) = z + Z_m,$$

where evidently $X_m = X + \Delta_1 X$, etc. From these equations we derive the following equations in the same manner as equations (4) to (11) were derived from equations (1)

$$\Delta_2 X = \cos \delta \sin \alpha p_a \rho_m + \sin \delta \cos \alpha p_\delta \rho_m; \quad \Delta_2 Y = -\cos \delta \cos \alpha p_a \rho_m + \sin \delta \sin \alpha p_\delta \rho_m$$

$$\Delta_2 Z = -\cos \delta p_\delta \rho_m. \quad (4a)$$

$$(\xi) = \rho_m \cos \delta \cos \alpha = x + X_m + \Delta_2 X; \quad (\eta) = \rho_m \cos \delta \sin \alpha = y + Y_m + \Delta_2 Y$$

$$(\zeta) = \rho_m \sin \delta = z + Z_m + \Delta_2 Z. \quad (5a)$$

$$(\xi)'' = \frac{X_m + \Delta_2 X - (\xi)}{r^3} + [X_m + \Delta_2 X]'' \quad (8a)$$

And, subject to later correction,

$$(X_m + \Delta_2 X)'' = -\frac{X_m + \Delta_2 X}{(R_m + \Delta_2 R)^3}, \quad (9aa)$$

where

$$R \Delta_2 R = X \Delta_2 X + Y \Delta_2 Y + Z \Delta_2 Z. \quad (10a)$$

$$(\xi)'' + \frac{(\xi)}{r^3} = (X_m + \Delta_2 X) \left[\frac{1}{r^3} - \frac{1}{(R_m + \Delta_2 R)^3} \right]. \quad (11a)$$

Equation (11a) is rigid except for the error committed in using the substitution (9aa) instead of (9a). The mass $m + m_1$ of the Earth-Moon system is negligible in preliminary orbit computations.

To determine the remaining error and to allow for the same, if necessary, we may write (9aa) by means of (9a)

$$-\frac{X_m}{R_m^3} + (\Delta_2 X)'' = -\frac{X_m}{R_m^3} - \frac{\Delta_2 X}{R_m^3} + 3 \frac{X_m \Delta_2 R}{R_m^4}, \quad (12)$$

where powers of $\Delta_2 R$ higher than the first and the product, $\Delta_2 X \Delta_2 R$, are neglected. Expression (11a), therefore, requires the correction

$$(\Delta_2 X)'' + \frac{\Delta_2 X}{R_m^3} - 3 \frac{X_m \Delta_2 R}{R_m^4}. \quad (13)$$

In this expression we may use the corrected values of the solar coördinates.

Denoting these by (X) , (Y) , (Z) , so that

$$(X) = (X_m + \Delta_2 X), \text{ etc. ;} \quad (R) = (R_m + \Delta_2 R), \quad (14)$$

we obtain for the corrected equation (11a)

$$(\xi)'' + \frac{(\xi)}{r^3} = (X) \left[\frac{1}{r^3} - \frac{1}{(R)^3} \left(1 + 3 \frac{\Delta_2 R}{(R)} \right) \right] + \frac{\Delta_2 X}{(R)^3} + (\Delta_2 X)'', \quad (15)$$

and analogous equations in the other two coördinates.

The numerical values of the $\Delta_2 X$, $\Delta_2 Y$ and $\Delta_2 Z$ depend by (4a) on the coördinates of the body and on the parallax factors. Hence, unless the observations have been made at stations differing widely in latitude, or unless the hour angles be very different, they may vary proportionally to the intervals for the three dates. If the proportionality hold true, then $(\Delta_2 X)''_0 = (\Delta_2 Y)''_0 = (\Delta_2 Z)''_0 = 0$. But we shall first consider the case where these accelerations cannot be neglected. Then $(\Delta_2 X)''_0$, etc., must be obtained from $\Delta_2 X$, $\Delta_2 X'' = \Delta_2 X_0$, and $\Delta_2 X'''$, etc., in the same manner as α''_0 or δ''_0 is obtained from α , $\alpha'' = \alpha_0$, and α''' , etc., in II, page 15, and the last two terms of (15) become for the middle date

$$\frac{1}{k^3} \frac{\Delta_2 X''' - \Delta_2 X_0}{t''' - t''} \frac{\Delta_2 X_0 - \Delta_2 X_i}{t'' - t_i} + \frac{\Delta_2 X_0}{(R)^3}. \quad (16)$$

Let

$$\Delta_2 X''' (t'' - t_i) + \Delta_2 X_i (t''' - t'') = \Delta_2 X_0 (t''' - t_i) (1 + d_x). \quad (17)$$

Then (16) reduces to

$$\left[-\frac{2d_x}{\theta_i \theta'''} + \frac{1}{(R)^3} \right] \Delta_2 X_0, \quad (18)$$

where $\theta_i = k(t''' - t'')$ and $\theta''' = k(t'' - t_i)$. After deriving analogous expressions in the second and third coördinates and letting

$$j \cos a = \left[\frac{1}{(R)^3} + \frac{2d_x}{\theta_i \theta'''} \right] \Delta_2 X_0; \quad j \sin a = \left[\frac{1}{(R)^3} + \frac{2d_y}{\theta_i \theta'''} \right] \Delta_2 Y_0; \quad j \tan d = \left[\frac{1}{(R)^3} + \frac{2d_z}{\theta_i \theta'''} \right] \Delta_2 Z_0, \quad (19)$$

equations (15) become

$$\begin{aligned} (\xi)'' + \frac{(\xi)}{r^3} &= (X) \left[\frac{1}{r^3} - \frac{1}{(R)^3} \left(1 + 3 \frac{\Delta_2 R}{(R)} \right) \right] + j \cos a \\ (\eta)'' + \frac{(\eta)}{r^3} &= (Y) \left[\frac{1}{r^3} - \frac{1}{(R)^3} \left(1 + 3 \frac{\Delta_2 R}{(R)} \right) \right] + j \sin a \\ (\zeta)'' + \frac{(\zeta)}{r^3} &= (Z) \left[\frac{1}{r^3} - \frac{1}{(R)^3} \left(1 + 3 \frac{\Delta_2 R}{(R)} \right) \right] + j \tan d. \end{aligned}$$

If, after introducing polar coördinates throughout in these equations, the indicated differentiations be performed, they may be written

$$\begin{aligned} \sigma'' \cos \alpha - 2 \sigma' \sin \alpha \alpha' - \sigma \sin \alpha \alpha'' - \sigma \cos \alpha \left[(\alpha')^2 - \frac{1}{r^3} \right] &= S \cos A \left[\frac{1}{r^3} - \frac{1}{(R)^3} \left(1 + 3 \frac{\Delta_2 R}{(R)} \right) \right] + j \cos a \\ \sigma'' \sin \alpha + 2 \sigma' \cos \alpha \alpha' + \sigma \cos \alpha \alpha'' - \sigma \sin \alpha \left[(\alpha')^2 - \frac{1}{r^3} \right] &= S \sin A \left[\frac{1}{r^3} - \frac{1}{(R)^3} \left(1 + 3 \frac{\Delta_2 R}{(R)} \right) \right] + j \sin a \\ \sigma'' \tan \delta + 2 \sigma' (\tan \delta)' + \sigma \left[(\tan \delta)'' + \frac{\tan \delta}{r^3} \right] &= S \tan D \left[\frac{1}{r^3} - \frac{1}{(R)^3} \left(1 + 3 \frac{\Delta_2 R}{(R)} \right) \right] + j \tan d, \end{aligned}$$

where the subscripts have been dropped, as all quantities refer to the second date.

If, as usual, the first two of these equations be added, on the one hand after multiplication of the first by $\cos \alpha$ and of the second by $\sin \alpha$, on the other hand after multiplication of the first by $-\sin \alpha$ and of the second by $\cos \alpha$, we have

$$\left. \begin{aligned} \sigma'' + \sigma \left[\frac{1}{r^2} - (\alpha')^2 \right] &= S \cos(A - \alpha) \left[\frac{1}{r^2} - \frac{1}{(R)^2} \left(1 + 3 \frac{\Delta_2 R}{(R)} \right) \right] + j \cos(\alpha - a) \\ 2\sigma' \alpha' + \sigma \alpha'' &= S \sin(A - \alpha) \left[\frac{1}{r^2} - \frac{1}{(R)^2} \left(1 + 3 \frac{\Delta_2 R}{(R)} \right) \right] + j \sin(\alpha - a) \\ \sigma'' \tan \delta + 2\sigma' (\tan \delta)' + \sigma \left[\frac{\tan \delta}{r^2} + (\tan \delta)'' \right] &= S \tan D \left[\frac{1}{r^2} - \frac{1}{(R)^2} \left(1 + 3 \frac{\Delta_2 R}{(R)} \right) \right] + j \tan d. \end{aligned} \right\} \quad (20)$$

The solution of these equations for σ and σ' , which is conveniently accomplished by means of determinants, gives, after replacing the subscripts

$$\left. \begin{aligned} \sigma_0 &= \rho_m \cos \delta_m = \kappa \left\{ \frac{1}{r^2} - \frac{1}{(R)^2} \left(1 + 3 \frac{\Delta_2 R}{(R)} \right) \right\} + \Delta \kappa \\ \sigma'_0 &= \lambda \left\{ \frac{1}{r^2} - \frac{1}{(R)^2} \left(1 + 3 \frac{\Delta_2 R}{(R)} \right) \right\} + \Delta \lambda, \end{aligned} \right\} \quad (21)$$

where

$$\left. \begin{aligned} \Delta \kappa &= -\frac{j}{N} \left\{ \left[\tan \delta_m \cos(a - \alpha_m) - \tan d \right] \alpha'_0 + \sin(a - \alpha_m) (\tan \delta)'_0 \right\} \\ \Delta \lambda &= \frac{j}{2N} \left\{ \left[\tan \delta_m \cos(a - \alpha_m) - \tan d \right] \alpha'_0 + \sin(a - \alpha_m) \left[(\alpha'_0)^2 \tan \delta_m + (\tan \delta)'_0 \right] \right\}, \end{aligned} \right\} \quad (22)$$

and where N , λ , and κ are defined as in III and V, pages 15 and 16.

Divide both sides of the first of (21) by $(R) \cos \delta_m$ and let

$$z = \frac{\rho_m}{(R)}; \quad (m) = -\frac{\kappa}{(R)^2 \cos \delta_m} \left(1 + 3 \frac{\Delta_2 R}{(R)} \right) + \frac{\Delta \kappa}{(R) \cos \delta_m}. \quad (23)$$

Then the first of (21) may be written

$$z - (m) = \frac{\kappa}{r^2 (R) \cos \delta_m} = (v). \quad (24)$$

The value of z is found from this equation by eliminating r by means of

$$r^2 = \rho_m^2 + (R)^2 - 2 \rho_m (R) \cos \psi,$$

or if, as before, we introduce $z = \frac{\rho_m}{(R)}$, by means of

$$r^2 = (R)^2 (z^2 - 2z \cos \psi + 1) = \mu (R)^2. \quad (25)$$

The elimination gives

$$r^2 = \mu^2 (R)^2 = \frac{\kappa^2}{(R)^2 \cos^2 \delta_m (v)^2},$$

or

$$\mu^2 (v)^2 = \frac{\kappa^2}{(R)^2 \cos^2 \delta_m} = m^2, \quad (26)$$

where m is defined as in III, page 15. Replacing μ and (v) by means of (25) and

$$(24) \text{ we obtain for the equation in } z = \frac{\rho_m}{(R)} \\ (z^2 - 2z \cos \psi + 1)^2 (z - (m))^2 - m^2 = 0, \quad (27)$$

which is of the same form as the corresponding equation (7), page 8, except that in (7) $(m) = \bar{m}$, the parallax being entirely neglected. Since (m) differs but little from \bar{m} , the solution of (27) is accomplished in exactly the same manner as that of (7),

page 8, except that in equations (10) and (11), page 8, we must write (ν) for r . Hence, if z_1 be a first approximation to z , taken from the tables at the end of this volume with $\frac{1}{m}$ or $\frac{1}{(m)}$ and ψ as arguments, then the correction Δz_1 which must be added to z_1 , so that $z_2 = z_1 + \Delta z_1$, shall satisfy (27) is given by

$$f(z_1) = \mu^3_1 (\nu)_1 - m^3 = M_1 \quad (28)$$

and

$$\Delta z_1 = \frac{-M_1}{2 \mu^3_1 (\nu)_1 [\mu_1 + 3 (\nu)_1 (z_1 - \cos \psi)]}. \quad (29)$$

The only additional labor, therefore, required for a complete elimination of the parallax in the determination of ρ_m consists in the computation of the auxiliary quantity (m) . By (23) and by (9), page 8, we may also write in a convenient form for computation

$$(m) = m \left(1 + 3 \frac{\Delta_2 R}{(R)} \right) + \frac{\Delta \kappa}{(R) \cos \delta}. \quad (30)$$

The computation of σ''_0 may be omitted, as it is not required in the further solution of the orbit. It enters only in the expressions for ρ , and ρ_m , but these distances are needed solely for the purpose of freeing the observed intervals from the planetary aberration (*cf.* V, page 16). As it is permissible in preliminary orbit calculations to neglect the second differences of the aberration corrections, the computation of σ''_0 may be avoided. The expression for σ''_0 , however, resulting from equations (20) may be written out by analogy from the corresponding formula in V, page 16, and is

$$\sigma''_0 = \frac{\sigma'_0}{\kappa} \left[(R) \cos D \cos (A - \alpha_m) - \sigma_0 \right] - \sigma_0 \left[\frac{1}{(R)^3} \left(1 + 3 \frac{\Delta_2 R}{(R)} \right) - (\alpha'_0)^2 \right] + j \cos (a - \alpha), \quad (31)$$

or it may be computed, if desired, from the first of equations (20) after r has become known.

Equations (21) give

$$\frac{\sigma'_0 - \Delta \lambda}{\sigma'_0 - \Delta \kappa} = \frac{\lambda}{\kappa}; \quad \sigma'_0 = \frac{\lambda}{\kappa} (\sigma_0 - \Delta \kappa) + \Delta \lambda, \quad (32)$$

from which σ'_0 may be computed after σ_0 has been found. From (32) we may also write

$$\frac{\sigma'_0}{\sigma_0} = \frac{\lambda}{\kappa} - \frac{1}{\sigma_0} \left[\frac{\lambda}{\kappa} \Delta \kappa - \Delta \lambda \right] = \frac{\lambda}{\kappa} - \frac{\beta}{\sigma_0}, \quad (33a)$$

which equation is in a convenient form for use in *all* preliminary orbit solutions.

In fact, in the solution of *parabolic* orbits the values of λ , κ , $\Delta \lambda$, and $\Delta \kappa$ are not themselves required, but only the value of the ratio $\frac{\sigma'_0}{\sigma_0}$ involving the value of the ratio $\frac{\lambda}{\kappa}$ and the value of β . Simple formulæ for these quantities may be deduced by the elimination of $\left[\frac{1}{r^3} - \frac{1}{(R)^3} \left(1 + 3 \frac{\Delta_2 R}{(R)} \right) \right]$ from the first of equations (21) by means of the second of equations (20). This elimination gives, if, for the present, the subscripts referring to the middle date be suppressed

$$\frac{\sigma'}{\sigma} = -\frac{\alpha''}{2 \alpha'} + \frac{S}{2 \alpha'} \sin (A - \alpha) + \frac{j}{2 \alpha' \sigma} \sin (a - \alpha) - \frac{S}{2 \alpha'} \sin (A - \alpha) \frac{\Delta \kappa}{\kappa \sigma}.$$

Replacing κ by its expression in III, page 15, and $\Delta\kappa$ by (22) and letting

$$\begin{aligned} C_1 &= \tan \delta \cos (A - \alpha) - \tan D ; & C_2 &= \sin (A - \alpha) \\ c_1 &= \tan \delta \cos (a - \alpha) - \tan d ; & c_2 &= \sin (a - \alpha) , \end{aligned}$$

we obtain

$$\frac{\sigma'}{\sigma} = -\frac{1}{2\alpha'} \frac{NC_2}{C_1\alpha' + C_2(\tan \delta)'} - \frac{\alpha''}{2\alpha'} \frac{j c_2}{2\alpha'\sigma} - \frac{j C_2}{2\alpha'\sigma} \frac{c_1\alpha' + c_2(\tan \delta)'}{C_1\alpha' + C_2(\tan \delta)'} ,$$

or introducing for N its expression by III, page 15, and letting $n = (\alpha')^2 \tan \delta + (\tan \delta)''$

$$\frac{\sigma'}{\sigma} = -\frac{1}{2} \frac{C_1\alpha'' + C_2n}{C_1\alpha' + C_2(\tan \delta)'} + \frac{j}{2\sigma} \frac{c_2 C_1 - c_1 C_2}{C_1\alpha' + C_2(\tan \delta)'} .$$

Let

$$\Gamma = \frac{C_2}{C_1} ; \quad \gamma = \frac{c_2}{c_1} ; \quad \Phi = \alpha' + \Gamma(\tan \delta)' ;$$

then

$$\frac{\sigma'}{\sigma} = -\frac{1}{2} \frac{\alpha'' + \Gamma n}{\Phi} + \frac{j}{2\sigma} \frac{1 - \frac{\Gamma}{\gamma}}{\Phi} .$$

By identifying terms independent and dependent of the parallax in this equation and in (33a), we find

$$\frac{\lambda}{\kappa} = -\frac{1}{2} \frac{\alpha'' + \Gamma n}{\Phi} ; \quad \beta = -\frac{j c_2}{2} \frac{1 - \frac{\Gamma}{\gamma}}{\Phi} ; \quad \frac{\sigma'_0}{\sigma_0} = \frac{\lambda}{\kappa} - \frac{\beta}{\sigma} . \quad (33b)$$

In practice $\Delta\kappa$, $\Delta\lambda$, and β which are given by (19), (22), and (33a, b) may generally be neglected. But if they be applied, their computation may be simplified in most cases. For, if by inspection the observed coördinates and the *parallax factors* be found to vary roughly as the time, then $d_x = d_y = d_z = 0$, or nearly so. In that case equations (19) become

$$j \cos a = \frac{\Delta X_0}{(R)^3} ; \quad j \sin a = \frac{\Delta Y_0}{(R)^3} ; \quad j \tan d = \frac{\Delta Z_0}{(R)^3} , \quad (34)$$

and by (4) we obtain the following expressions for $j c_1$ and $j c_2$:

$$j \tan \delta_{\alpha} \cos (a - \alpha_{\alpha}) - j \tan d = \frac{1}{(R)^3} \sec \delta_{\alpha} p_{\delta} \rho_m ; \quad j \sin (a - \alpha_{\alpha}) = -\frac{1}{(R)^3} \cos \delta_{\alpha} p_{\alpha} \rho_m . \quad (35)$$

Hence equations (22) reduce to

$$\left. \begin{aligned} \Delta\kappa &= -\frac{1}{N(R)^3} \left\{ \sec \delta_{\alpha} p_{\delta} \rho_m \alpha'_0 - \cos \delta_{\alpha} p_{\alpha} \rho_m (\tan \delta)'_0 \right\} \\ \Delta\lambda &= \frac{1}{2N(R)^3} \left\{ \sec \delta_{\alpha} p_{\delta} \rho_m \alpha''_0 - \cos \delta_{\alpha} p_{\alpha} \rho_m \right\} , \end{aligned} \right\} \quad (36a)$$

and

$$\beta = \frac{1}{2(R)^3} \frac{\cos \delta_{\alpha} p_{\alpha} \rho_m + \Gamma \sec \delta_{\alpha} p_{\delta} \rho_m}{\Phi} . \quad (36b)$$

In these expressions the numerical values of all quantities involved are available from previous computations.

Collecting results, we have the following directions for a *partial* and for a *complete* elimination of the parallax in σ_0 and σ'_0 .

For a *partial* elimination compute (4) and (10), and retain the corrected solar coördinates throughout. In (30) let $\Delta\kappa = 0$ and solve (27) for z . In (33a, b) let $\beta = 0$ and solve for σ'_0 .

For a *complete* elimination of the parallax, proceed in the same manner by (4a), and (10a), except that β , and in general solutions also $\Delta\kappa$, must be computed and retained, the necessary expressions being either (33a) or (33b) or simply (36a, b). σ''_0 may be neglected entirely or, if taken into account, may be computed by (31) or by the first of equations (20).

Instead of computing ΔR from (10a, b), its value may be obtained by taking the difference between (R) as derived by I, page 15, from the corrected solar coördinates or by $(R)^2 = (X)^2 + (Y)^2 + (Z)^2$ and R as interpolated from an astronomical ephemeris.

To allow for the parallax in the heliocentric velocities x'_0, y'_0 , and z'_0 , we have from (5a) by differentiation and omitting the subscripts referring to the middle date

$$x' = (\xi)' - X'_m - (\Delta_2 X)' = (\rho_m \cos \delta \cos \alpha)' - X' - (\Delta_1 X + \Delta_2 X)',$$

or

$$x' = (\sigma_0 \cos \alpha)' - X' - (\Delta X)' = \sigma_0 \left[\frac{\sigma'_0}{\sigma_0} \cos \alpha - \sin \alpha \alpha' \right] - X' - (\Delta X)',$$

or by (33a) or (33b),

$$x' = \sigma_0 \left[\frac{\lambda}{\kappa} \cos \alpha - \sin \alpha \alpha' \right] - \beta \cos \alpha - X' - (\Delta X)'. \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \quad (37)$$

Similarly

$$y' = \sigma_0 \left[\frac{\lambda}{\kappa} \sin \alpha + \cos \alpha \alpha' \right] - \beta \sin \alpha - Y' - (\Delta Y)'$$

$$z' = \sigma_0 \left[\frac{\lambda}{\kappa} \tan \delta + (\tan \delta)' \right] - \beta \tan \delta - Z' - (\Delta Z)',$$

where $(\Delta X)', (\Delta Y)', (\Delta Z)'$ must be obtained from $\Delta X', \Delta X'',$ and $\Delta X'''$, etc., in the same manner as α'_0 and δ'_0 are obtained from $\alpha', \alpha'',$ and α''' , etc., in II, page 15. In the *partial* elimination of the parallax $(\Delta X)', (\Delta Y)',$ and $(\Delta Z)'$ may be entirely neglected.

In all other respects the computation remains the same as in Part I.

In the comparison between theory and observation (VI, page 16), either the corrected or geocentric solar coördinates may be used. In the former case the observed places require no further correction, while in the latter the *geocentric* parallax is applied, as usual, to the observed places.

The procedure outlined above for the elimination of the parallax is extremely simple in its application and, as the quantities involved are small, requires little extra computation, even if $(\Delta X)''_0, (\Delta Y)''_0$, and $(\Delta Z)''_0$ be appreciable.

In correcting a preliminary orbit on the basis of a longer arc, it is generally assumed that the initial geocentric distances are known with sufficient accuracy for deriving the parallax corrections. But if the final distances involve a change in the parallax, then, although the observations as originally corrected for parallax may be properly represented, residuals remain which are equal to the difference of

the parallax corrections corresponding to the final and initial geocentric distances. In the differential formulæ developed in the First Paper (VII, page 17), any change in the parallax may be taken up in the course of the calculation by applying small corrective terms to the expressions for A , A' , B , and B' in formulæ (23), page 12. Let α , δ ; α_g , δ_g ; and α_c , δ_c be the observed, the geocentric, and the computed place, that is, the place computed from the initial fundamental data, and let ρ and $\rho + \partial\rho$ be the initial and the final geocentric distance, respectively, at either the first or third date. Then the true residuals of the initial orbit are

$$(\partial\alpha) = (\cos\delta\partial\alpha) = \cos\delta(\alpha_g - \alpha_c) = \cos\delta\left(\alpha + \frac{p_a^2\rho}{\rho + \partial\rho} - \alpha_c\right); (\partial\delta) = \delta_g - \delta_c = \delta + \frac{p_b^2\rho}{\rho + \partial\rho} - \delta_c, \quad (38)$$

and these are the residuals which must be removed in the differential correction of the initial fundamental quantities ρ_0 , x'_0 , y'_0 , and z'_0 . But as the final or true geocentric distance $\rho + \partial\rho$ is not known, we may write $\frac{1}{\rho + \partial\rho} = \frac{1}{\rho} - \frac{\partial\rho}{\rho^2}$, neglecting higher powers of $\partial\rho$. The residuals (38) will then take the form

$$(\partial\alpha) = \cos\delta\left[\left(\alpha + \frac{p_a^2\rho}{\rho}\right) - \alpha_c\right] - \cos\delta\frac{p_a^2\rho}{\rho^2}\partial\rho; (\partial\delta) = \left[\left(\delta + \frac{p_b^2\rho}{\rho}\right) - \delta_c\right] - \frac{p_b^2\rho}{\rho^2}\partial\rho. \quad (39)$$

In these expressions $\cos\delta\left[\left(\alpha + \frac{p_a^2\rho}{\rho}\right) - \alpha_c\right]$ and $\left[\left(\delta + \frac{p_b^2\rho}{\rho}\right) - \delta_c\right]$ represent the residuals of the initial orbit on the basis of the initial parallax corrections, and if we denote these residuals by $\partial\alpha$ and $\partial\delta$ as on pages 11 and 12, then

$$(\partial\alpha) = \partial\alpha - \cos\delta\frac{p_a^2\rho}{\rho^2}\partial\rho, (\partial\delta) = \partial\delta - \frac{p_b^2\rho}{\rho^2}\partial\rho. \quad (40)$$

By replacing in (24) and (25), page 12, $\partial\alpha$ and $\partial\delta$ by $(\partial\alpha)$ and $(\partial\delta)$ as given in (40) and transferring the terms containing the parallax factors to the right-hand sides of (24) and (25), we observe that these terms, which contain the unknown $\partial\rho$ may be combined with $A\partial\rho$ and $B\partial\rho$. Hence in order to allow for a possible change in parallax, it is only necessary to write in (23)

$$\text{for } A, \quad A' = A + \cos\delta\frac{p_a^2\rho}{\rho^2}; \quad \text{for } B, \quad B' = B + \frac{p_b^2\rho}{\rho^2}. \quad (41)$$

It should be observed, however, that the correction $\partial\rho$ in (40) pertains to ρ , or ρ_{III} while $\partial\rho_0$ in (24) and (25), page 12, pertains to ρ_0 . While formulæ to take account of the differences $\partial\rho_{III} - \partial\rho_0$ and $\partial\rho_0 - \partial\rho$, might readily be deduced, such refinement would be unwarranted.

The corrections to A and B , as defined in (41) are so easily computed that it is advisable to take them into account, unless there be reason to assume that the initial orbit is very close.

These considerations regarding a possible change in the parallax from one approximation of an orbit to the next, apply also if the parallax has been eliminated as above in the solution of the initial orbit, although this does not seem to be the case at first sight. For, the values of α_g and δ_g , occurring in (4) must correspond to the parallax corrections based on the same unknown value of ρ as that which appears in (1) to (3). Otherwise we could not identify in (3) the pro-

ducts of ρ into ρ'_x and ρ'_y with the given parallax factors. This applies to all three places. This conclusion is readily verified, if we remember that in the comparison between theory and observation for the first and third places, the same residuals will result, whether we use the uncorrected solar coördinates and correct the observed places by parallaxes based on the geocentric distances resulting from the initial orbit (*cf.* VI, page 16), or whether, in accordance with the directions for eliminating the parallax, we use the corrected solar coördinates and leave the observed places as they are. Hence, even if the parallax has been eliminated in the preliminary solution, it is advisable to use the values A' and B' , given in (41), instead of A and B , unless it be safe to assume that the first orbit is so close that the parallaxes are definitely determined from the initial distances.

It will be observed that for the determination of the heliocentric coördinates and velocities by VI, page 16, σ'_0 is not needed. For by (37) we have

$$x'_0 = a_x \sigma_0 - [X]', \text{ etc., where } a_x = \cos \alpha_x \frac{\lambda}{\kappa} - \sin \alpha_x, \alpha'_0 \text{ and } [X]' = X' + (\Delta X)' + \beta \cos \alpha, \text{ etc.}$$

Hence, beside σ_0 , only the ratio $\frac{\lambda}{\kappa}$ and β appear in the velocities. x'_0, y'_0, z'_0 are independent of the corrections (4) applied to the solar coördinates, except in the rare cases where these corrections appear in σ_0 and $[X]'$, etc. But even then we may avoid the computation of σ'_0 by using (33a, b). As far as σ_0 is concerned, the corrections enter only into the auxiliary quantity (m). Similarly $\frac{\lambda}{\kappa} - \frac{\beta}{\sigma_0}$ may be introduced for $\frac{\sigma'_0}{\sigma_0}$ in the expressions for ρ , and ρ_{∞} in V, page 16.

The foregoing elimination of the parallax has a simple geometrical interpretation. Draw the arc of a circle about the body through M , the center of mass, until it intersects the line of sight. The radius of this circle is the distance of the body from M . Let P be the point of intersection. Or P may be considered the projection of M on the line of sight. Then the coördinates $(\xi), (\eta), (\zeta)$ in (5a) will be the coördinates of the body referred to P . The parallax p is the angle at the body subtended by $MP = \rho_m p$. The components of the parallax perpendicular to and in the body's hour circle are $\cos \delta p_a$ and p_δ , and $\overline{MP}^2 = (\rho_m \cos \delta p_a)^2 + (\rho_m p_\delta)^2$, MP being the resultant displacement of $\rho_m \cos \delta p_a$ and $\rho_m p_\delta$. The latter, lying in the hour circle, makes the angle $90^\circ - \delta$ with the equator, while the former is parallel to the equator, so that its angles of projection on the x and y axes are $90^\circ - \alpha$ and α , respectively. The projection of $\rho_m p_\delta$ on the equator is $\sin \delta \rho_m p_\delta$, and as this component lies in the hour circle, its angles of projection on the x and y axes are α and $90^\circ - \alpha$, respectively. If the two components, $\rho_m \cos \delta p_a$ and $\rho_m p_\delta$, be thus projected upon the three axes with due regard to the algebraical signs, then we shall obtain the coördinates of P referred to M , and these will be found identical with $-\Delta_2 X$, $-\Delta_2 Y$, and $-\Delta_2 Z$, as defined in (4a). Thus these corrections signify that for the purpose of eliminating the parallax and preserving the distance of the body from M , the origin of its coördinates is changed from the center of mass to its projection on the line of sight. Since the coördinates $(\xi), (\eta), (\zeta)$ of the body referred to P are equal to the sum of its heliocentric coördinates x, y, z ,

of the barycentric coördinates X_m, Y_m, Z_m of the Sun, and of the coördinates $\Delta_2 X, \Delta_2 Y, \Delta_2 Z$ of M referred to P , equations (4a) and (5a) might have been set up independently on the basis of these geometrical considerations. The line joining the Sun and P is (R) and ΔR is the difference between (R) and the projection of R upon it.

Reference has already been made to the fact that the planetary aberration is fully accounted for within the accuracy of the solution of the preliminary orbit by combining in the equation $\xi = x + X$, the heliocentric place of the body at the time $(t - \alpha\rho)$, with the place of the Sun and the observed or apparent place of the body at the time t , freed of the annual aberration in the reduction to the beginning of the year. The distance ρ , involved in ξ, η, ζ , is then the distance between the Earth at the time t , and the true place of the body at the time $t - \alpha\rho$. As the heliocentric coördinates and velocities are thus referred to the time $t - \alpha\rho$, or the true time, the epoch will also be the true time. The effect of the aberration is thereby eliminated from the values of x, y, z , resulting for the time $t - \alpha\rho$, provided that it can be shown that the distance ρ , as defined above and as determined in the course of the solution, is free from other errors. But it is evident that whatever errors may remain in the resulting values of σ_0 and σ'_0 , and therefore, in the heliocentric coördinates and velocities derived from them, must be of the order $k\alpha^1$, the factor k being accounted for by the fact that all intervals must be expressed in the proper unit, which is $\frac{1}{k}$ mean solar days. The error involved in the solution of σ_0, σ'_0 , and σ''_0 , on account of neglecting third and higher derivatives in the initial determination of the geocentric velocities and accelerations $\alpha'_0, \alpha''_0, \delta'_0$ and δ''_0 , is of the second order if the epoch chosen be the mean of the observed dates, and of the first order, if the epoch be identified with the middle date, unless the intervals be equal, in which case it is again of the second order. If it be of the second order, then it is of the order of the product of the intervals, expressed in mean solar days, multiplied by k^2 . With intervals as short as one day, the errors will therefore be of the order k^2 and larger with longer intervals. The error which may arise in the heliocentric coördinates and velocities, on account of aberration through errors in σ_0 and σ'_0 , are of the order αk , as stated above, and since $\alpha k < k^2$, it is not necessary to take account of the aberration further than is provided for by the foregoing choice of coördinates.

¹ Expressions for corrections to allow for these errors have been produced by POINCARÉ, *Bulletin Astronomique*, t. XXIII, pp. 175-177. In these expressions $k^2 = 1$, so that, in numerical application, the aberration factor α must be multiplied by k . The errors might also be taken up in the course of the solution in a manner quite similar to that employed above in eliminating the parallax.

which applies to positive and negative values of θ . Hence if θ be always taken positive, the signs of the odd power terms must be reversed for dates before the epoch. This form is very convenient for ephemeris purposes when a large number of positions are to be computed.

If only a few positions are required, as in the comparison between theory and observation, the following expressions, arranged by PROFESSOR CRAWFORD, will prove advantageous. They contain terms inclusive of the fifth power of θ in f and g and are readily produced from the preceding equations. Let

$$\alpha = \frac{(r r')^2}{r}; \quad \beta = \frac{\theta}{r^2 r'}; \quad \gamma = \frac{\theta^2}{2 r^2}. \quad (5)$$

Then

$$\begin{aligned} f &= 1 - \gamma \left\{ 1 - \alpha \beta \left[1 + \frac{\beta}{3} \left(1 - \frac{15}{4} \alpha - \frac{3}{4} \frac{r}{a} - 3 \alpha \beta \left[1 - \frac{7}{4} \alpha - \frac{3}{4} \frac{r}{a} \right] \right) \right] \right\} \dots \\ g &= \theta \left\{ 1 - \frac{\gamma}{3} \left[1 - \frac{3}{2} \alpha \beta \left(1 + \frac{\beta}{3} \left[1 - \frac{9}{2} \left(\alpha + \frac{1}{5} \frac{r}{a} \right) \right] \right) \right] \right\} \dots \end{aligned} \quad (6)$$

For the parabola, $a = \infty$. Hence

$$\begin{aligned} f &= 1 - \gamma \left\{ 1 - \alpha \beta \left[1 + \frac{\beta}{3} \left(1 - \frac{15}{4} \alpha \left[1 + \frac{4}{5} \beta \left(1 - \frac{7}{4} \alpha \right) \right] \right) \right] \right\} \dots \\ g &= \theta \left\{ 1 - \frac{\gamma}{3} \left[1 - \frac{3}{2} \alpha \beta \left(1 + \frac{\beta}{3} \left[1 - \frac{9}{2} \alpha \right] \right) \right] \right\} \dots \end{aligned} \quad (7)$$

As soon as r , r' , and a have been computed, the magnitude of the higher power terms in f and g may be estimated from (3) and (4). It will not be convenient to use the series if the convergence, judged from θ and r , is slow, and if $\theta^5 f_5$ and $\theta^5 g_5$ are appreciable within the last decimal of the calculation. The choice for the computation of the places then lies between the closed expressions for f and g , and the constants for the equator. But, as in preliminary orbit computations the intervals are moderate, the series will, in general, prove sufficient even for small values of r . Long intervals are used in practice only when it is desired to improve an approximate orbit, and then closed expressions for f and g and ∂f and ∂g are readily derived on the basis of the preliminary elements. The computation of the constants for the equator may be deferred until a satisfactory agreement between theory and observation has been reached by means of the f and g .

KUHNERT'S closed expressions are

$$\begin{aligned} f &= 1 - \frac{2a}{r_0} \sin^2 \bar{g} = 1 + \frac{a}{r_0} (\cos 2\bar{g} - 1); & 2\bar{g} &= E - E_0 \\ g &= \theta a^{3/2} (2\bar{g} - \sin 2\bar{g}) = a^{1/2} r_0 \sin 2\bar{g} + 2 a_0 r_0 r'_0 \sin^2 \bar{g}. \end{aligned} \quad (8)$$

We may, however, also compute f from g by the formula¹

$$f = 1 - \frac{g^2}{2 r r_0^2 \cos^2 \frac{1}{2} (v - v_0)}. \quad (9)$$

¹V. OPPOLZER, *Lehrbuch zur Bahnbestimmung*, Vol. I, p. 97.

The computation of the true anomalies may be avoided by a few simple transformations, with the aid of the formula

$$1 - r/r_0 \sin \frac{1}{2} (v - v_0) = a \sqrt{1 - e^2} \sin \bar{g}. \quad (10)$$

Let

$$\gamma = \sqrt{1 - r/r_0 (1 - f)}. \quad (11)$$

Then the first of (8) may be written

$$\gamma = \sqrt{2a} \sin \bar{g}. \quad (12)$$

Eliminating $\sin \bar{g}$ from (10) by means of (12), we have

$$1 - r/r_0 \sin \frac{1}{2} (v - v_0) = a \sqrt{1 - e^2} \frac{\gamma}{\sqrt{2a}} = \sqrt{\frac{p}{2}} \gamma. \quad (13)$$

From (9) we obtain by (11)

$$r/r_0 \cos^2 \frac{1}{2} (v - v_0) = \frac{g^2}{2\gamma^2}, \quad (14)$$

which, added to the square of (13) gives

$$r/r_0 = \frac{g^2}{2\gamma^2} + \frac{p\gamma^2}{2}, \quad (15)$$

or

$$g^2 = 2r/r_0 \gamma^2 - p\gamma^4. \quad (16)$$

g is negative for dates before and positive for dates after the epoch t_0 of E_0 or v_0 .

After γ has been computed by (12), f and g are readily obtained from (11) and (16). The eccentric anomaly E involved in γ may be obtained in the usual way by the solution of KEPLER'S equation; p is given by (7), page 252, and r , which also appears in (16), by $r = a(1 - e \cos E)$.

Numerous other general relations involving g and f might be deduced. Thus, for instance, by squaring and adding

$$x = fx_0 + gx'_0; \quad y = fy_0 + gy'_0; \quad z = fz_0 + gz'_0,$$

we easily derive

$$r^2 = fr_0^2 + 2fg(r_0 r'_0) + g^2(r_0'^2),$$

which may serve as a check equation. Also, since $g = \frac{rr_0 \sin(v - v_0)}{1 - p}$, we obtain from (9), after some simple reductions,

$$p\gamma^2 = rr_0 - rr_0 \cos(v - v_0).$$

But KUEHNERT gives, *loco citato*,

$$rr_0 \cos(v - v_0) = fr_0^2 + g(r_0 r'_0).$$

So that

$$p\gamma^2 = r_0(r - r_0) + r_0\gamma^2 - g(r_0 r'_0).$$

For the parabola we may derive γ as follows. Since

$$g = \frac{rr_0 \sin(v - v_0)}{1 - p} = \frac{rr_0 2 \sin \frac{1}{2} (v - v_0) \cos \frac{1}{2} (v - v_0)}{1 - 2q}, \quad (17)$$

and since

$$r = \frac{q}{\sec^2 \frac{1}{2} v}; \quad r_0 = \frac{q}{\sec^2 \frac{1}{2} v_0}, \quad (18)$$

we obtain from (9) and (11)

$$\gamma = \sqrt{q} (\tan \frac{1}{2} v - \tan \frac{1}{2} v_0), \quad (19)$$

or since

$$\sqrt{q} \tan \frac{1}{2} v = \sqrt{r - q}; \quad \sqrt{q} \tan \frac{1}{2} v_0 = \sqrt{r_0 - q}, \quad (20)$$

therefore, also,

$$\gamma = \sqrt{r - q} - \sqrt{r_0 - q}, \quad (21)$$

where the algebraical signs of the square roots are the same as those of the corre-

sponding $\tan \frac{1}{2} v$. The true anomaly v may be obtained in the usual way with the aid of BARKER'S table. r_0 may be derived from

$$\sin r_0 = 2 \sin \frac{1}{2} r_0 \cos \frac{1}{2} r_0 = \sqrt{2q} r'_0, \quad (22)$$

or, after division of this expression by $\cos^2 \frac{1}{2} r_0 = \frac{q}{r}$,

from

$$2 \tan \frac{1}{2} r_0 = \frac{\sqrt{2q} r_0 r'_0}{q}, \quad (23)$$

or

$$\sqrt{2q} \tan \frac{1}{2} r_0 = r_0 r'_0. \quad (24)$$

For the circle the series and the closed expressions for f and g assume very simple forms. Thus, if in (4) we let $r = a$ and $r' = 0$, we obtain for the series

$$f = 1 - \frac{\theta^2}{2a^2} + \frac{\theta^4}{24a^4} \dots ; \quad g = \theta - \frac{\theta^3}{6a^2} + \frac{\theta^5}{120a^4} \dots \quad (25)$$

The corresponding closed expressions are derived from (8) by letting $a = r_0$, and $r' = 0$, as above. Since for the circle the mean, eccentric, and true anomalies are identical, we have

$$2\bar{g} = \frac{\theta}{a^{3/2}} ; \quad f = \cos 2\bar{g} ; \quad g = a^{3/2} \sin 2\bar{g}, \quad (26)$$

and by (12), $\gamma = \sqrt{2a} \sin \bar{g}$.

Since the series (25) represent the cosine and sine series, the relations (26) might also have been written out at once.

THE CONSTANTS FOR THE EQUATOR EXPRESSED IN TERMS OF THE HELIOCENTRIC RECTANGULAR COÖRDINATES AND THEIR VELOCITIES.

It is customary in orbit computations to derive the constants for the equator from the elements which define the position of the orbit plane in space. In VIII, page 18, the elements are, therefore, derived first, and as these elements refer to the equator, the computation of the constants becomes quite simple. But to conform to the general usage in stating the final results, it is necessary to transform the elements from the equator to the ecliptic (*cf.* page 14).

In many cases it will be found advantageous to compute the constants for the equator as soon as the heliocentric coördinates and velocities have become known, and then to derive the elements referred to the elliptic directly from the constants. This obviously renders the computation of the equatorial elements altogether unnecessary and admits of perfecting the agreement between theory and observation, before proceeding to the computation of the final elements even if the residuals are computed by the aid of the constants for the equator instead of the series or closed expressions for f and g .

The constants a and A' may be expressed in terms of the heliocentric coördinates and velocities by means of the equation

$$r \sin a \sin (A' + v) = x \quad (1)$$

and an equation defining $\sin a \cos (A' + v)$, which latter we shall derive on the basis of the following well known relations:

$$\begin{aligned} x &= r \{ \cos u \cos \Omega - \sin u \sin \Omega \cos i \}; & y &= r \{ \cos u \sin \Omega + \sin u \cos \Omega \cos i \}, \\ z &= r \sin u \sin i \end{aligned} \quad (2)$$

and

$$\left. \begin{aligned} \sin a \sin A &= \cos \Omega & \sin b \sin B &= \sin \Omega; & \sin c \sin C &= 0; \\ \sin a \cos A &= -\sin \Omega \cos i; & \sin b \cos B &= \cos \Omega \cos i; & \sin c \cos C &= \sin i, \end{aligned} \right\} \quad (3)$$

where x, y, z, Ω , and i are referred to the equator, and where $u = \omega + v$ is the argument of latitude. Since $A' + v = A + \omega + A = u$, we have

$$\sin a \cos (A' + v) = \sin a \cos A \cos u - \sin a \sin A \sin u, \quad (4)$$

or by (3)

$$\sin a \cos (A' + v) = -\sin \Omega \cos i \cos u - \cos \Omega \sin u.$$

Eliminating from this equation $\sin \Omega \cos u$ by means of the second, and subsequently $\sin u \sin i$ by means of the third of (2), we obtain

$$\begin{aligned} \sin a \cos (A' + v) &= \sin u \cos \Omega \cos^2 i - \frac{y}{r} \cos i - \cos \Omega \sin u, \\ &= -\sin u \cos \Omega \sin^2 i - \frac{y}{r} \cos i, \\ &= -\frac{z}{r} \sin i \cos \Omega - \frac{y}{r} \cos i. \end{aligned}$$

or, replacing by VIII, page 18, $\sin i \cos \Omega$ and $\cos i$ in terms of the coördinates and velocities,

$$\begin{aligned} \sin a \cos (A' + v) &= -\frac{z}{r} \frac{xz' - zx'}{\sqrt{p}} - \frac{y}{r} \frac{xy' - yx'}{\sqrt{p}} \\ &= \frac{1}{r \sqrt{p}} \left[x' (y^2 + z^2) - x (yy' + zz') \right], \end{aligned}$$

or, since

$$x^2 + y^2 + z^2 = r^2; \quad xx' + yy' + zz' = rr', \quad (5)$$

$$\begin{aligned} \sin a \cos (A' + v) &= \frac{1}{r \sqrt{p}} \left[x' (r^2 - x^2) - x (rr' - xx') \right] \\ &= \frac{1}{r \sqrt{p}} \left[x' r^2 - x rr' \right], \end{aligned}$$

or, finally

$$\sin a \cos (A' + v) = \frac{r x' - x r'}{\sqrt{p}},$$

and, similarly,

$$\sin b \cos (B' + v) = \frac{r y' - y r'}{\sqrt{p}}; \quad \sin c \cos (C' + v) = \frac{r z' - z r'}{\sqrt{p}}. \quad (6)$$

Equations (6), together with the three equations of the form (1), determine a , b , c , A' , B' , and C' in terms of the heliocentric rectangular coördinates and their velocities and of r , r' , p , and v . The radius vector r and its velocity r' are given by (5), and p is determined by the equation

$$p = r^2 G^2 - (rr')^2 = r^2 (G^2 - (r')^2), \quad (7)$$

where G^2 is the velocity in the orbit, viz.:

$$G^2 = x'^2 + y'^2 + z'^2 = \frac{2}{r} - \frac{1}{a}. \quad (8)$$

For the parabola (8) becomes

$$G^2 = \frac{2}{r}, \quad (8a)$$

and

$$p = 2q = 2r - (rr')^2. \quad (7a)$$

Equation (7) is readily verified by squaring and adding side for side the expressions for $e \sin v$ and $e \cos v$ in VIII, page 18.

The mean anomaly v is determined as in VIII, page 18. For the parabola, since $e = 1$, we may also write

$$e \sin v = 2 \sin \frac{1}{2} v \cos \frac{1}{2} v = r' \sqrt{p} = r' \sqrt{2q}.$$

Eliminating $\cos \frac{1}{2} v$ by means of $r = \frac{q}{\cos^2 \frac{1}{2} v}$

$$\sin \frac{1}{2} v = r' \sqrt{\frac{r}{2}}, \quad (9a)$$

or, by (8a),

$$\sin \frac{1}{2} v = \frac{r'}{G}. \quad (9b)$$

It remains to arrange the relations existing between the elements referred to the ecliptic and the constants for the equator in a convenient form for the computation of the former from the latter.

When the elements are referred to the ecliptic, the constants for the equator are given in terms of the elements by

$$\left. \begin{aligned} \sin a \sin A &= \cos \Omega; & \sin b \sin B &= \sin \Omega \cos \varepsilon \\ \sin a \cos A &= -\sin \Omega \cos i; & \sin b \cos B &= \cos \Omega \cos i \cos \varepsilon - \sin i \sin \varepsilon \\ \sin c \sin C &= \sin \Omega \sin \varepsilon \\ \sin c \cos C &= \cos \Omega \cos i \sin \varepsilon + \sin i \cos \varepsilon. \end{aligned} \right\} \quad (10)$$

By analogy with (4) we may write

$$\left. \begin{aligned} \sin b \cos (B' + v) &= \sin b \cos B \cos u - \sin b \sin B \sin u \\ \sin c \cos (C' + v) &= \sin c \cos C \cos u - \sin c \sin C \sin u \\ \sin b \sin (B' + v) &= \sin b \sin B \cos u + \sin b \cos B \sin u \\ \sin c \sin (C' + v) &= \sin c \sin C \cos u + \sin c \cos C \sin u. \end{aligned} \right\} \quad (11)$$

If on the right-hand side of equations (11) the constants for the equator be replaced by (10) in terms of the ecliptical elements, then equations (11) will serve for the determination of these elements in terms of the left-hand members of (11), the numerical values of which have already become known by (1) and (6).

Multiplying the first and third equations of (11) by $\tan \varepsilon$ and subtracting the products from the second and fourth, respectively, we have by (10)

$$\begin{aligned} \sin c \cos (C' + v) - \sin b \cos (B' + v) \tan \varepsilon &= \\ (\cos \Omega \cos i \sin \varepsilon + \sin i \cos \varepsilon) \cos u - \sin \Omega \sin \varepsilon \sin u \\ - (\cos \Omega \cos i \cos \varepsilon - \sin i \sin \varepsilon) \cos u \tan \varepsilon + \sin \Omega \cos \varepsilon \sin u \tan \varepsilon \\ \sin c \sin (C' + v) - \sin b \sin (B' + v) \tan \varepsilon &= \\ \sin \Omega \sin \varepsilon \cos u + (\cos \Omega \cos i \sin \varepsilon + \sin i \cos \varepsilon) \sin u \\ - \sin \Omega \cos \varepsilon \cos u \tan \varepsilon - (\cos \Omega \cos i \cos \varepsilon - \sin i \sin \varepsilon) \sin u \tan \varepsilon, \end{aligned}$$

or, transposing the right- and left-hand members,

$$\left. \begin{aligned} \frac{\sin i}{\cos \varepsilon} \cos u &= \sin c \cos (C' + v) - \sin b \cos (B' + v) \tan \varepsilon \\ \frac{\sin i}{\cos \varepsilon} \sin u &= \sin c \sin (C' + v) - \sin b \sin (B' + v) \tan \varepsilon, \end{aligned} \right\} \quad (12)$$

which determine $\sin i$ and u . We have further

$$\omega = u - v; \quad A = A' - \omega; \quad B = B' - \omega; \quad C = C' - \omega. \quad (13)$$

Finally Ω may be derived from the first and the third or fifth equation of (10), and $\cos i$ from the second. As a partial check we may use the well known relation

$$\tan i = \frac{\sin b \sin c \sin (C - B)}{\sin a \cos A}. \quad (14)$$

It will be observed that the semi-major axis a is defined by (8) and the parameter p by (7). The eccentricity is given by

$$e^2 = \frac{a - p}{a}. \quad (15)$$

The residuals in VI, page 16, may be computed either by means of the f and g series, or by means of the closed expressions for f and g , or with the aid of the constants for the equator. In any case, it is most convenient to compute the constants first and from them the elements. If the constants for the equator are to be used in the comparison between theory and observation, then they should be computed as soon as the heliocentric coördinates and their velocities and r and r' have been derived by VI; otherwise their computation may be deferred until the observations have been properly represented. It is best to defer the computation of the elements to the last in any case.

Collecting results for the computation of the constants for the equator, and incidentally of a and e , which are not needed but reveal the general character of the orbit, we arrive at the following order of computation:

From the heliocentric coördinates and velocities, obtained in VI, page 16, compute G^2 and a by (8) or G^2 by (8a), and p or q by (7) or (7a); then v as in VIII, page 18, or for the parabola by (9a) or (9b); finally the constants for the equator by (1) and (6) and e from (15).

The elements which determine the position of the orbit plane in space, with reference to the ecliptic, are given in terms of the constants by formulæ (12) to (14).

THE DIFFERENTIAL CORRECTION OF THE FUNDAMENTAL DATA.

The arrangement of the differential formulæ developed on pages 11 to 13 and collected in [VII], page 17, is intended for the correction of the direct solution without regard to the eccentricity. A slight change in the arrangement of these formulæ will facilitate their application to parabolic and circular orbits and will permit also of ready transition from parabolic and circular to general orbits and *vice versa*. Thus, if in the original solution an assumption has been made regarding the eccentricity, and if subsequently it becomes apparent that the observations can not be represented under such assumption, then practically no extra labor will be required for the determination of the general orbit.

If we solve equations (25), page 12, for $\partial x'_0$ and $\partial y'_0$, respectively, and let

$$P_x = \frac{C_{\infty} \cos \alpha_{\infty} \partial \alpha - C \cos \alpha \partial \alpha_{\infty}}{C, C_{\infty} \sin (\alpha_{\infty} - \alpha)} ; \quad Q_x = \frac{A, C_{\infty} \cos \alpha_{\infty} - A_{\infty} C \cos \alpha}{C, C_{\infty} \sin (\alpha_{\infty} - \alpha)} \quad (1)$$

$$P_y = \frac{C_{\infty} \sin \alpha_{\infty} \partial \alpha - C \sin \alpha \partial \alpha_{\infty}}{C, C_{\infty} \sin (\alpha_{\infty} - \alpha)} ; \quad Q_y = \frac{A, C_{\infty} \sin \alpha_{\infty} - A_{\infty} C \sin \alpha}{C, C_{\infty} \sin (\alpha_{\infty} - \alpha)} ,$$

then

$$\partial x'_0 = P_x - Q_x \partial \rho_0 ; \quad \partial y'_0 = P_y - Q_y \partial \rho_0 . \quad (2)$$

Substituting these values for $\partial x'_0$ and $\partial y'_0$ in the equations (25), page 12, and letting

$$P_z = \frac{\partial \delta + C \sin \delta (\cos \alpha P_x + \sin \alpha P_y)}{C \cos \delta} ; \quad Q_z = \frac{B + C \sin \delta (\cos \alpha Q_x + \sin \alpha Q_y)}{C \cos \delta} \quad (3)$$

$$P_{z_{\infty}} = \frac{\partial \delta_{\infty} + C_{\infty} \sin \delta_{\infty} (\cos \alpha_{\infty} P_x + \sin \alpha_{\infty} P_y)}{C_{\infty} \cos \delta_{\infty}} ; \quad Q_{z_{\infty}} = \frac{B_{\infty} + C_{\infty} \sin \delta_{\infty} (\cos \alpha_{\infty} Q_x + \sin \alpha_{\infty} Q_y)}{C_{\infty} \cos \delta_{\infty}} ,$$

we obtain

$$\partial z'_0 = P_z - Q_z \partial \rho_0 ; \quad \partial z'_{\infty} = P_{z_{\infty}} - Q_{z_{\infty}} \partial \rho_0 . \quad (4)$$

Formulæ (1) to (4) may be applied in all classes of orbits. If no assumption be made regarding the eccentricity in determining the corrections to the fundamental data ρ_0 , x'_0 , y'_0 , and z'_0 , then equations (4) give

$$\partial \rho_0 = \frac{P_{z_{\infty}} - P_z}{Q_{z_{\infty}} - Q_z} , \quad (5)$$

and $\partial z'_0$, $\partial x'_0$, $\partial y'_0$ are obtained from (4) and (2). It will be observed that this process remains the same no matter whether an assumption regarding the eccentricity was made in the previous approximation or not.

If the corrections to the fundamental data are to yield a parabolic orbit, then, irrespective of the character of the previous approximation, the corrected fundamental data must satisfy the equation

$$x'^2_0 + y'^2_0 + z'^2_0 = \frac{2}{r_0} . \quad (6)$$

Five conditions therefore are available for the determination of the unknowns, namely, equations (2), (4), and (6). If these equations are inconsistent, then a parabolic solution is not possible. The five equations referred to may be applied in various ways for the computation of a parabola and for testing its sufficiency.

In the case of a circular orbit the corrected fundamental data must satisfy the equation

$$x_0'^2 + y_0'^2 + z_0'^2 = \frac{1}{r_0}, \quad (7)$$

and similar considerations prevail as for the parabola. Convenient formulæ for determining parabolic and circular orbits and testing their sufficiency are developed for the parabola on page 279, and for the circle on page 305.

As it may become necessary to pass from one class of orbit to another, it is advisable always to use the foregoing formulæ in determining the differential corrections to the fundamental data in place of the expression for $\partial\rho_0$ and the formulæ following it in [VII], (page 17). In this connection it will be observed that the last of formulæ [VII] afford a check only on the computation of the differential corrections from the adopted linear relations. The sufficiency of the linear relations and of the final results may be rigidly tested by computing the residuals as in VI, (page 16).

In the derivation of the formulæ for the differential correction of our first hypothesis or of any preliminary orbit, (formulæ [VII], page 17), the squares of the variations are neglected (*cf.* page 11), and the linear variations ∂f and ∂g are developed in series progressing in ascending powers of the intervals ((16), page 11). These series are obtained from the series for f and g ((13), page 10). The applicability of the f and g series to longer arcs is discussed on page 19. A comparison of the ∂f and ∂g series with the f and g series shows that the applicability of the former to longer arcs rests on the same conditions as that of the latter.

If an inspection of θ , r , and r' indicates that the number of terms of the f and g series given in (13), page 10, is not sufficient, the simple remedy exists of computing as many additional terms as are found to be appreciable from formulæ (1) to (7), pages 247-248, or we may use the closed expressions for f and g . But in the ∂f and ∂g series some of the neglected terms depend upon $\partial r'$, which must remain unknown until the differential corrections of the fundamental quantities ρ_0 , x'_0 , y'_0 , z'_0 have been found, and therefore these terms can not be computed in advance, except in circular orbits where $r' = 0$. In general, it will then become necessary to have recourse to closed expressions for ∂f and ∂g , and these will be derived below. But in some cases it may be found more convenient to make allowance for the neglected terms in the course of the differential correction, particularly so if the number of additional terms to be considered is very limited.

When the slow convergence of the ∂f and ∂g series arises from a small value of r rather than from long intervals, then it is advisable to combine with the additional terms in the ∂f and ∂g series a correction so as to express ∂r in terms of $\partial\rho$ more accurately than is accomplished by the linear relation $\partial r = \cos\beta \partial\rho$

given in (24) and (25), page 12, except that the following additional terms will appear:

$$\begin{aligned}
 & \text{in } \partial \alpha_i, -\frac{\cos \alpha_i}{\rho_i} y_0 \frac{\theta_{iii}^3 \partial r'}{2 r_0^4} - \frac{\cos \alpha_i}{\rho_i} \frac{3 \theta_{iii}^2 \cos \beta}{r_0^4} \left[\frac{y_0 \cos \beta}{r_0} - \frac{1}{2} \frac{\eta_0}{\rho_0} \right] (\partial \rho_0)^2 \\
 & \quad + \frac{\sin \alpha_i}{\rho_i} x_0 \frac{\theta_{iii}^3 \partial r'}{2 r_0^4} + \frac{\sin \alpha_i}{\rho_i} \frac{3 \theta_{iii}^2 \cos \beta}{r_0^4} \left[\frac{x_0 \cos \beta}{r_0} - \frac{1}{2} \frac{\xi_0}{\rho_0} \right] (\partial \rho_0)^2 ; \\
 & \text{in } \partial \alpha_{iii}, +\frac{\cos \alpha_{iii}}{\rho_{iii}} y_0 \frac{\theta_i^3 \partial r'}{2 r_0^4} - \frac{\cos \alpha_{iii}}{\rho_{iii}} \frac{3 \theta_i^2 \cos \beta}{r_0^4} \left[\frac{y_0 \cos \beta}{r_0} - \frac{1}{2} \frac{\eta_0}{\rho_0} \right] (\partial \rho_0)^2 \\
 & \quad - \frac{\sin \alpha_{iii}}{\rho_{iii}} x_0 \frac{\theta_i^3 \partial r'}{2 r_0^4} + \frac{\sin \alpha_{iii}}{\rho_{iii}} \frac{3 \theta_i^2 \cos \beta}{r_0^4} \left[\frac{x_0 \cos \beta}{r_0} - \frac{1}{2} \frac{\xi_0}{\rho_0} \right] (\partial \rho_0)^2 ; \\
 & \text{in } \partial \delta_i, -\frac{\sin \delta_i}{\rho_i} \left\{ -\sin \alpha_i y_0 \frac{\theta_{iii}^3 \partial r'}{2 r_0^4} - \sin \alpha_i \frac{3 \theta_{iii}^2 \cos \beta}{r_0^4} \left[\frac{y_0 \cos \beta}{r_0} - \frac{1}{2} \frac{\eta_0}{\rho_0} \right] (\partial \rho_0)^2 \right. \\
 & \quad \left. - \cos \alpha_i x_0 \frac{\theta_{iii}^3 \partial r'}{2 r_0^4} - \cos \alpha_i \frac{3 \theta_{iii}^2 \cos \beta}{r_0^4} \left[\frac{x_0 \cos \beta}{r_0} - \frac{1}{2} \frac{\xi_0}{\rho_0} \right] (\partial \rho_0)^2 \right\} \\
 & \quad + \frac{\cos \delta_i}{\rho_i} \left\{ -z_0 \frac{\theta_{iii}^3 \partial r'}{2 r_0^4} - \frac{3 \theta_{iii}^2 \cos \beta}{r_0^4} \left[\frac{z_0 \cos \beta}{r_0} - \frac{1}{2} \frac{\zeta_0}{\rho_0} \right] (\partial \rho_0)^2 \right\} ; \\
 & \text{in } \partial \delta_{iii}, -\frac{\sin \delta_{iii}}{\rho_{iii}} \left\{ +\sin \alpha_{iii} y_0 \frac{\theta_i^3 \partial r'}{2 r_0^4} - \sin \alpha_{iii} \frac{3 \theta_i^2 \cos \beta}{r_0^4} \left[\frac{y_0 \cos \beta}{r_0} - \frac{1}{2} \frac{\eta_0}{\rho_0} \right] (\partial \rho_0)^2 \right. \\
 & \quad \left. + \cos \alpha_{iii} x_0 \frac{\theta_i^3 \partial r'}{2 r_0^4} - \cos \alpha_{iii} \frac{3 \theta_i^2 \cos \beta}{r_0^4} \left[\frac{x_0 \cos \beta}{r_0} - \frac{1}{2} \frac{\xi_0}{\rho_0} \right] (\partial \rho_0)^2 \right\} \\
 & \quad + \frac{\cos \delta_{iii}}{\rho_{iii}} \left\{ +z_0 \frac{\theta_i^3 \partial r'}{2 r_0^4} - \frac{3 \theta_i^2 \cos \beta}{r_0^4} \left[\frac{z_0 \cos \beta}{r_0} - \frac{1}{2} \frac{\zeta_0}{\rho_0} \right] (\partial \rho_0)^2 \right\} .
 \end{aligned} \tag{9}$$

In these expressions ∂r has been eliminated by the relation $\partial r = \cos \beta \partial \rho$. But when r is small, it is advisable to introduce for $\frac{\partial r}{\partial \rho}$ a more accurate expression than $\cos \beta$. Since $r^2 = x^2 + y^2 + z^2$, omitting the subscript zero for the present, we have rigidly

$$(r + \partial r)^2 = (x + \partial x)^2 + (y + \partial y)^2 + (z + \partial z)^2,$$

or by (17), (18), and (19), page 11,

$$2r \partial r + (\partial r)^2 = 2r \cos \beta \partial \rho + (\partial \rho)^2. \tag{10}$$

Let

$$\frac{\partial r}{\partial \rho} = \frac{2r \cos \beta + \partial \rho}{2r + \partial r} = \Gamma, \tag{11}$$

then, by replacing ∂r in the denominator of (11) by $\Gamma \partial \rho$,

$$2r \cos \beta + \partial \rho = \Gamma (2r + \Gamma \partial \rho).$$

After multiplying both sides of this equation by $\partial \rho$, we obtain the following quadratic equation in $\Gamma \partial \rho$,

$$(\Gamma \partial \rho)^2 + 2r \Gamma \partial \rho = (2r \cos \beta + \partial \rho) \partial \rho,$$

the solution of which gives

$$\Gamma \partial \rho = \partial r = -r \pm \sqrt{r^2 + (2r \cos \beta + \partial \rho) \partial \rho}, \tag{12}$$

or

$$\partial r = -r \pm \sqrt{(r + \cos \beta \partial \rho)^2 + \sin^2 \beta (\partial \rho)^2}.$$

Since by differentiation $\partial r = \cos \beta \partial \rho$ is the first term in the expansion of ∂r in terms

of the heliocentric coördinates, as in (15), page 11. The variations of the latter are expressed rigidly in terms of the variations of the fundamental quantities ρ_0, x'_0, y'_0, z'_0 and of f and g , except for the term $\partial g \partial \omega_0$, which is of the order θ^4 multiplied by a variation, that is, of a higher order than the terms retained in the variations ∂f and ∂g . In expressing ∂f and ∂g in terms of the variations of the fundamental quantities, linear relations, correct inclusively to terms of the order θ^3 multiplied by a variation, are adopted, except that ∂r_0 is expressed rigidly in terms of $\partial \rho_0$. By this procedure all terms are included which numerically may become comparable with the smallest terms retained in ∂f and ∂g , as far as this is possible without complicating the resulting expressions.

On the basis of the foregoing considerations, we may now determine the most advantageous course to pursue if after the direct solution or first hypothesis it becomes necessary to improve the orbit. The correction of the fundamental quantities may be performed by means of differential relations in which the variations ∂f and ∂g are expressed either, as above, in series, or in closed form. Closed expressions for ∂f and ∂g are to be adopted when the ∂f and ∂g series do not converge rapidly. The convergency of these series depends in the main on θ and r . But as all the quantities necessary for estimating the magnitude of the various coefficients in ∂f and ∂g are available from the direct solution, a mere inspection of the θ^3 terms in (16) page 11, or of the terms in (9), will reveal at once whether the largest of these theoretically small coefficients are greatly in excess of numbers that may be neglected in a five- or six-place computation, as the case may be. If the magnitude of any of these terms does not exceed 50 units of the last decimal, the series in ∂f and ∂g may be considered satisfactory, although it is obviously not possible to fix the limit definitely, as it also depends upon the magnitude of the residuals which are to be removed. If these are small, the limit may be taken considerably smaller, etc.

The relation $\partial r_0 = \Gamma \partial \rho_0$ enters into the differential formulæ whether closed expressions are adopted or not, but for closed expressions, which are generally employed only for longer arcs, when fairly accurate elements are available, we may let $\Gamma = \cos \beta$.

Assuming now that the series for ∂f and ∂g are to be adopted in the differential correction of the fundamental quantities, we estimate first of all the two terms in θ^3 in the formulæ for f_x, f_y, f_z , page 12, in the manner indicated above. They should be included if they count within the last decimal adopted for the computation. For Γ we use the approximate value $\cos \beta$. The differential correction is then performed as in [VII], page 17, or more conveniently as in (1) to (5). As soon as $\partial \rho_0$ has been computed, we may test the convergency of the series for ∂r by (13). If the series be divergent, the method of differential correction fails, and we must resort to the method of arbitrary variation. But as then $\partial \rho > \frac{r}{\sqrt{2}}$, it is easily recognized that this case scarcely will occur in practice. If the series converges

very slowly, we compute Γ accurately to the required number of decimals by (14). If Γ differ considerably from $\cos \beta$, we may conclude that the differential correction will not remove the residuals entirely without taking into account the additional terms (9) or (16) in the expressions for the residuals, and it is advisable to compute these terms by (15) and (16) before proceeding any further in the computation of the differential corrections than is necessary for the determination of $\partial r'$, which is required in (16). By variation of the formula for rr' in VI, page 16, we easily derive, omitting the subscript zero,

$$r \partial r' = x \partial x' + y \partial y' + z \partial z' + (x' \xi + y' \eta + z' \zeta) \frac{\partial \rho}{\rho} - r' \Gamma \partial \rho, \quad (17)$$

from which $\partial r'$ may be derived, after $\partial x'$, $\partial y'$, $\partial z'$ have been computed by (2) and (4).

If now the terms given in (16) be transposed to the left-hand sides of (24) and (25), page 12, and be applied as corrections to the residuals, then it will only be necessary to recompute P_x , P_y , P_z with the corrected residuals, all other auxiliary quantities remaining the same.

The computation of the corrections to the residuals by (14), (15), (16), and (17), presents no difficulties and requires but little computation as the terms are small, and all the quantities involved are available from the previous computations.

With the improved values of P_x , P_y , P_z the computation of the differential corrections to the fundamental quantities ρ_0 , x'_0 , y'_0 , z'_0 is carried through in the usual manner.

It will be observed that in the foregoing directions for taking account of higher terms in the differential corrections, no allowance for the difference between $\cos \beta$ and Γ has been made in the expressions for f_x , f_y , and f_z . This difference may, in general, be neglected in these expressions, but should it involve a considerable change in the three terms multiplied by Γ , then it will be advisable to recompute also the f_x , f_y , f_z and all auxiliary quantities which depend on them.

From the preceding considerations we conclude that the computation of the additional terms in the expressions for the residuals will be required only for large values of $\partial \rho$, combined with slow convergence of the ∂r series. But as such cases chiefly arise from excessively small values of r , the ∂f and ∂g series will then also converge slowly, and, as stated above, these cases are met by the use of closed expressions for ∂f and ∂g .

But it may happen that after the differential correction of the fundamental data has been carried through in its simplest form, the resulting residuals point toward a possible slight improvement of the orbit. This is then generally accomplished as outlined above, and in a far more convenient manner than by a second application of the differential formulæ.

Closed expressions for ∂f and ∂g are given by KUEHNERT in *A. N.* 2266 together with his closed expressions for f and g . These expressions apply to elliptic

orbits and become indeterminate for the parabola. Special formulæ for the circle and the parabola are deduced below (for parabola *cf.* also page 300). These may be derived from KUEHNERT'S expressions, for the circle at once, and for both, after γ has been introduced in place of the product $\sqrt{2a} \sin \bar{g}$ by (12), page 249. This product is indeterminate for the parabola, but its equivalent γ is definitely given by (19) or (21), page 249.

The expression for γ also gives $\sin 2\bar{g} = \left(\frac{2}{a}\right)^{1/2} \gamma \sqrt{1 - \frac{\gamma^2}{2a}}$. In the notation adopted in formulæ (8) to (12), pages 248-249, KUEHNERT'S expressions for ∂f and ∂g now become, after some simple reductions,

$$\left. \begin{aligned} M &= -\frac{a\gamma}{rr_0} \left[r\gamma + \sqrt{1 - \frac{\gamma^2}{2a}} \left(2r_0 \gamma \sqrt{1 - \frac{\gamma^2}{2a}} + \frac{r_0 r'_0 \gamma^2}{\sqrt{2}} - \sqrt{2} \frac{3}{2} \theta \right) \right] \\ N &= -a \left[\frac{3}{2} \left(\theta \left(1 - \frac{\gamma^2}{r} \right) - g \right) + \frac{\gamma^3}{r} \left(\sqrt{2} r_0 \sqrt{1 - \frac{\gamma^2}{2a}} + \frac{r_0 r'_0 \gamma}{2} \right) \right], \end{aligned} \right\} \quad (18)$$

where g is introduced by means of (8), page 248, to eliminate $2\bar{g} - \sin 2\bar{g}$ from N ;

$$\left. \begin{aligned} \partial f &= \left\{ \frac{\gamma^2}{rr_0} \left[\frac{r}{r_0} + 2 \left(1 - \frac{\gamma^2}{2a} \right) \right] + \frac{2M}{r_0^2} \right\} \partial r_0 + M \partial (G_0^2) + \frac{\sqrt{2} \gamma^3}{rr_0} \sqrt{1 - \frac{\gamma^2}{2a}} \partial (r_0 r'_0) \\ \partial g &= \left\{ \frac{\sqrt{2} \gamma^3}{r} \sqrt{1 - \frac{\gamma^2}{2a}} + \frac{2N}{r_0^2} \right\} \partial r_0 + N \partial (G_0^2) + \frac{\gamma^4}{r} \partial (r_0 r'_0), \end{aligned} \right\} \quad (19)$$

where $G_0^2 = \frac{2}{r_0} - \frac{1}{a}$ represents the velocity in the orbit.

It will be observed that the expression $\sqrt{1 - \frac{\gamma^2}{2a}}$ may be computed directly from \bar{g} , since $\sqrt{1 - \frac{\gamma^2}{2a}} = \cos \bar{g} = \cos \frac{1}{2} (E - E_0)$.

For the parabola $G_0^2 = \frac{2}{r_0}$, and $\partial (G_0^2) = -\frac{2}{r_0^2} \partial r_0$. Hence the terms containing M and N destroy one another in ∂f and ∂g respectively. If, then, we also let $a = \infty$, we obtain for the parabola

$$\left. \begin{aligned} \partial f &= \frac{\gamma^2}{rr_0} \left[\frac{r}{r_0} + 2 \right] \partial r_0 + \frac{\sqrt{2} \gamma^3}{rr_0} \partial (r_0 r'_0) \\ \partial g &= \frac{\sqrt{2} \gamma^3}{r} \partial r_0 + \frac{\gamma^4}{r} \partial (r_0 r'_0), \end{aligned} \right\} \quad (20)$$

in accordance with the expressions derived below directly and given in (34) and (35), page 300.

For the circle $G_0^2 = \frac{1}{r_0} = \frac{1}{a}$, and $\partial G_0^2 = -\frac{1}{r_0^2} \partial r_0$. Hence the terms containing M and N in ∂f and ∂g may be combined into $\frac{M}{r_0^2} \partial r_0$ and $\frac{N}{r_0^2} \partial r_0$, respectively. If then we let $r = a$ in (18) and (19) and introduce (18) in (19), the latter reduce to

$$\partial f = \frac{\gamma}{a^3} \sqrt{1 - \frac{\gamma^2}{2a}} \sqrt{2} \frac{3}{2} \theta \partial r_0; \quad \partial g = -\frac{3}{2a} \left(\theta \left(1 - \frac{\gamma^2}{a} \right) - g \right) \partial r_0, \quad (21)$$

or, if we eliminate γ by (12), page 249, and introduce g by (26), page 250, and also write ∂a for ∂r_0 ,

$$\partial f = \frac{3}{2} \frac{\theta g}{a^4} \partial a ; \quad \partial g = -\frac{3}{2a} (\theta f - g) \partial a . \quad (22)$$

These expressions may also be derived directly by differentiation of equations (26), page 250.

In order to derive, on the basis of closed expressions for ∂f and ∂g in an elliptic orbit, such corrections to the fundamental data ρ_0 , x'_0 , y'_0 , and z'_0 , as will remove the residuals the foregoing equations (19) must be introduced into the developments on page 11 *et seq.* in place of the last two equations in (16), page 11. For the sake of brevity let in (18) and (19)

$$\gamma_c = \sqrt{1 - \frac{\gamma^2}{2a}} = \cos \bar{g} = \cos \frac{1}{2} (E - E_0) , \quad (23)$$

where the algebraical sign of γ_c is the same as that of $\cos \bar{g}$. Introducing, then, the expressions (19) for ∂f and ∂g into the first of equations (16), page 11, where $\omega = x, y, z$, and noting that by (17), page 11, $\partial \omega_0 = \frac{w_0}{\rho_0} \partial \rho_0$, where $w_0 = \xi_0, \eta_0, \zeta_0$, we obtain, since $\partial r_0 = \cos \beta \partial \rho_0$,

$$\begin{aligned} \partial \omega = & \left\{ f \frac{w_0}{\rho_0} + \frac{2 \cos \beta}{r_0^2} (\omega_0 M + \omega'_0 N) + \frac{\cos \beta \gamma^2}{r r_0} \left(\omega_0 \left[\frac{r}{r_0} + 2 \gamma^2 \right] + \sqrt{2} \omega'_0 r_0 \gamma \gamma_c \right) \right\} \partial \rho_0 \\ & + g \partial \omega'_0 + (\omega_0 M + \omega'_0 N) \partial (G_0^2) + \left\{ \frac{\sqrt{2} \omega_0 \gamma^2}{r r_0} \gamma_c + \frac{\omega'_0 \gamma^4}{r} \right\} \partial (r_0 r'_0) . \end{aligned} \quad (24)$$

But since $G_0^2 = x_0'^2 + y_0'^2 + z_0'^2$ and $r_0 r'_0 = x_0 x'_0 + y_0 y'_0 + z_0 z'_0$, therefore, also

$$\partial (G_0^2) = 2 \sum \omega'_0 \partial \omega'_0 ; \quad \partial (r_0 r'_0) = \sum \omega_0 \partial \omega'_0 + \sum \omega'_0 \partial \omega_0 , \quad (25)$$

or if $\partial \omega_0$ be expressed as above in terms of $\partial \rho_0$

$$\partial (r_0 r'_0) = \sum \omega'_0 \partial \omega'_0 + (\sum \omega'_0 w_0) \frac{\partial \rho_0}{\rho_0} . \quad (26)$$

Let

$$\varphi_0 = \frac{\sum \omega'_0 w_0}{\rho_0} = \frac{x'_0 \xi_0 + y'_0 \eta_0 + z'_0 \zeta_0}{\rho_0} , \quad (27)$$

and eliminate $\partial (G_0^2)$ and $\partial (r_0 r'_0)$ from (24) by means of the foregoing relations. Then, after some simple reductions, (24) becomes

$$\begin{aligned} \partial \omega = & \left\{ f \frac{w_0}{\rho_0} + \frac{\cos \beta}{r_0^2} [2 (\omega_0 M + \omega'_0 N) + \omega_0 \gamma^2] + \frac{1}{r r_0} \cos \beta \gamma_c \frac{\gamma^2}{r_0} [\omega'_0 r_0 \gamma + \sqrt{2} \omega_0 \gamma_c] + \right. \\ & \left. \frac{\varphi_0 \gamma^2}{r r_0} [\omega'_0 r_0 \gamma + \sqrt{2} \omega_0 \gamma_c] \right\} \partial \rho_0 + g \partial \omega'_0 + \frac{\gamma^2}{r r_0} [\omega'_0 r_0 \gamma + \sqrt{2} \omega_0 \gamma_c] \sum \omega_0 \partial \omega'_0 \\ & + 2 (\omega_0 M + \omega'_0 N) \sum \omega'_0 \partial \omega'_0 . \end{aligned} \quad (28)$$

Let

$$\begin{aligned} g_\omega &= \frac{\gamma^2}{r r_0} [\omega'_0 r_0 \gamma + \sqrt{2} \omega_0 \gamma_c] ; \quad m_\omega = \omega_0 M + \omega'_0 N \\ f_\omega &= f \frac{w_0}{\rho_0} + \frac{\cos \beta}{r_0^2} [2 m_\omega + \omega_0 \gamma^2] + g_\omega \left[\varphi_0 + \frac{\sqrt{2} \cos \beta \gamma_c}{\gamma} \right] . \end{aligned} \quad (29)$$

Then

$$\partial\omega = f_{\omega}\partial\rho_0 + g\partial\omega'_0 + g_{\omega}\Sigma\omega_0\partial\omega'_0 + 2m_{\omega}\Sigma\omega'_0\partial\omega'_0. \quad (30)$$

Equation (30) gives ∂x , ∂y , or ∂z in a closed form for any date, the auxiliary quantities g_x, g_y, g_z ; m_x, m_y, m_z ; f_x, f_y, f_z being defined by (29). To obtain the values of ∂x , ∂y , or ∂z for a particular date, the values of $f, g, \gamma, \gamma_c, r, M$, and N for that date must be introduced in (29) and (30). γ and γ_c are given by (12), page 249, and by (23); f and g in terms of γ by (11) and (16), page 249; M and N in terms of γ and γ_c by (18), while $r = a(1 - e \cos E)$. Thus we have, for instance, for the first date

$$\left. \begin{aligned} g_{x_0} &= \frac{\gamma_0^2}{r_0 r'_0} [x'_0 r_0 \gamma_0 + \sqrt{2} x_0 \gamma'_0]; & m_{x_0} &= x_0 M_0 + x'_0 N_0, \\ f_{x_0} &= f_0 \frac{\xi_0}{\rho_0} + \frac{\cos \beta}{r_0^2} [2m_{x_0} + x_0 \gamma_0^2] + g_{x_0} \left[\varphi_0 + \frac{1}{\gamma_0} \frac{\cos \beta \gamma'_0}{\gamma_0} \right] \end{aligned} \right\} \quad (31)$$

$$\partial x_0 = f_{x_0} \partial \rho_0 + [g_0 + g_{x_0} x_0 + 2 m_{x_0} x'_0] \partial x'_0 + [g_{x_0} y_0 + 2 m_{x_0} y'_0] \partial y'_0 + [g_{x_0} z_0 + 2 m_{x_0} z'_0] \partial z'_0,$$

and analogous expressions in y and z for the first date and for all three coördinates for the third date. Thus $f_{y_{III}}$ is written out by changing x into y and the subscript one into three in (31), etc.

If, now, equations (15), page 11, be applied to the first and third places, we obtain, by (29) and (30),

$$\begin{aligned} \partial \alpha_0 &= \frac{1}{\rho_0} [\cos \alpha_0 f_{y_0} - \sin \alpha_0 f_{x_0}] \partial \rho_0 \\ &\quad - \sin \alpha_0 \left[\frac{g_0}{\rho_0} + x_0 \frac{g_{x_0}}{\rho_0} + x'_0 2 \frac{m_{x_0}}{\rho_0} \right] \partial x'_0 + \cos \alpha_0 \left[x_0 \frac{g_{y_0}}{\rho_0} + x'_0 2 \frac{m_{y_0}}{\rho_0} \right] \partial x'_0 \\ &\quad - \sin \alpha_0 \left[y_0 \frac{g_{x_0}}{\rho_0} + y'_0 2 \frac{m_{x_0}}{\rho_0} \right] \partial y'_0 + \cos \alpha_0 \left[\frac{g_0}{\rho_0} + y_0 \frac{g_{y_0}}{\rho_0} + y'_0 2 \frac{m_{y_0}}{\rho_0} \right] \partial y'_0 \\ &\quad - \sin \alpha_0 \left[z_0 \frac{g_{x_0}}{\rho_0} + z'_0 2 \frac{m_{x_0}}{\rho_0} \right] \partial z'_0 + \cos \alpha_0 \left[z_0 \frac{g_{y_0}}{\rho_0} + z'_0 2 \frac{m_{y_0}}{\rho_0} \right] \partial z'_0 \\ \partial \delta_0 &= -\frac{1}{\rho_0} [\sin \delta_0 (\sin \alpha_0 f_{y_0} + \cos \alpha_0 f_{x_0}) - \cos \delta_0 f_{z_0}] \partial \rho_0 \\ &\quad - \sin \delta_0 \cos \alpha_0 \left[\frac{g_0}{\rho_0} + x_0 \frac{g_{x_0}}{\rho_0} + x'_0 2 \frac{m_{x_0}}{\rho_0} \right] \partial x'_0 - \sin \delta_0 \sin \alpha_0 \left[x_0 \frac{g_{y_0}}{\rho_0} + x'_0 2 \frac{m_{y_0}}{\rho_0} \right] \partial x'_0 \\ &\quad - \sin \delta_0 \cos \alpha_0 \left[y_0 \frac{g_{x_0}}{\rho_0} + y'_0 2 \frac{m_{x_0}}{\rho_0} \right] \partial y'_0 - \sin \delta_0 \sin \alpha_0 \left[\frac{g_0}{\rho_0} + y_0 \frac{g_{y_0}}{\rho_0} + y'_0 2 \frac{m_{y_0}}{\rho_0} \right] \partial y'_0 \\ &\quad - \sin \delta_0 \cos \alpha_0 \left[z_0 \frac{g_{x_0}}{\rho_0} + z'_0 2 \frac{m_{x_0}}{\rho_0} \right] \partial z'_0 - \sin \delta_0 \sin \alpha_0 \left[z_0 \frac{g_{y_0}}{\rho_0} + z'_0 2 \frac{m_{y_0}}{\rho_0} \right] \partial z'_0 \\ &\quad + \cos \delta_0 \left[x_0 \frac{g_{x_0}}{\rho_0} + x'_0 2 \frac{m_{x_0}}{\rho_0} \right] \partial x'_0 \\ &\quad + \cos \delta_0 \left[y_0 \frac{g_{x_0}}{\rho_0} + y'_0 2 \frac{m_{x_0}}{\rho_0} \right] \partial y'_0 \\ &\quad + \cos \delta_0 \left[\frac{g_0}{\rho_0} + z_0 \frac{g_{z_0}}{\rho_0} + z'_0 2 \frac{m_{z_0}}{\rho_0} \right] \partial z'_0, \end{aligned}$$

need here to be given. The computer will choose from the available methods of solution the one which for his particular problem permits of the most convenient determination of the unknowns on the basis of the numerical values of the coefficients. Nor is it necessary in this case to arrange the solution in a form suitable for the transition from one class of orbit to another as in (2), (4) and (5), since special formulæ will be derived below for the transition from circular and parabolic to elliptic (or hyperbolic) orbits, and since the corrected orbit will prove elliptic (or hyperbolic) if in the preliminary orbit calculations the circle or parabola proves insufficient.

Linear differential relations have been adopted throughout in the foregoing developments on the basis of closed expressions for ∂f and ∂g , as these latter will generally be used only for long arcs for which close approximations to the orbit are almost always available.

Quite frequently the differential correction of ρ_0, x'_0, y'_0, z'_0 may be considerably simplified. For short intervals and not too small a value of r_0 , inspection of formulæ (22), page 12, may show that the terms in f_x, f_y, f_z depending on θ^2 and θ^3 are very small in comparison with the first term of each. We may then write

$$f_x = f \cos \delta_u \cos \alpha_u; \quad f_y = f \cos \delta_u \sin \alpha_u; \quad f_z = f \sin \delta_u.$$

Introducing these values into formulæ (23), page 12, we obtain for the first and third date,

$$\left. \begin{aligned} A_i &= \frac{f_i}{\rho_i} \sin (\alpha_u - \alpha_i) \cos \delta_u; & A_{iii} &= \frac{f_{iii}}{\rho_{iii}} \sin (\alpha_u - \alpha_{iii}) \cos \delta_u \\ B_i &= \frac{f_i}{\rho_i} [\sin (\delta_u - \delta_i) + 2 \sin \delta_i \cos \delta_u \sin^2 \frac{1}{2} (\alpha_u - \alpha_i)] \\ B_{iii} &= \frac{f_{iii}}{\rho_{iii}} [\sin (\delta_u - \delta_{iii}) + 2 \sin \delta_{iii} \cos \delta_u \sin^2 \frac{1}{2} (\alpha_u - \alpha_{iii})], \end{aligned} \right\} \quad (35)$$

where in B_i and B_{iii} the second term frequently may be neglected in comparison with the first.

If the inspection proves these approximate values of $A_i, A_{iii}, B_i, B_{iii}$ to be sufficient, they may be used in formulæ (1) and (3) on page 255 in place of their more rigid expressions. The subsequent computation remains as indicated on page 255.

To determine the error in σ , we must determine the error in κ in II, page 15. Substituting α' for α'_0 , δ for δ'_0 , etc., from (2) and (3) into κ , we have

$$\begin{aligned} \kappa = -R \cos D \frac{[\tan \delta_{\alpha} \cos (A - \alpha_{\alpha}) - \tan D] \alpha'_0 + \sin (A - \alpha_{\alpha}) (\tan \delta)_0' -}{(\alpha'_0)^2 \tan \delta_{\alpha} - \alpha''_0 (\tan \delta)_0' + \alpha'_0 (\tan \delta)_0'' - \left\{ \frac{\theta_1 \theta_{111}}{6} [3 (\alpha'_0)^2 \tan \delta_{\alpha} + (\tan \delta)_0''] + \right.} \\ \left. \frac{\theta_1 \theta_{111}}{6} \right\} [\tan \delta_{\alpha} \cos (A - \alpha_{\alpha}) - \tan D] \alpha''' + \sin (A - \alpha_{\alpha}) (\tan \delta)''' \} \\ \frac{\theta_{111} - \theta_1}{3} (\tan \delta)_0' \left\{ \alpha''' + \left\{ \frac{\theta_{111} - \theta_1}{3} \alpha'_0 + \frac{\theta_1 \theta_{111}}{6} \alpha''_0 \right\} (\tan \delta)''' \right\} \end{aligned} \quad (4)$$

or, denoting the values of N and κ in III, page 15, where third derivatives are neglected, by N_0 and κ_0

$$\begin{aligned} \kappa = \kappa_0 + \frac{\theta_1 \theta_{111}}{6 N_0} \{ [\tan \delta_{\alpha} \cos (A - \alpha_{\alpha}) - \tan D] \alpha''' + \sin (A - \alpha_{\alpha}) (\tan \delta)''' \} R \cos D \\ - \left\{ \frac{\theta_1 \theta_{111}}{6 N_0^2} [3 (\alpha'_0)^2 \tan \delta_{\alpha} + (\tan \delta)_0''] + \frac{\theta_{111} - \theta_1}{3 N_0^2} (\tan \delta)_0' \right\} R \cos D \alpha''' \\ + \left\{ \frac{\theta_{111} - \theta_1}{3 N_0^2} \alpha'_0 + \frac{\theta_1 \theta_{111}}{6 N_0^2} \alpha''_0 \right\} R \cos D (\tan \delta)''' \dots \dots \dots \end{aligned} \quad (4a)$$

where the remaining terms may be omitted for the purpose of estimating the error in κ . This error is of the second order for equal or nearly equal intervals and of the first for unequal intervals. It becomes inappreciable for nearly uniform geocentric motion.

Since

$$\sigma = \kappa \left\{ \frac{1}{r^3} - \frac{1}{R^3} \right\}$$

the error in σ is of the same order as the error in κ . By inspection of the formulæ for σ' and σ'' in V, page 16, the same orders of error will be found for these quantities. In a similar manner the errors depending on the fourth derivatives, etc., may be deduced.

In the first hypothesis the derivatives beyond the second are neglected. In order to indicate how a second hypothesis may be based on the first, we shall develop expressions for α''' and $(\tan \delta)'''$. By successive differentiation of

$$\sigma \cos \alpha = x + X; \quad \sigma \sin \alpha = y + Y; \quad \sigma \tan \delta = z + Z$$

and remembering that

$$x''' = \frac{1}{k} \frac{d}{dt} x'' = \frac{1}{k} \frac{d}{dt} \left(-\frac{x}{r^3} \right) = -\frac{x'}{r^3} + 3 \frac{x r'}{r^4}; \quad y''' = \text{etc.}$$

we obtain the equations

$$\left. \begin{aligned} \sigma''' \cos \alpha - 3 \sigma'' \sin \alpha \alpha' - 3 \sigma' [\cos \alpha (\alpha')^2 + \sin \alpha \alpha''] - \sigma \{ 3 \cos \alpha \alpha' \alpha'' + \sin \alpha [\alpha''' - (\alpha')^3] \} \\ = -\frac{x'}{r^3} + 3 \frac{x r'}{r^4} + X''' \\ \sigma''' \sin \alpha + 3 \sigma'' \cos \alpha \alpha' - 3 \sigma' [\sin \alpha (\alpha')^2 - \cos \alpha \alpha''] - \sigma \{ 3 \sin \alpha \alpha' \alpha'' - \cos \alpha [\alpha''' - (\alpha')^3] \} \\ = -\frac{y'}{r^3} + 3 \frac{y r'}{r^4} + Y''' \\ \sigma''' \tan \delta + 3 \sigma'' (\tan \delta)' + 3 \sigma' (\tan \delta)'' + \sigma (\tan \delta)''' = -\frac{z'}{r^3} + 3 \frac{z r'}{r^4} + Z''' \end{aligned} \right\} \quad (5)$$

which determine α''' and $(\tan \delta)'''$. Multiply the first of these by $\cos \alpha$, the second by $\sin \alpha$ and add the results. Then

$$\sigma''' = 3\sigma'(\alpha')^2 + 3\sigma\alpha'\alpha'' + \frac{3r'}{r^4} [r \cos \alpha + y \sin \alpha] - \frac{1}{r^3} [r' \cos \alpha + y' \sin \alpha] + X''' \cos \alpha + Y''' \sin \alpha,$$

or since by VI, page 16,

$$x \cos \alpha + y \sin \alpha = \sigma - [X \cos \alpha + Y \sin \alpha]; \quad x' \cos \alpha + y' \sin \alpha = \sigma' - [X' \cos \alpha + Y' \sin \alpha],$$

therefore, also

$$\sigma''' = \sigma \left(\frac{3r'}{r^4} + \alpha' \alpha'' \right) - \sigma' \left(\frac{1}{r^3} - 3(\alpha')^2 \right) - \frac{3r'}{r^4} [X \cos \alpha + Y \sin \alpha] + \frac{1}{r^3} [X' \cos \alpha + Y' \sin \alpha] + X''' \cos \alpha + Y''' \sin \alpha. \quad (6)$$

Substituting this expression for σ''' in the first and second of (5), we solve

$$\begin{aligned} \alpha''' = (\alpha')^3 - \frac{\alpha'}{r^3} - \frac{3}{\sigma} [\sigma' \alpha'' + \sigma'' \alpha'] + \frac{3r'}{\sigma r^4} [X \sin \alpha - Y \cos \alpha] - \frac{1}{\sigma r^3} [X' \sin \alpha - Y' \cos \alpha] \\ - \frac{1}{\sigma} [X''' \sin \alpha - Y''' \cos \alpha]. \end{aligned} \quad (7)$$

By similarly eliminating σ''' from the third of (5) by means of (6), we obtain

$$\begin{aligned} (\tan \delta)''' = -3 \tan \delta \alpha' \alpha'' + \frac{(\tan \delta)'}{r^3} - \frac{3}{\sigma} [\sigma'' (\tan \delta)' + \sigma' ((\tan \delta)'' + \tan \delta (\alpha')^2)] \\ + \frac{3r'}{\sigma r^4} [(X \cos \alpha + Y \sin \alpha) \tan \delta - Z] - \frac{1}{\sigma r^3} [(X' \cos \alpha + Y' \sin \alpha) \tan \delta - Z'] \\ - \frac{1}{\sigma} [(X''' \cos \alpha + Y''' \sin \alpha) \tan \delta - Z'']. \end{aligned} \quad (8)$$

These expressions give α''' and $(\tan \delta)'''$ in terms of $\sigma, \alpha, \tan \delta$, their velocities and accelerations, and of r and r' . For a second hypothesis, all these quantities are taken from the first hypothesis. It should be observed, however, that α and δ and the solar coördinates remain constant. After α''' and $(\tan \delta)'''$ have been computed by (7) and (8), we may either compute $\alpha', \alpha'', (\tan \delta)', (\tan \delta)''$ by (2) and (3) and then repeat the solution by the formulæ of the first hypothesis, or we may directly compute the corrected value of x by (4) or (4a).

As stated in the introduction, however, an expression for σ may be derived which permits of performing a second hypothesis without the necessity of actually determining the corrections to the initial velocities and accelerations, or of computing α''' and $(\tan \delta)'''$ for substitution in (4) or (4a). This may evidently be accomplished by eliminating α''' from (2) and (7) and $(\tan \delta)'''$ from (3) and (8) by successive substitution so as to express α', α'' and $(\tan \delta)', (\tan \delta)''$ in series of ascending powers of the initial velocities and accelerations α'_0, α''_0 , etc. By introducing these series for the geocentric velocities and accelerations into the formula given for x in II, page 15, we shall obtain an expression for the second hypothesis of x and, therefore, also of σ , etc. This expression will involve the unknown velocity r' in addition to the unknown quantities occurring in the equation for σ in the first hypothesis. The terms distinctive of the second hypothesis will also

contain as factors the first and higher powers of the products and the differences of the intervals.

As soon, therefore, as the velocity r' has been derived in the first hypothesis, the complete expressions of the second hypothesis may be computed. But this procedure would lead to the usual complicated numerical operations, even in minor planet orbits for which r' is generally small. The process would become still more complicated, if extended to higher derivatives than the third of the geocentric coördinates.

As stated above, we have therefore chosen the plan of determining from differential relations such corrections to the fundamental quantities ρ, x', y', z' as will remove whatever residuals may remain after our first hypothesis or direct solution.

With reference to the errors of $\sigma, \sigma', \sigma''$ in the first hypothesis, POINCARÉ has shown, as stated in the introduction, that even with unequal intervals they are of the second order, if the mean of the three dates be chosen for the epoch.

The method of determining the error of an orbit solution by general considerations based on the order of neglected terms is, at best, unsatisfactory, since account is not taken of the special conditions which may exist. It is far more satisfactory to estimate the error *numerically* and thus to arrive at the accuracy of the result in each particular case. It is even possible at the outset to compare the relative effect of the various sources of errors and to plan the solution accordingly by eliminating some of the errors and neglecting others. This method of procedure will preclude an indiscriminate mechanical application of the formulæ for the solution of an orbit. But it makes it possible to obtain the best orbit derivable from the given data. The accuracy of the orbits computed in the Berkeley Astronomical Department is in a large measure due to such an analysis of the conditions of the problem in each individual case.

The accuracy of the preliminary orbit solution depends in the main on the following four conditions:

- (1) The magnitude of the geocentric motion.
- (2) The magnitude of the errors of observation.
- (3) The magnitude of the parallax, if this be neglected.
- (4) The magnitude of the neglected terms containing the third and higher derivatives of α and δ .

As these effects are independent of each other, they may be considered separately; nor is it necessary to study the accuracy of the resulting solution both for solutions with and without assumption regarding the eccentricity. It is sufficient to determine the accuracy of the solution for the general case, that is, without assumption regarding the eccentricity. The feasibility of a special solution, that is, with assumption regarding the eccentricity, then depends on whether the special solution falls within the range of uncertainty of the general solution.

With reference to the *geocentric motion* it is evident at once that the number of figures to which the velocities $\alpha'_1, \alpha'_{111}, \alpha'_0, \delta'_1, \delta'_{111}, \delta'_0, (\tan \delta)'_0$, and the accelerations α''_0 and $(\tan \delta)''_0$ can be determined, depends upon the number of places in the differences of the observed α and δ , expressed in seconds of arc, from which these velocities and accelerations are computed. As the tenths of a second of arc are practically always uncertain, even in the most accurate observations, we may consider these differences to be accurately known to the nearest second of arc, except for other errors of observations the effect of which is to be considered separately. No special calculation therefore, is necessary to determine the number of places accurately known in each velocity and acceleration. This number of places is obtained at once by a mere inspection of the given differences of the observed α and δ .

In a general orbit solution $z = \frac{\rho}{R}$ is to be obtained by interpolation from the tables at the end of this paper with ψ and $\frac{1}{m}$ as arguments. The accuracy to which z can be determined depends, therefore, on the accuracy of $\frac{1}{m}$, but by formulæ III, page 15,

$$\frac{1}{m} = \frac{N}{C_1 \alpha' + C_2 (\tan \delta)'} \frac{\cos \delta R^4}{S}, \quad (9)$$

where

$$N = \alpha'^3 - \alpha'' (\tan \delta)' + \alpha' (\tan \delta)''; \quad C_1 = \tan \delta \cos (A - \alpha) - \tan D; \quad C_2 = \sin (A - \alpha). \quad (10)$$

In this formula the subscripts referring to the middle place have been suppressed. The accuracy of $\frac{1}{m}$ depends, therefore, upon the accuracy to which the ratio

$$\frac{N}{C_1 \alpha' + C_2 (\tan \delta)'} \quad (11)$$

can be computed. The number of known places in $\frac{1}{m}$ may thus be obtained by inspection on the basis of the number of places to which the velocities and accelerations are known. It may, however, occur that the ratio (11) becomes indeterminate. In such cases the intelligent computer will transform his coördinates and velocities from the equator to an arbitrary fundamental plane so chosen as to give the most definite value of $\frac{1}{m}$.

As indicated above, after $\frac{1}{m}$ has been computed and the number of places to which $\frac{1}{m}$ is known has been decided by inspection, the interpolation of z from the tables reveals at once the number of places to which z is known.

The effect of *errors of observation* can not, of course, be determined with absolute precision. Observations of the point images of minor planets, of satellites, or of comets with a well-defined nucleus are always more reliable than observations of diffused comets. For objects observed at low altitudes the observations often are affected more than is necessary by errors arising in the correction for refraction. But, as the nature of the object and the conditions under which the observations were made are always known, it is possible to fix an upper limit for the error of observation. No account can, of course, be taken of gross errors of

observation and reduction. No computer can be held responsible for an inaccuracy in his result due to an uncalled for error on the part of the observer.

Let u , u'' , u''' represent the correct values of the observed α or δ ; e , e'' , e''' the corresponding errors of observation; $\partial_1 u'$ and $\partial_1 u''$ the errors arising in the velocity u' and the acceleration u'' from the errors of observation. Then by substituting $u + e$ for either α or δ ; $u'' + e''$ for either α'' or δ'' ; etc., in the formulæ for the velocities and accelerations given in III, page 15, and omitting for the present the factors $\frac{15 \sin 1''}{k}$ and $\frac{\sin 1''}{k}$, so that the velocities are expressed in seconds of arc per mean solar day, we obtain

$$\partial_1 u' = \frac{\frac{t''' - t''}{t'' - t'}(e''' - e') + \frac{t'' - t'}{t''' - t''}(e''' - e'')}{t''' - t'} = \frac{\left[\frac{t''' - t''}{t'' - t'} - \frac{t'' - t'}{t''' - t''} \right] e'' - \frac{t''' - t''}{t'' - t'} e' + \frac{t'' - t'}{t''' - t''} e'''}{t''' - t'}.$$

To obtain the maximum effect of these errors, let the sign of the term containing e be changed from $-$ to $+$. Let also $e' = e'' = e''' = e$, where e represents the maximum error of observation. Then the error of the velocity u' will be less than

$$\partial_1 u' = 2 \frac{t''' - t''}{t'' - t'} \frac{e}{t''' - t'}. \quad (12)$$

For equal or nearly equal intervals, the error of u' is, therefore, less than e , divided by the constant interval. For unequal intervals we have by division of $\partial_1 u'$ by u' , where, as stated above, u' stands for the velocity in seconds of arc per mean solar day,

$$\frac{\partial_1 u'}{u'} = \frac{t''' - t''}{t'' - t'} \frac{e}{(t''' - t'') u''' + (t'' - t') u'} = \frac{t''' - t''}{t'' - t'} \frac{e}{d'} = a'_1, \quad (13)$$

where the divisor $d' = (t''' - t'') u''' + (t'' - t') u'$ has been previously computed in the derivation of α' and δ' .

In a similar manner we obtain for the upper limit of the error of the acceleration u'' , expressed in seconds of arc per mean solar day per mean solar day,

$$\partial_1 u'' = 2 \frac{\frac{e''' - e''}{t''' - t''} - \frac{e'' - e'}{t'' - t'}}{t''' - t'} = 2 \frac{e''' - e'' \left[\frac{1}{t''' - t''} + \frac{1}{t'' - t'} \right] + \frac{e'}{t'' - t'}}{t''' - t'}.$$

By changing the sign of e'' and equating the errors to e , as before, we find that the error in the acceleration u'' can not exceed

$$\partial_1 u'' = 4e \frac{\frac{1}{t''' - t''} + \frac{1}{t'' - t'}}{t''' - t'}.$$

For equal intervals $\partial_1 u''$ is less than $4e$ divided by the square of the constant interval. For unequal intervals

$$\frac{\partial_1 u''}{u''} = \frac{2e \left[\frac{1}{t''' - t''} + \frac{1}{t'' - t'} \right]}{u''' - u''} = a''_1, \quad (14)$$

where all quantities involved are known from previous computation.

As soon as the maximum value of the errors of observation has been fixed it is possible, therefore, to compute the maximum errors of the velocities and accelerations by formulæ (13) and (14).

The parallax may be dealt with in three different ways:

(a) It may be *neglected entirely*. In that case its effect on the velocities and accelerations is the same as that of the errors of observation. The parallax errors may, therefore, be included in the estimate of the maximum error ϵ . It is, of course, not possible to know anything definite about the parallax until after the geocentric distances have become known. By choosing, however, the error of observation sufficiently large to include the probable parallax as judged from the observational data, the effect of the parallax may be determined, at least in a preliminary manner. The estimate of the parallax can be verified and, if necessary, corrected as soon as the geocentric distances have become known.

(b) Or the parallax may be *eliminated partially*, by applying to the solar coördinates at the middle date the corrections $\Delta_2 X$, $\Delta_2 Y$ and $\Delta_2 Z$, given by equations (4), page 234. In the partial elimination of the parallax the coördinates are referred to the projection of the center of the Earth upon the line of sight as origin. It should be remembered that the partial elimination of the parallax does not eliminate the barycentric parallax from the coördinates. Nor does it eliminate the effect of the parallax on the velocities and accelerations. The error in these latter must therefore be derived as though the parallax had been entirely neglected.

(c) Or the parallax may be *eliminated completely* by means of the formulæ derived on pages 236 and following.

The decision as to which of the three methods of dealing with the parallax shall be applied, depends upon the observational data. If the estimated error of observation and the error introduced by neglecting the third and higher derivatives of the coördinates, which will be discussed below, are comparatively large, then little or nothing can be gained by completely eliminating the parallax. For the convenience, however, of subsequent computation it is advisable to apply the partial elimination of the parallax in all cases. This convenience arises from the fact that the partial elimination of the parallax leads to the *geocentric* distance at the middle date and that the parallax need not be considered further in computing the heliocentric coördinates.

Whether, in case of an accurately observed object and of small errors due to the neglect of the higher derivatives, the parallax is to be eliminated partially or completely, will depend on whether the complete elimination will greatly enhance the accuracy of the solution. In general, the observations may be represented with the same degree of accuracy with or without elimination of the parallax, but by completely eliminating the parallax a more accurate orbit will result so that a better representation of the *subsequent* motion is attained.

But since all preliminary orbits must necessarily be approximate on account of the shortness of the observed arc and since practical considerations demand that all extra computation, however small, be avoided, the complete elimination of the parallax should be undertaken only in those rare cases in which a solution otherwise can not be accomplished. Such cases may occur for an accurately observed object, if the motion is very small, if the intervals are quite short, and if the parallax factors (barycentric plus geocentric) are very different for the three observations. In all other cases the partial elimination of the parallax will suffice for practical purposes. If, for instance, the parallax factors for the three observations are nearly constant, as would be the case for observations made at moderate intervals at the same observatory at nearly the same hour angle, then the neglect of the parallax considered as an error of observation would not affect the velocities and accelerations, since these depend on the differences of the observed α and δ .

It would, of course, be entirely useless to eliminate the parallax completely if the parallax factors indicate that it is much smaller than the possible errors of observation and the errors due to the neglect of the higher derivatives of α and δ .

(d) In determining the effect of *neglecting the higher derivatives* of α and δ we shall confine ourselves to the consideration of the third derivatives. According to the theory of numerical differentiation, if the intervals are constant, a third derivative may be expressed in a series, the first term of which is the mean of the two third differences on either side of the date. Let this mean be denoted by f_u^m , where u stands for either α or δ .

By equation (2), omitting the algebraical sign, the error of the velocities u' and accelerations u'' due to neglecting the third derivatives, is therefore

$$\partial_2 u' = \frac{\theta_i \theta_{iii}}{6} \frac{f_u^m}{\theta^2}; \quad \partial_2 u'' = \frac{\theta_{iii} - \theta_i}{3} \frac{f_u^m}{\theta^2}, \quad (15)$$

while

$$u' = \frac{f_u^1}{\theta}; \quad u'' = \frac{f_u^m}{\theta^2}, \quad (16)$$

where f_u^1 and f_u^m are the means of the first and third differences on either side of the middle date corresponding to the constant interval θ . Dividing (15) by (16) we obtain

$$\frac{\partial_2 u'}{u'} = \frac{\theta_i \theta_{iii}}{\theta^2} \frac{\frac{f_u^m}{6}}{\frac{f_u^1}{\theta}} = a'_2; \quad \frac{\partial_2 u''}{u''} = \frac{\theta_{iii} - \theta_i}{\theta} \frac{\frac{f_u^m}{3}}{\frac{f_u^m}{\theta^2}} = a''_2. \quad (17)$$

For equal intervals $\theta_i = \theta_{iii} = \theta$ and

$$\partial_2 u' = \frac{\frac{f_u^m}{6}}{f_u^1} u'; \quad u'' = 0, \quad (18)$$

except for the effect of fourth and higher differences.

If four observations at equal intervals be available, the f_u^m are given at once by the observations and the velocities derived from three observations may be

mined from the observations on the basis that the observations are correct to the nearest second of arc. Equations (20) and (21) now further determine how many of the places given by the geocentric motion are lost on account of the various errors involved, so that the number of places accurately given in $\frac{1}{m}$ has finally become known in any particular case and, as stated above, the accuracy of $\frac{1}{m}$ determines the accuracy to which z can be interpolated from the tables. In this manner a fairly close estimate of the accuracy of the resulting orbit computation may be obtained.

An orbit becomes indeterminate when $\frac{1}{m}$ becomes indeterminate. As stated above, in some cases inspection will show that the indeterminateness may be removed by referring the coördinates to a new fundamental plane. But if the indeterminateness is due entirely to errors of observation then no solution can be made. In this case, which is not probable in practice, the errors of observation would be comparable to the motion.

Finally the indeterminateness may arise from the neglect of the third and higher derivatives. Too much weight has been attached in the past to this possibility. In fact it has been used as the chief argument against the convergence of the LAPLACEAN methods. Experience, however, as well as the foregoing discussion, are sufficient proof that indeterminateness will rarely arise from this cause. In fact, with short intervals, such as are always available, and particularly for slow motion, the first direct solution has, in most cases, yielded a satisfactory representation of the observations.

If an indeterminateness should arise on account of the neglect of the third and higher differences, this fact can be inferred from the differences of the observed α and δ , at the outset, before the solution is undertaken. The remedy then lies in the use of at least the third differences by basing the computation on more than three observations by the formula derived on page 230 and following, or by having recourse to shorter intervals, if such be available. For equation (19) shows that for one-half the interval the third differences will be reduced to approximately one-eighth of their value, etc., so that, theoretically at least, it is always possible to choose intervals so small that the third and higher differences will not affect the solution at all. But it should be remembered that the geocentric arc will be less and the number of decimals to which the orbit can be computed will be reduced in the ratio in which the number of seconds in the geocentric arc is reduced by the shorter intervals.

It is safe to say that, by carefully judging the conditions of the problem at the outset in every case, the computer will never run the risk of performing a fruitless computation if he apply the LAPLACEAN methods in the form in which they are here presented. This has been the experience in all orbit computations at the Students' Observatory of the University of California in recent years.

The order in which the effect of the various errors referred to above should be considered must be decided by the computer in each individual case. Very

this most unfavorable case, we shall assume, as above, that the errors e are numerically equal, but that their algebraical signs are such that every error counts fully in one or the other error sum. The error sums will then reduce to $\frac{3}{2}e$ and $\frac{16}{3}e$. It is clear, of course, from equations (24) that in practice they can not reach their maximum for both the velocities and the accelerations at the same time. In order to judge the relative error made by neglecting the third and higher differences in the case of three observations, we might let $e_1 = e_3 = 0$, but if five observations are made the basis of computation, then we shall also have to deal with five errors of observation. On that account it is more accurate to base the comparison on the error sums as given above. For equal intervals, therefore, we must compare

$$\text{in } \theta u' : \frac{1}{6} f'''(a + iw) \text{ with } \frac{3}{2}e; \text{ in } \theta^2 u'' : \frac{1}{12} f^{iv}(a + iw) \text{ with } \frac{32}{6}e.$$

It follows, therefore, that little can be gained by the use of more than three observations if

$$f'''(a + iw) \leq 9e; \quad f^{iv}(a + iw) \leq 64e.$$

As was found above, the accuracy of an acceleration is diminished in a greater degree by the errors of observation than that of a velocity. A detailed discussion of the velocities may therefore be dispensed with. In order to gain anything then, by basing the computation on more than three observations, f^{iv}_u would have to be considerably greater than $64e$; that is, for every second of error of observation, the fourth difference would have to exceed $1'$. In the case of comets, the errors of observations are quite frequently greater than $5''$. To gain anything then, the fourth differences in α and δ would have to be in excess of $5'$.

From these considerations it appears that the criticisms that have been applied to the LAPLACEAN methods, and which were refuted above, are still more unjustifiable when the errors of observations are taken into consideration. In general, one element of uncertainty in any method of orbit solution depends upon the percentage error in the motion. This element is necessarily about the same for all methods. It is brought out here very distinctly, that in attacking the LAPLACEAN methods altogether too much emphasis has been laid in the past on the effect of the third and higher differences. The conclusion drawn above, is, therefore, fully justified.

In the case of unequal intervals, the error in the accelerations was found to be approximately $\frac{f'''}{3}$. Comparing this error with the error sum $\frac{16}{3}e$, which method of comparison, although not as rigid as the foregoing comparisons, is sufficient for our purposes, it is found that in order to gain anything f''' would have to be in excess of $16e$, so that in the case of unequal intervals the inclusion of a fourth observation would be more justifiable. It is, however, more advantageous, even in the case of unequal intervals, to base the computation, whenever possible, merely on three observations and to remove, if necessary, any inaccuracies in the resulting elements by the simple methods of differential correction to be discussed later.

which is the desired equation of the sixth degree, since $\sigma_0 = \rho_0 \cos \delta_0$. Expanding the squares within the second parenthesis and introducing the auxiliary quantities

$$a^2 = (a_x^2 + a_y^2 + a_z^2) \cos^2 \delta_0; \quad b = (a_x X'_0 + a_y Y'_0 + a_z Z'_0) \cos \delta_0; \quad G^2 = X_0'^2 + Y_0'^2 + Z_0'^2, \quad (11)$$

where, provided the parallax be entirely neglected, the velocity G , may also be computed from the well-known relation

$$G^2 = \frac{2}{R} - 1, \quad (11a)$$

we may write equation (10) in the form

$$(\rho_0^2 - 2 \rho_0 R \cos \psi + R^2) (a^2 \rho_0^2 - 2 b \rho_0 + G^2) = 4 \quad (12)$$

or

$$F(\rho_0) = \Gamma \gamma^2 - 4 = 0 \quad (13)$$

where

$$\Gamma = \rho_0^2 - 2 \rho_0 R \cos \psi + R^2; \quad \gamma = a^2 \rho_0^2 - b \rho_0 + G^2. \quad (14)$$

A first approximation to the value of ρ_0 in equation (13) may be obtained by taking $z = \frac{\rho_0}{R}$ from the tables at the end of this paper with the arguments ψ and $\frac{1}{m}$. The formulæ for ψ and $\frac{1}{m}$ are given on page 15, formulæ III. Let the approximate value of ρ_0 obtained with the aid of the tables be denoted by ρ_1 and let

$$(\rho_1^2 - 2 \rho_1 R \cos \psi + R^2) (a^2 \rho_1^2 - 2 b \rho_1 + G^2) - 4 = M_1, \quad (15)$$

or, by equations (13) and (14),

$$\Gamma_1 \gamma_1^2 - 4 = M_1. \quad (16)$$

Then it will be necessary to determine a correction $\Delta \rho_1$ to ρ_1 in such a manner that $\rho_2 = \rho_1 + \Delta \rho_1$ will satisfy (12), or (13) and (14). We may write approximately

$$\Delta F(\rho_1) = \left[\frac{dF(\rho_0)}{d\rho_0} \right]_{\rho_0=\rho_1} \Delta \rho_1 = -M_1. \quad (17)$$

But from (13) and (14) we have

$$\left[\frac{dF(\rho_0)}{d\rho_0} \right]_{\rho_0=\rho_1} = 2 \gamma_1 [2 \Gamma_1 (a \rho_1 - b) + \gamma_1 (\rho_1 - R \cos \psi)]. \quad (18)$$

Hence from (17) and (18)

$$\Delta \rho_1 = - \frac{M_1}{2 \gamma_1 [2 \Gamma_1 (a \rho_1 - b) + \gamma_1 (\rho_1 - R \cos \psi)]} \quad (19)$$

and

$$\rho_2 = \rho_1 + \Delta \rho_1. \quad (20)$$

If ρ_2 does not satisfy (12), or (13) and (14), then we let

$$F(\rho_2) = M_2$$

and repeat the approximations until a value of ρ is found which satisfies (12), or (13) and (14). In general, not more than two approximations will be required.

If we derive the equation in ρ_0 in a slightly different form, approximate values of the roots may be found graphically with more convenience than with the aid of the tables at the end of this paper. The modified form also seems to be preferable for numerical computation.

Equation (3) gives

$$r_0 = + [\rho_0^2 - 2 \rho_0 R \cos \psi + R^2]^{\frac{1}{2}}, \quad (21)$$

Let $f(z_2) = M_1$ and continue the approximations until $f(z) = 0$.

To obtain a first approximation to z from the tables, we enter them as before in case of equations (12) or (13) and (14) with the arguments ψ and $\frac{1}{m}$.

To obtain the roots of equation (24) graphically, we determine the values of z for the intersections of the parabola

$$y = (z - p')^2 \quad (28)$$

and the curve

$$y = \frac{h}{[(z - c)^2 + s^2]^{1/2}} - q'^2, \quad (29)$$

where we shall take the z as abscissæ and the y as ordinates.

The parabola is the same in all cases. Its axis coincides with the line $z = p'$, and $y = 0$ is a tangent at the vertex. We may, therefore, once for all, plot the parabola $y = z^2$ on a convenient scale.

The curve (29) is symmetrical with respect to $z = c$ and lies wholly above the line $y = -q'^2$, because the negative sign of the square root is excluded, r always being positive. For $z = \pm \infty$, $y = -q'^2$; hence the line $y = -q'^2$ is an asymptote to the curve. At $z = c$, $y = \frac{h}{s} - q'^2$ the curve has its maximum. By letting $\frac{d^2 y}{dz^2} = 0$ we readily find for the points of inflection $z - c = \frac{\pm s}{\sqrt{2}}$, $y = \sqrt{\frac{2}{3}} \frac{h}{s} - q'^2$. The distance of the points of inflection above the line $y = -q'^2$ is therefore $\sqrt{\frac{2}{3}}$ or .816... times the distance of the vertex of the curve above the same line.

Since the parabola lies wholly above the z axis, we are concerned only with that part of the asymptotic curve which lies above $y = 0$. For $y = 0$ we find from (29)

$$\frac{h}{[(z - c)^2 + s^2]^{1/2}} = q'^2; \quad (z - c)^2 = \frac{h^2}{q'^4} - s^2,$$

or

$$z - c = \pm \sqrt{\frac{h^2}{q'^4} - s^2}. \quad (30)$$

An approximate value of $z - c$ may be obtained from (30) by formulæ (23), if in the latter we let $G^2 = \frac{2}{R} - 1$ and $R = 1$. Then $G^2 = 1$ and $q'^2 = \frac{1}{a^2} - \frac{b^2}{a^4} = \frac{1}{a^2} \left(1 - \frac{b^2}{a^2}\right)$, while $h = \frac{2}{a^2}$.

Since q'^2 is positive, we have $b^2 < a^2$.

We may now write (30) in the form

$$z - c = \pm \sqrt{\left(\frac{4}{1 - \frac{b^2}{a^2}}\right)^2 - s^2} = \pm \frac{1}{1 - \frac{b^2}{a^2}} \sqrt{4 - s^2 \left(1 - \frac{b^2}{a^2}\right)^2}, \quad (31)$$

or

$$z - c = \pm \frac{1}{a^2 q'^2} \sqrt{4 - s^2 a^4 q'^4}. \quad (32)$$

Denote this value of $z - c$ by $(z - c)_m$. Then

$$\text{Mod. } (z - c)_m = 2_m < \frac{2}{a^2 q'^2}.$$

duce the two halves of the curve lying on either side of $z=0$. But, since only that portion of the curve which lies above $y=0$ is under consideration, we may plot to every $\pm z$ the corresponding value $y = \eta - q'^2$. Owing to the adopted limiting value S_∞ of S , negative values of y will not occur.

In practice it is convenient to plot the parabola on tracing-paper once for all and the asymptotic curve on a separate sheet for each individual case. It is, however, to be remembered that only the positive portion of this curve needs to be drawn. If, then, in the plot of the curve we also draw the line $z=p'$, the intersections of the parabola and the asymptotic curve may be found by superposing the parabola on the curve in such a manner that the axes $y=0$ of the two plots coincide and that the axis of the parabola coincides with the line $z=p'$. The abscissæ of the points of intersection then represent the required values of z .

As the asymptotic curve may become quite steep for large values of $\frac{h}{s}$, it is advisable to plot the abscissæ on a larger scale than the ordinates. The scale two to one has been adopted in the applications that have so far been made of this graphical method of determining the roots of the equation of the sixth degree.

The number of *mathematical* solutions of the problem will be the same as the number of intersections for which z is positive. It can be shown that the number of positive roots of the equation of the sixth degree will be either one or three.

For practical purposes it is sufficient to study the behavior of the roots by considering the intersections of the parabola and the asymptotic curve, which latter, for the sake of brevity, we shall simply call the "curve."

The number of possible intersections of the parabola and the curve depends on the number of possible contacts. If the two curves can have but one point of contact, then there can be but two intersections. The case of no contact, corresponding to a single real root, is excluded, as the equation in z is of the sixth degree and therefore the real roots must occur in pairs. For the same reason any even number of contacts is excluded. If three contacts are possible, then there may be four intersections and therefore four real roots. To determine the number of contacts we write equations (28) and (29) in the form

$$y = z'^2 ; \quad y = \frac{h}{\sqrt{(z' - c')^2 + s^2}} = q'^2, \quad (39)$$

where

$$z' = z - p' ; \quad c' = c - p'. \quad (40)$$

The axis of the parabola and the curve then coincides with $z'=0$ and $z'=c'$ respectively, c' being the distance between these axes. The axis of the parabola lies to left of the axis of the curve if c' is positive, and to the right if c' is negative.

Differentiating equations (39) with respect to z' we have

$$\frac{dy}{dz'} = 2z' ; \quad \frac{dy}{dz'} = - \frac{h(z' - c')}{[(z' - c')^2 + s^2]^{3/2}}. \quad (41)$$

Since the absolute term of this equation is negative, there always exists one positive and one negative root. The other two real roots must either be both positive or both negative. The equation may, therefore, have either one or three positive roots. For instance, if both p' and c are negative, then every coefficient in equation (47) becomes positive, excepting the absolute term. There being in that case only a single variation of sign, the number of positive roots or of solutions is only *one*. But if both p' and c are positive, then the coefficients are alternately positive and negative, excepting the coefficient of z and the absolute term. This condition of signs admits of at most five positive roots and one negative root. But as at least two roots must be imaginary and one real root must be negative, the number of positive roots can not exceed three. The same results would have been obtained if (46) had been expanded as it stands, that is, without introducing the simplification $R = 1$, but equation (47) would then have become very complicated.

Three positive roots, or what is the same, four real roots, can exist only if the curves have three points of contact, that is, if equation (43) or (44) has three real roots. Whether the number of real roots be one or three, the substitution of each real root in (41a) gives the condition which the constants must satisfy in order that the curves may be in contact for the particular point defined by such root z' . We might thus determine the limits within which the roots must lie. But in practice the number and the approximate value of the positive roots is more conveniently determined graphically from the intersections of the parabola and the curve. Equation (44) has one or three real roots according to whether

$$\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 > \text{or} < 0. \quad (48)$$

Hence for positive values of p more than one solution can never exist. But if p is negative, the criterion (48) must be applied to decide the number of contacts.

Let us now consider the case that

$$\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 < 0,$$

so that we shall have three points of contact. The number of real roots will be four. Of these one or three may be positive. In order that there may be three positive roots z or three solutions, all three points of contact must lie on the positive side of $z = 0$. We may determine the conditions which must be satisfied so that there may be three positive roots with the aid of equation (43). In this equation $z' = z - p$, so that the roots z' are measured from the axis of the parabola. Now let $c' = c - p$ be negative. Then every coefficient in (43) will be positive and, therefore, all three roots will be negative. If, however, c' is positive, then all three roots will be positive. In the first case the axis of the parabola lies to the right of the axis of the curve and the points of contact are

contained between $z = c$ and $z = p'$. In the second case the axis of the parabola lies to the left of $z = c$ and the points of contact lie between $z = p'$ and $z = c$. Both cases apply to positive and negative values of c and p' . Let us designate for both parabola and curve that part which lies to the right and the left of their axes by the positive and the negative half, respectively. Then in the first case, when c' is negative, the points of contact lie on the positive half of the curve and on the negative half of the parabola, while in the second case they lie on the negative half of the curve and on the positive half of the parabola. The following cases may occur:

I $c' = c - p'$ is negative

(a) p' negative, c negative

(b) [p' negative, c positive]

(c) p' positive, c negative

(d) p' positive, c positive ($p > c$)

II $c' = c - p'$ is positive

(a) p' negative, c negative ($p' > c$)

(b) p' negative, c positive

(c) [p' positive, c negative]

(d) p' positive, c positive.

The cases (Ib) and (IIc) must necessarily be excluded in this grouping, as the first is provided for under IIb and the second under Ic. Cases Ia and IIa need not be considered here, it having been shown above (page 286) that only one positive root exists, if both p' and c are negative. It remains to consider the cases Ic, Ib, IIb, and IIId. For Ic and Id the points of contact lie between $z = c$ and $z = p'$. The axis $z = 0$ lies to right of $z = c$ for Ic and to the left for Id. In the latter case, therefore, the three points of contact lie on the positive side of $z = 0$ and there are three positive roots. In the former case, Ic, the number of positive roots will be one or three according to whether the point of inflection on the positive half of the curve lies on the negative or the positive side of the axis $z = 0$. For this point of inflection $z = c + \frac{s}{1-2}$. Hence, there will be one or three positive roots according to whether $c + \frac{s}{1-2} < 0$ or > 0 . Let $\tan \psi' = \sqrt{2}$. Then, since $c = \cos \psi$ is negative, there will be one solution if $180 - \psi' < \psi < 180^\circ$, and three solutions if $90^\circ < \psi < 180^\circ - \psi'$.

Similarly, we observe that for cases IIb and IIId the points of contact lie between $z = p'$ and $z = c$, while the axis $z = 0$ lies to the right of $z = p'$ for IIb and to the left for IIId. In the latter case, therefore, the three points of contact lie to the right of $z = 0$ and there are three positive roots, while in the former case, IIb, there will be one or three positive roots according to whether the axis $z = 0$ lies to the right or the left of the point of inflection of the negative branch of the curve. For this point of inflection $z = c - \frac{s}{1-2}$. The number of positive roots will, therefore, be one or three according to whether $c - \frac{s}{1-2} < 0$ or > 0 . Hence, since $c = \cos \psi$ is positive, there will be one positive root, if $\psi' < \psi < 90^\circ$ and three positive roots if $0 < \psi < \psi'$, where $\tan \psi' = \sqrt{2}$ and $\psi' = 54^\circ 44'$.

Collecting results, we find that three positive roots exist if *first*:

$$\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 < 0,$$

where by (45a)

$$\frac{p}{3} = \frac{1}{81} [9(2s^2 + q'^2) - 7c'^2]; \quad \frac{q}{2} = \frac{5c'}{9} \left[\frac{p}{3} + \frac{1}{9} \left(m^2 - \frac{11}{5} q'^2 \right) \right]$$

and if *secondly*:

$$\begin{aligned} m^2 &= c'^2 + s^2; & c' &= c - p' \\ \text{either } p' &> 0, & c &> 0; \\ \text{or } p' &> 0, & c &< 0, & 90^\circ < \psi < 125^\circ 16'; \\ \text{or } p' &< 0, & c &> 0, & 0^\circ < \psi < 54^\circ 44'. \end{aligned} \quad (49)$$

In all other cases the equation of the sixth degree in ρ_0 or z ((12) or (46)) will have only one positive root.

In applying these criteria it suffices to use approximate values of the quantities involved. If either p' or c are negative, the *first* criterion needs to be applied only if ψ lies between the limits specified for the existence of three solutions. It should also be remembered that only one solution exists, if p is positive, that is, if $9(2s^2 + q'^2) - 7c'^2$ is positive. The sign of p is readily ascertained from a mere inspection of the numerical values of s , q' , and c' , all of which are available. The value of q needs to be computed only if p is negative and then only with sufficient accuracy to determine the algebraical sign of $\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2$. Thus, the number of positive roots is determined practically without additional computation. In most cases a mere inspection of the available auxiliary quantities will suffice.

Having thus disposed of the number of positive roots, we may proceed to a further consideration and a recapitulation of the graphical determination of the numerical values of z' , from which $z = \frac{\rho}{R} = z' + p'$. By (39) the roots z' are defined as the intersections of

$$y = z'^2 \quad \text{and} \quad y = \frac{h}{[(z' - c')^2 + s^2]^{1/2}} - q'^2, \quad (50)$$

where by (36) and (37), (35) and (40) we plot the asymptotic curve for equidistant values of \mathfrak{D} from

$$y = \eta - q'^2 = \frac{h}{s} \cos \mathfrak{D} - q'^2; \quad z' = \mathfrak{E} + c'. \quad (51)$$

Thus the asymptotic curve may be plotted directly with reference to the axes of the parabola, which are $y = 0$, $z' = 0$.

Formulae (38) gives the maximum positive value \mathfrak{D}_m of \mathfrak{D} required in (51), and on page 284 it was pointed out that only that part of the curve needs to be plotted which lies between $z = 0$ and $z = c + \mathfrak{E}_m$, where $\mathfrak{E}_m = s \tan \mathfrak{D}_m$. Expressed in terms of z' we are, therefore, concerned only with that part of the curve which lies between $z' = -p'$ and $z' = s \tan \mathfrak{D}_m + c'$. Now, as it has been decided that \mathfrak{D} should always be chosen positive, we must plot the portions of the curve to the right and the

left of $z' = 0$ from $z' = \mathcal{Z} + c'$ and $z' = -\mathcal{Z} + c'$, respectively, together with the value of η corresponding to \mathcal{Z} . Hence, the largest value of \mathcal{Z} which must be used with the negative sign is obtained from $z' = -\mathcal{Z} + c' = -p'$ or from $\mathcal{Z} = c' + p' = c$. This result is also at once evident from the fact that the curve is symmetrical with reference to $\mathcal{Z} = 0$ and that $\mathcal{Z} = -c$ for $z = 0$, c being the distance from $z = 0$ to $\mathcal{Z} = 0$.

We may also easily derive the numerical value of \mathcal{S} which corresponds to the identical conditions $z = 0$ or $z' = -p$ or $\mathcal{Z} = -c$. Expressing \mathcal{Z} in terms of \mathcal{S} we have $s \tan \mathcal{S} = -c$ or $\tan \mathcal{S} = -\frac{c}{s} = -\cot \psi = \tan(\psi - 90^\circ)$. Hence, if $\psi > 90^\circ$, the values of \mathcal{S} required in (51) are contained between $\mathcal{S} = \psi - 90^\circ$ and $\mathcal{S} = \mathcal{S}_m$ and η and \mathcal{Z} need not be computed for values of \mathcal{S} outside of this range; that is, all roots correspond to values of \mathcal{S} within the range. Similarly, if $\psi < 90^\circ$, we again have the limits $\mathcal{S} = \psi - 90^\circ$ and $\mathcal{S} = \mathcal{S}_m$. The negative value of the lower limit in this case signifies that negative values of \mathcal{Z} may be determined from $\mathcal{Z} = s \tan(-\mathcal{S})$. In every case, then, the values of \mathcal{S} required in (51) are contained between $\mathcal{S} = \psi - 90^\circ$ and $\mathcal{S} = \mathcal{S}_m$.

In determining the intersections of the parabola and the curve it is necessary to plot only a short arc of each on either side of the intersections. The scale may be chosen sufficiently large to suit any desired degree of accuracy.

For the intersections the coördinates of the curve satisfy the condition $(\mathcal{Z} + c')^2 = \eta - q'^2$. Let $y' = (\mathcal{Z} + c')^2 - (\eta - q'^2)$. Then at the intersections $y' = 0$. Remembering the definition of η and \mathcal{Z} in terms of \mathcal{S} by (36) and (37), we may find the region of an intersection with the aid of a table of natural tangents and cosines and of a multiplication table like CRELLÉ'S *Rechentafeln* or mechanism like a slide rule, by noting down approximate values of $z' = \mathcal{Z} + c'$ and $y = \eta + q'^2$ for a few values of \mathcal{S} between the limits $\mathcal{S} = \psi - 90^\circ$ and $\mathcal{S} = \mathcal{S}_m$ and estimating the value of \mathcal{S} for which y' changes sign. Let this value of \mathcal{S} be called \mathcal{S}_0 . Then compute η and \mathcal{Z} , etc., for a few values of \mathcal{S} , greater and less than \mathcal{S}_0 , to the degree of accuracy to which it is desired to find z' at the intersection from the plot. Having determined z' at the intersection from the plot, we finally have $z = z' + p'$.

But the graphical solution may be simplified still further by plotting

$$y' = (\mathcal{Z} + c')^2 - (\eta - q'^2) = f(\mathcal{S}) \quad (52)$$

for a few values of \mathcal{S} greater and less than \mathcal{S}_0 with \mathcal{S} as abscissa and y' as ordinate. The solution is then given by the intersection of the curve $y' = f(\mathcal{S})$ with the \mathcal{S} axis. In case of three positive roots y' will change sign for three values of \mathcal{S}_0 between $\psi - 90^\circ$ and \mathcal{S}_m , and each of the three roots is obtained in the same manner as in the case of a single root. The intersections of the curve $y' = f(\mathcal{S})$ with the \mathcal{S} axis give the accurate values of \mathcal{S}_0 corresponding to the positive roots. For each intersection \mathcal{S}_0 the value of $z = \frac{p}{R}$ is then given by (35), or

$$z = \mathcal{Z}_0 + c = s \tan \mathcal{S}_0 + c. \quad (53)$$

Instead of plotting the values $y' = f(\mathcal{S})$, the values of y' may be arranged in tabular form with \mathcal{S} as argument. From this table the value of \mathcal{S}_0 corresponding to $y' = 0$ may then be interpolated. Graphical constructions are thereby entirely avoided.

The investigation of the number of positive roots and the various methods¹ of deriving their numerical values having been disposed of, it remains to consider, (a) whether the parabolic hypothesis is feasible, and (b), if so, whether in case of three positive roots any of these roots may be discarded as fictitious. It is evident that a parabolic solution is feasible only if it is also a general solution. In the general solution the positive roots of the fundamental equation (7), page 8, are obtained by interpolation from the tables at the end of this paper with ψ and $\frac{1}{m}$ as arguments. In a parabolic solution $\frac{1}{m}$ is required only if the first approximation of the geocentric distance is obtained from the tables, but even if the geocentric distance is determined graphically, it is necessary to perform the simple computation of $\frac{1}{m}$ and to interpolate the geocentric distance from the tables for the general solution in order to test the feasibility of the parabolic solution. The maximum number of positive roots in the general solution² is two. If none of the positive roots for the parabolic solution agree, within the uncertainty of the solution, with a positive root for the general solution as taken from the tables, then the parabolic hypothesis must be abandoned.

(a) On pages 270-278 it was shown how the uncertainty of $\frac{1}{m}$ and, therefore, by means of the tables also of z , may be expressed numerically in any given case. If we denote by Δz the uncertainty of z , then the parabolic hypothesis is infeasible, if the value of z for the parabolic solution differs from the value of z for the general solution by more than Δz , or in case of multiple roots, if none of the values of z for the parabolic solutions agree with the values of z for the general solutions within the uncertainty Δz for each general solution. The parabolic hypothesis, therefore, may be abandoned in the course of the computation, and it is not necessary to proceed to the representation of the first and third places. But the parabolic hypothesis should not be abandoned at this stage of the computation, if the uncertainty Δz is caused chiefly by the neglected third differences of the coördinates as estimated from the first and second differences. The rejection of the parabolic hypothesis, therefore, should be based only on that part of the uncer-

¹Since writing this paper Mr. B. A. BERNSTEIN of the University of California and I have derived another very simple graphical construction of the roots, which dispenses with the numerical calculation of the coördinates. Mr. BERNSTEIN has applied the same principle to the determination of the roots in deriving an orbit without hypothesis regarding the eccentricity, but for practical purposes the roots in the general case are more readily obtained by interpolation from the tables at the end of this paper. A. O. LEUSCHNER and B. A. BERNSTEIN, *Note on the Graphical Solutions of the Fundamental Equations in the Short Methods of Determining Orbits*, *Bulletin of the American Mathematical Society*, Vol. XVIII, Number 6.

²For one of the most complete discussions of the number of solutions in the general case compare Mrs. YOUNG (GRACE CHISHOLM), *Monthly Notices of the Royal Astronomical Society*, Vol. LVII, page 379.

tainty Δz which is due to the errors of observation and to the limitation of the number of available places expressed in seconds of arc in the first and second differences of the observed coördinates.

Whether the parabolic hypothesis be abandoned or not, any residuals arising from the neglected third and higher differences may be removed by differential correction, but the accuracy of the resulting orbit still will be determined by the accuracy of z as conditioned by the errors of observation and by the number of places available in the first and second differences of the coördinates. The value of Δz referred to here is the maximum uncertainty of z . Hence $z_1 = z + \Delta z$ and $z_2 = z - \Delta z$ may be considered the maximum and minimum values of z consistent with the observations. To each of these limits corresponds a limiting value of the elements. The difference between these elements represents *the range of numerical solutions*. It is frequently of interest to compute this range for at least the period and eccentricity and thus to obtain a fairly accurate estimate of the accuracy of the adopted orbit.

(b) In case of multiple solutions it is easily shown that only one of the three roots for the parabolic solution can agree with a root for the general solution, so that the other two parabolic roots may be discarded as fictitious. This is of importance because it will obviate the possibility of adopting one of the fictitious roots as the physical solution. As stated above, it is evident that a parabolic solution in order to be feasible must agree with a general solution. Since the general case admits of at most two solutions, the case of three parabolic solutions is at once excluded. The case of two parabolic solutions does not exist, as it was shown above that the number of positive roots for the parabolic solution is *either three or one*. Hence at most one of the parabolic roots can agree with a general root.

But if the tables indicate the existence of two general solutions, whether one of them be parabolic or not, then it is not possible to discriminate between the roots in the course of the computation, for both roots must correspond to values of r which are either both smaller or both greater than R , according to whether in the first of the equations (1) κ is positive or negative, since in the nature of the solution not only the fundamental equation (7), page 8, but also the equations (1) and (3), page 279, from which it is derived, are satisfied.

It may happen that a value of z cannot be found in the tables to correspond to the given arguments $\frac{1}{m}$ and ψ . The value of $\frac{1}{m}$ is then affected by a serious error, which may be due either to an unusual error of observation or computation or to the error produced by neglecting the third and higher differences. Since, however, the magnitude of the third differences can be estimated with a fair degree of accuracy (page 274), the case where a serious error in $\frac{1}{m}$ would arise from the neglect of the third differences can be avoided. It is, therefore, in general always

possible to determine the cause of the inaccuracy of $\frac{1}{m}$ and to discard the observations if necessary. If care be taken in the application of these methods, the neglected third differences will never lead to a value of $\frac{1}{m}$ that can not be found in the tables together with the corresponding argument ψ . Any discrepancy that may exist will be slight, and then the value of z may be taken from the tables for the tabular value of $\frac{1}{m}$ corresponding most nearly to the computed value of $\frac{1}{m}$.

If the parallax has been neglected entirely, then it contributes to the error of observation. In the rare cases in which the possibility of the solution of the orbit depends upon the elimination of the parallax, the values of m and (m) in equation (27), page 240, may differ so considerably that they may not be considered equal for the purpose of interpolating a first approximation of z from the tables. But in such cases the tables no longer apply, because if m and (m) differ considerably, the equation referred to no longer has one root equal to zero and, therefore, does not reduce to the equation of the seventh degree which the tables are intended to solve.¹

After an initial value of z or ρ , where $z = \frac{\rho}{R}$, has been found in accordance with the foregoing directions, it remains to determine their final value z_f or ρ_f . Let the initial value of z or ρ be denoted by z_1 or ρ_1 ; then the final value ρ_f or z_f is obtained by successive application of either formulæ (15), (16), (19), and (20) or of formulæ (24), (25), (27a) and (27b).

The elimination of the parallax and aberration is accomplished in accordance with the directions given on pages 233 to 246 in the general solution of an orbit. But it should be observed that m and (m) do not enter into the solution of a parabolic orbit and that m needs to be computed only if a first approximation of $z = \frac{\rho}{R}$ is to be taken from the tables at the end of this paper. But the parallax enters into the ratio $\frac{\sigma'}{\sigma}$ which is required in (6) and which by (7) may be replaced by $\frac{\lambda}{\kappa}$ in the auxiliary quantities (8), if the parallax be neglected. To allow for the parallax in $\frac{\sigma'}{\sigma}$, we make use of equations (37), page 243, in place of equation (9).

By (8) and if we let

$$\left. \begin{aligned} [X]' &= X' + (\Delta X)' + \beta \cos \alpha \\ [Y]' &= Y' + (\Delta Y)' + \beta \sin \alpha \\ [Z]' &= Z' + (\Delta Z)' + \beta \tan \delta, \end{aligned} \right\} \quad (54)$$

equations (37), page 243, may be written

$$x' = \alpha_x \sigma_0 - [X]'; \quad y' = \alpha_y \sigma_0 - [Y]'; \quad z' = \alpha_z \sigma_0 - [Z]'. \quad (55)$$

These equations then take the place of equations (9) if it be intended to perform a complete elimination of the parallax, so that the only change is in the numerical

¹The solution, however, is easily accomplished by the graphical method, referred to in footnote 2, page 290.

values of the solar coördinates, while the nature of the solution remains the same as if the parallax were neglected.

The final value ρ_f or $z_f = \frac{\rho_f}{R_0}$ obtained in the preceding solution is the value of ρ_0 or z_0 , that is, the value of z or ρ at the middle or normal date, and

$$\sigma_0 = \rho_0 \cos \delta_{00} = \frac{z_0}{R_0} \cos \delta_{00}. \quad (56)$$

The derivation of the heliocentric velocities x'_0, y'_0, z'_0 and the representation of the first and third places are accomplished practically in the same manner as in the general case, Part I. The necessary formulæ are given under V and VI, page 16. It should be observed, however, that some minor modifications in the use of these formulæ have been proposed in the preceding pages. These modifications are introduced in the Synopsis of Formulæ preceding the tables.

As a check on the parabolic solution we have

$$\frac{1}{a} = \frac{2}{r_0} - x_0'^2 + y_0'^2 + z_0'^2, \quad (57)$$

where $\frac{1}{a}$ should be equal to zero within the uncertainty of the numerical operations.

If the residuals of the first and third places are found to be satisfactory, the elements, constants for the equator, and an ephemeris may be computed by VIII, page 18, and in accordance with the additional directions given on pages 251 to 254. If the residuals are not sufficiently small for the purpose in hand, then it will be necessary to determine such corrections to the fundamental data $\sigma_0, x'_0, y'_0, z'_0$, as will remove the residuals. Any small difference $\frac{1}{a}$, resulting from (1), may be removed at the same time and the observations may thus be forced to conform as closely as possible to a strictly parabolic orbit. Differential formulæ for the removal of the residuals and of $\frac{1}{a}$ are developed in the next paragraph. If $\frac{1}{a}$ should exceed the uncertainty of the computation, that is, if it should be larger than a few units of the last decimal employed in the calculation, it follows that a mistake has been made in the previous computation and the same must be revised.

The differential formulæ developed in the next paragraph will always produce a parabola, but the residuals can be reduced to zero only if the orbit is accurately parabolic.

THE DIFFERENTIAL CORRECTION OF PARABOLIC ORBITS.

The first approximation of the orbit of a comet may be made either with or without assumption regarding the eccentricity. If it does not yield a satisfactory representation of the observations, and if it is intended to base the second approximation on the assumption that the orbit is parabolic, then, as stated on page 279, the corrected fundamental data ρ , x' , y' , and z' must satisfy the equation

$$x'^2 + y'^2 + z'^2 = \frac{2}{r}. \quad (1)$$

Let $\partial\rho$, ∂r , $\partial x'$, $\partial y'$, and $\partial z'$ be the required corrections to the initial quantities ρ_0 , r_0 , x'_0 , y'_0 , and z'_0 , so that $\rho = \rho_0 + \partial\rho$, $r = r_0 + \partial r$, $x' = x'_0 + \partial x'$, etc. Then the corrections $\partial\rho$, ∂r , $\partial y'$, and $\partial z'$ must be determined so as to remove the residuals of the first approximation and satisfy equation (1). Equation (1) may be written rigidly

$$(x'_0 + \partial x')^2 + (y'_0 + \partial y')^2 + (z'_0 + \partial z')^2 = \frac{2}{r_0 + \partial r_0} = \frac{2}{r_0} - \frac{2 \partial r}{r_0^2} + \frac{2 (\partial r)^2}{r_0^3} - \frac{2 (\partial r)^3}{r_0^4} + \dots,$$

or, since $x_0'^2 + y_0'^2 + z_0'^2 = \frac{2}{r_0} - \frac{1}{a_0}$,

$$2 x'_0 \partial x' + 2 y'_0 \partial y' + 2 z'_0 \partial z' + (\partial x')^2 + (\partial y')^2 + (\partial z')^2 = \frac{1}{a_0} - \frac{2 \partial r}{r_0^2} + \frac{2 (\partial r)^2}{r_0^3} + \frac{2 (\partial r)^3}{r_0^4} + \dots \quad (2)$$

If the first approximation was made on the assumption that the orbit is parabolic, then $\frac{1}{a_0}$ will be small and of the order of the uncertainty of the numerical operations. The equations which determine the corrections $\partial\rho$, $\partial x'$, $\partial y'$, and $\partial z'$ so as to remove the residuals are given in (2) and (4), page 255, and are

$$\partial x' = P_x - Q_x \partial \rho; \quad \partial y' = P_y - Q_y \partial \rho; \quad \partial z' = P_z - Q_z \partial \rho = P_{z,\dots} - Q_{z,\dots} \partial \rho. \quad (3)$$

Since ∂r may be expressed rigidly in terms of $\partial\rho$ by formulæ (11) or (12a), pages 258-259, equations (2) and (3) furnish five conditions for the determination of the four unknowns $\partial\rho$, $\partial x'$, $\partial y'$, and $\partial z'$. To reduce the conditions to the necessary and sufficient number, any one of conditions (3) may be discarded. But the greatest symmetry of the resulting formulæ will be attained by discarding one of the expressions for $\partial z'$ or by combining the two expressions for $\partial z'$ into one, by forming their mean either with or without the assignment of weights. If the observations can be represented by a parabola, then it is immaterial which of conditions (3) be discarded, but if the orbit is not parabolic, then residuals will appear in the α or δ of the first and third places according to the plan adopted in reducing the number of conditions from five to four. Thus, if either of the expressions for $\partial z'$ be discarded, then a residual will appear in the corresponding declination, since P_z , depends on $\partial\delta$, and $P_{z,\dots}$ on $\partial\delta_{\dots}$. By combining the two expressions for $\partial z'$

preliminary orbit calculations. In the solution of equation (6), an approximate value of $\partial\rho$ is first obtained from $Q\partial\rho = P$. With this the remaining terms on the right-hand side of (6) are computed with the aid of (3). If the sum of these terms be denoted by ΔP , then a more accurate value of $\partial\rho$ results from $Q\partial\rho = P + \Delta P$. With the new value of $\partial\rho$, the corrections $\partial x'$, $\partial y'$, and $\partial z'$ are recomputed by (3), etc., and this process is continued until the corrections remain constant.

At this point a test may be applied to ascertain whether a parabolic solution is feasible on the basis of the given geocentric places. For this purpose we substitute the final corrections to the fundamental data into both of the equations (25), page 12, and compute $\partial\delta$, and $\partial\delta_{\text{true}}$. If the computed values of these residuals agree within the probable errors of observation with the original values which are to be removed by the process of differential correction, then a parabolic orbit is feasible and the solution of the differential formulæ is checked at the same time. It will be observed that whatever discrepancy may exist will reveal itself in the residual corresponding to the equation which has been discarded in the parabolic solution, or in accordance with the weights, if both expressions for $\partial z'$ in (4), page 255, have been retained. If the corrections defined in (16), page 259, were taken into account in the process of differential correction, then the test is to be applied to the residuals $\partial\delta$, and $\partial\delta_{\text{true}}$ as modified by these corrections.

If the test, as applied by equations (25), page 12, indicates that a parabolic solution is infeasible, then we may pass at once from the parabola to a general orbit by solving equation (5), page 255, for $\partial\rho_0$ and then equations (2) and (4), page 255, for $\partial x'_0$, $\partial y'_0$, $\partial z'_0$.

After final corrections to ρ_0 , x'_0 , y'_0 , and z'_0 have been adopted, the corrections to x_0 , y_0 , and z_0 are obtained in the same manner as in the general case, viz.:

$$\partial x = \frac{\xi_0}{\rho_0} \partial\rho ; \quad \partial y = \frac{\eta_0}{\rho_0} \partial\rho ; \quad \partial z = \frac{\zeta_0}{\rho_0} \partial\rho .$$

With the corrected heliocentric coördinates and velocities, the comparison between theory and observation and the computation of the elements is then performed by formulæ VI and VIII, Part I, whereby, however, the modifications outlined in the preceding pages should be observed.

The auxiliary quantities P and Q arise in the solution of equations (24) and (25), page 12, and are defined in (1) and (3), page 255. On pages 256 and following it has been shown that in extreme cases, namely, for long intervals or small heliocentric distances, or both, these equations may not be sufficiently rigid to give the final results. The accuracy of the differential solution may then be increased and a repetition of the differential correction avoided, the same as in the general case, by subtracting expressions (16), page 259, from the residuals, as soon as approximate values of $\partial\rho$, $\partial r'$, and Γ become known through the solution of equations (6),

pendent relation. But as this equation when differentiated involves the additional variation ∂r , it is necessary to provide one more equation by means of which ∂r may be eliminated. Before deriving this equation we shall differentiate (10) and (11) above and (11), page 249. Then we shall obtain

$$\text{from (10):} \quad \partial r = \partial r_0 + 2 \left[\gamma + \frac{r_0 r'_0}{\sqrt{2}} \right] \partial \gamma + \sqrt{2} \gamma \partial (r_0 r'_0),$$

or by (9a)

$$\partial r = \partial r_0 + 2 \sqrt{r-q} \partial \gamma + \sqrt{2} \gamma \partial (r_0 r'_0); \quad (12)$$

from (11)

$$\partial g = [\sqrt{2} r_0 + 2 (r_0 r'_0) \gamma] \partial \gamma + \sqrt{2} \gamma \partial r_0 + \gamma^2 \partial (r_0 r'_0); \quad (13)$$

from (11), page 249, by logarithmic differentiation after squaring both sides:

$$\frac{2 \partial \gamma}{\gamma} = \frac{\partial r_0}{r_0} - \frac{\partial f}{1-f} = \frac{\partial r_0}{r_0} - \frac{r_0}{\gamma^2} \partial f,$$

or

$$2 \partial \gamma = \frac{\gamma}{r_0} \partial r_0 - \frac{r_0}{\gamma} \partial f. \quad (14)$$

After $\partial \gamma$ has been expressed in terms of ∂r_0 and $\partial (r_0 r'_0)$, equations (13) and (14) give ∂g and ∂f in terms of the same variations. $\partial \gamma$ is given by (12) after ∂r has been eliminated. To derive the additional equation needed for the elimination of ∂r we differentiate the well-known equation

$$\frac{\theta}{q^{3/2}} = \sqrt{2} \left\{ (\tan \frac{1}{2} v - \tan \frac{1}{2} v_0) + \frac{1}{3} (\tan^3 \frac{1}{2} v - \tan^3 \frac{1}{2} v_0) \right\}, \quad (15)$$

which gives

$$-3 \frac{\theta}{q^{5/2}} \partial q = \sqrt{2} \left\{ \sec^4 \frac{1}{2} v \partial v - \sec^4 \frac{1}{2} v_0 \partial v_0 \right\}, \quad (16)$$

where ∂v and ∂v_0 may be expressed in terms of ∂r and ∂r_0 after differentiation of the equation $r = q \sec^2 \frac{1}{2} v$, etc. This gives

$$\partial r = \sec^2 \frac{1}{2} v \partial q + q \sec^2 \frac{1}{2} v \tan \frac{1}{2} v \partial v,$$

or, after both sides have been multiplied by $\frac{1}{q} \sec^2 \frac{1}{2} v \cot \frac{1}{2} v$,

$$\sec^4 \frac{1}{2} v \partial v = \frac{\sec^2 \frac{1}{2} v \cot \frac{1}{2} v}{q} \partial r - \frac{\sec^4 \frac{1}{2} v \cot \frac{1}{2} v}{q} \partial q,$$

or, since

$$\sec^2 \frac{1}{2} v = \frac{r}{q} \quad \text{and} \quad \cot \frac{1}{2} v = \frac{\sqrt{q}}{\sqrt{r-q}},$$

$$\sec^4 \frac{1}{2} v \partial v = \frac{r}{q^{3/2} \sqrt{r-q}} \partial r - \frac{r^2}{q^{5/2} \sqrt{r-q}} \partial q, \quad (17)$$

and an analogous expression for $\sec^4 \frac{1}{2} v_0 \partial v_0$. Equation (16) then reduces to

$$\frac{r \partial r}{\sqrt{r-q}} = \frac{r_0 \partial r_0}{\sqrt{r_0-q}} + \left[\frac{r^2}{\sqrt{r-q}} - \frac{r_0^2}{\sqrt{r_0-q}} - \frac{3}{\sqrt{2}} \theta \right] \frac{\partial q}{q}, \quad (18)$$

which is the equation required for the elimination of ∂r from (12). Equation (18) contains ∂q , but the differentiation of the square of (9) gives, by (9) and by (20), page 249,

$$\partial r_0 - \partial q = (r_0 r'_0) \partial (r_0 r'_0) = \sqrt{2} \sqrt{r_0 - q} \partial (r_0 r'_0) = \sqrt{2q} \tan \frac{1}{2} v_0 \partial (r_0 r'_0), \quad (19)$$

by means of which ∂q may be expressed in terms of ∂r_0 and $\partial(r_0 r'_0)$. We shall, however, for the present retain ∂q in (18) and introduce ∂q for $\partial(r_0 r'_0)$ in (12) by (19). After γ has been replaced by (21), page 249, and both sides of (12) have been divided by $\sqrt{r-q}$, (12) may be written

$$\frac{\partial r}{\sqrt{r-q}} = \frac{\partial r_0}{\sqrt{r_0-q}} + 2 \partial \gamma + \frac{\sqrt{r-q} - \sqrt{r_0-q}}{\sqrt{r-q}} \left[\frac{\partial r_0}{\sqrt{r_0-q}} - \frac{\partial q}{\sqrt{r_0-q}} \right],$$

or

$$\frac{\partial r}{\sqrt{r-q}} = 2 \partial \gamma + \frac{\partial r_0}{\sqrt{r_0-q}} - \left[\frac{1}{\sqrt{r_0-q}} - \frac{1}{\sqrt{r-q}} \right] \partial q. \quad (20)$$

Equating now (18) and (20), after both sides of the former equation have been divided by r , we obtain for the required equation in $\partial \gamma$

$$2 \partial \gamma = \frac{r_0 - r}{r \sqrt{r_0 - q}} \partial r_0 + \left[\frac{1}{q} - \frac{r - q}{r q \sqrt{r_0 - q}} - \frac{3\theta}{\sqrt{2} r q} \right] \partial q. \quad (21)$$

In order to obtain the desired expressions for ∂f and ∂g it remains to eliminate $\partial \gamma$ from (14) and (13) by (21) and to replace ∂q by $\partial(r_0 r'_0)$ by (19). Thus equating (14) and (21), after γ has been replaced in the former by (21), page 249, we have

$$\begin{aligned} \frac{r_0}{\sqrt{r-q} - \sqrt{r_0-q}} \partial f = & \left[\frac{1}{r_0} \frac{r-q - \sqrt{r_0-q}}{\sqrt{r_0-q}} - \frac{r_0 - r}{r \sqrt{r_0-q}} \right] \partial r_0 \\ & - \left[\frac{1}{\sqrt{r-q}} + \frac{r q - r_0^2}{r \sqrt{r_0-q}} - \frac{3\theta}{\sqrt{2} r} \right] \frac{\partial q}{q}. \end{aligned} \quad (22)$$

If in this equation we eliminate θ by (15) and ∂q by (19) and introduce true anomalies and q throughout by (20), page 249, and by $r = q \sec^2 \frac{1}{2} v$, etc., we obtain without further reduction

$$\begin{aligned} \frac{q \sec^2 \frac{1}{2} v_0}{\sqrt{q} \tan \frac{1}{2} v - \sqrt{q} \tan \frac{1}{2} v_0} \partial f = & \left[\frac{1}{q \sec^2 \frac{1}{2} v_0} \frac{q \sec^2 \frac{1}{2} v_0 - q \sec^2 \frac{1}{2} v}{q \sec^2 \frac{1}{2} v} \frac{1/\sqrt{q} \tan \frac{1}{2} v}{q} \right. \\ & - \frac{q^2 \sec^2 \frac{1}{2} v - q^2 \sec^2 \frac{1}{2} v_0}{q^2 \sec^2 \frac{1}{2} v \sqrt{q} \tan \frac{1}{2} v_0} + 3 \frac{\sqrt{q} (\tan \frac{1}{2} v - \tan \frac{1}{2} v_0)}{q \sec^2 \frac{1}{2} v} + \frac{1/\sqrt{q} (\tan^3 \frac{1}{2} v - \tan^3 \frac{1}{2} v_0)}{q \sec^2 \frac{1}{2} v} \left. \right] \partial r_0 \\ & + \frac{\sqrt{2}}{\sqrt{q}} \tan \frac{1}{2} v_0 \left[\frac{1/\sqrt{q} \tan \frac{1}{2} v}{q \sec^2 \frac{1}{2} v} + \frac{q^2 \sec^2 \frac{1}{2} v - q^2 \sec^2 \frac{1}{2} v_0}{q \sec^2 \frac{1}{2} v \sqrt{q} \tan \frac{1}{2} v_0} - q^{1/2} \left\{ \frac{3 (\tan \frac{1}{2} v - \tan \frac{1}{2} v_0)}{q \sec^2 \frac{1}{2} v} \right. \right. \\ & \left. \left. + \frac{\tan^3 \frac{1}{2} v - \tan^3 \frac{1}{2} v_0}{q \sec^2 \frac{1}{2} v} \right\} \right] \partial(r_0 r'_0). \end{aligned} \quad (23)$$

After multiplying both sides of this equation by $\sqrt{q} \sec^2 \frac{1}{2} v$, we note that q disappears from the coefficient of ∂r_0 , while the coefficient of $\partial(r_0 r'_0)$ retains the factor \sqrt{q} . Of the six terms of the coefficient of ∂r_0 the sum of the second and fourth becomes

$$\frac{\sec^2 \frac{1}{2} v_0 (\sec^2 \frac{1}{2} v_0 - 1)}{\tan \frac{1}{2} v_0} = \sec^2 \frac{1}{2} v_0 \tan \frac{1}{2} v_0.$$

Then the sum of the second and fourth and the third terms reduces to

$$\sec^2 \frac{1}{2} v_0 \tan \frac{1}{2} v_0 - \sec^2 \frac{1}{2} v \tan \frac{1}{2} v = -(\tan \frac{1}{2} v - \tan \frac{1}{2} v_0) - (\tan^3 \frac{1}{2} v - \tan^3 \frac{1}{2} v_0).$$

The latter of these two terms cancels against the last term in the coefficient of ∂r_0 , while the former may be combined with the fifth term. The coefficient of ∂r_0 then becomes

$$(\tan \frac{1}{2} v - \tan \frac{1}{2} r_0) \left[\frac{\sec^2 \frac{1}{2} v}{\sec^2 \frac{1}{2} r_0} + 2 \right]. \quad (24)$$

The coefficient of $\sqrt{2} q \partial (r_0 r'_0)$ in (23) is

$$\begin{aligned} \tan \frac{1}{2} r_0 \tan \frac{1}{2} v \sec^2 \frac{1}{2} v + \sec^2 \frac{1}{2} v - \sec^2 \frac{1}{2} r_0 - 3 \tan \frac{1}{2} r_0 (\tan \frac{1}{2} v - \tan \frac{1}{2} r_0) \\ - \tan \frac{1}{2} r_0 (\tan^2 \frac{1}{2} v - \tan^2 \frac{1}{2} r_0). \end{aligned} \quad (25)$$

The sum of the first three terms of this expression may be written

$$\tan \frac{1}{2} v \tan \frac{1}{2} r_0 + \tan^2 \frac{1}{2} v \tan \frac{1}{2} r_0 + 1 + \tan^2 \frac{1}{2} v - 1 - 2 \tan^2 \frac{1}{2} r_0 - \tan^4 \frac{1}{2} r_0. \quad (26)$$

Of the five trigonometrical terms of this expression the sum of the second and fifth is

$$\tan \frac{1}{2} r_0 (\tan^2 \frac{1}{2} v - \tan^2 \frac{1}{2} r_0),$$

and this cancels against the last term of (25). The sum of the first term and one-half the fourth of (26) gives

$$\tan \frac{1}{2} r_0 (\tan \frac{1}{2} v - \tan \frac{1}{2} r_0), \quad (27)$$

while the remainder of (26) becomes

$$(\tan \frac{1}{2} v + \tan \frac{1}{2} r_0) (\tan \frac{1}{2} v - \tan \frac{1}{2} r_0). \quad (28)$$

Finally the sum of (27), (28), and the fourth term of (25) gives for the coefficient of $\sqrt{2} q \partial (r_0 r'_0)$:

$$(\tan \frac{1}{2} v - \tan \frac{1}{2} r_0)^2. \quad (29)$$

After introducing (24) and (29) in (23) we obtain, therefore,

$$\frac{q \sec^2 \frac{1}{2} r_0 \sec^2 \frac{1}{2} v}{\tan \frac{1}{2} v - \tan \frac{1}{2} r_0} \partial f = (\tan \frac{1}{2} v - \tan \frac{1}{2} r_0) \left[\frac{\sec^2 \frac{1}{2} v}{\sec^2 \frac{1}{2} r_0} + 2 \right] \partial r_0 + \sqrt{2} q (\tan \frac{1}{2} v - \tan \frac{1}{2} r_0)^2 \partial (r_0 r'_0). \quad (30)$$

or by (19), page 249, and if we introduce $r = q \sec^2 \frac{1}{2} v$, etc.,

$$\partial f = \frac{r^2}{r_0 r} \left[\frac{r}{r_0} + 2 \right] \partial r_0 + \frac{\sqrt{2} r^2}{r_0 r} \partial (r_0 r'_0). \quad (31)$$

The substitution of this value of ∂f in (14) gives

$$\partial \gamma = - \frac{\gamma}{r} \partial r_0 - \frac{\gamma^2}{1 - 2r} \partial (r_0 r'_0). \quad (32)$$

Eliminating $\partial \gamma$ from (13) by means of (32) we have

$$\partial g = \frac{\sqrt{2} \gamma}{r} [r - r_0 - \sqrt{2} (r_0 r'_0) \gamma] \partial r_0 + \frac{\gamma^2}{r} [r - r_0 - \sqrt{2} (r_0 r'_0) \gamma] \partial (r_0 r'_0), \quad (33)$$

and by (10), and collecting results from (31) and (33), we finally obtain in accordance with (20), page 262,

$$\partial f = \frac{\cos \beta \gamma^2}{r_0 r} \left[\frac{r}{r_0} + 2 \right] \partial \rho_0 + \frac{\sqrt{2} \gamma^2}{r_0 r} \partial (r_0 r'_0) \quad (34)$$

$$\partial g = \frac{\cos \beta}{r} \sqrt{2} \gamma^2 \partial \rho_0 + \frac{\gamma^2}{r} \partial (r_0 r'_0), \quad (35)$$

where $\cos \beta \partial \rho_0 = \partial r_0$.

Introducing these values of ∂f and ∂g into the first of equations (16), page 11,

where $\omega = x, y, z$, and noting that by (17), page 11, $\partial \omega_0 = \frac{w_0}{\rho_0} \partial \rho_0$, where $w_0 = \xi_0, \eta_0, \zeta_0$, we obtain

$$\partial \omega = \left\{ f \frac{w_0}{\rho_0} + \omega_0 \frac{\cos \beta}{r_0 r} \left[\frac{r}{r_0} + 2 \right] + \omega'_0 \frac{\cos \beta}{r} \sqrt{2} \gamma^2 \right\} \partial \rho_0 + g \partial \omega'_0 \\ + \left\{ \frac{\sqrt{2} \omega_0 \gamma^2}{r_0 r} + \frac{\omega'_0 \gamma^2}{r} \right\} \partial (r_0 r'_0),$$

or

$$\partial \omega = \left\{ f \frac{w_0}{\rho_0} + \frac{\cos \beta}{r_0 r} \gamma^2 \left[\omega_0 \left(\frac{r}{r_0} + 2 \right) + \sqrt{2} \omega'_0 r_0 \gamma \right] \right\} \partial \rho_0 + g \partial \omega'_0 \\ + \frac{\gamma^2}{r_0 r} \left\{ \omega'_0 r_0 \gamma + \sqrt{2} \omega_0 \right\} \partial (r_0 r'_0). \quad (36)$$

But, since

$$r_0 r'_0 = x_0 x'_0 + y_0 y'_0 + z_0 z'_0 = \Sigma \omega_0 \omega'_0,$$

therefore also

$$\partial (r_0 r'_0) = \Sigma \omega_0 \partial \omega'_0 + \Sigma \omega'_0 \partial \omega_0$$

or, if $\partial \omega_0$ be expressed as above in terms of $\partial \rho_0$

$$\partial (r_0 r'_0) = \Sigma \omega_0 \partial \omega'_0 + (\Sigma \omega'_0 w_0) \frac{\partial \rho_0}{\rho_0}. \quad (37)$$

Let

$$\varphi_0 = \frac{\Sigma \omega'_0 w_0}{\rho_0} = \frac{x'_0 \xi_0 + y'_0 \eta_0 + z'_0 \zeta_0}{\rho_0}, \quad (38)$$

and eliminate $\partial (r_0 r'_0)$ from (36) by means of (37). Then

$$\partial \omega = \left\{ f \frac{w_0}{\rho_0} + \frac{\cos \beta}{r_0 r} \gamma^2 \left[\omega_0 \left(\frac{r}{r_0} + 2 \right) + \sqrt{2} \omega'_0 r_0 \gamma \right] + \frac{\varphi_0 \gamma^2}{r_0 r} \left[\omega'_0 r_0 \gamma + \sqrt{2} \omega_0 \right] \right\} \partial \rho_0 \\ + g \partial \omega'_0 + \frac{\gamma^2}{r_0 r} \left[\omega'_0 r_0 \gamma + \sqrt{2} \omega_0 \right] \Sigma \omega_0 \partial \omega'_0,$$

or if

$$g_\omega = \frac{\gamma^2}{r_0 r} \left[\omega'_0 r_0 \gamma + \sqrt{2} \omega_0 \right] \quad (39)$$

and

$$f_\omega = f \frac{w_0}{\rho_0} + \frac{\cos \beta}{r_0 r} \gamma^2 \left[\omega_0 \left(\frac{r}{r_0} + 2 \right) + \sqrt{2} \omega'_0 r_0 \gamma \right] + \varphi_0 g_\omega, \quad (40)$$

then

$$\partial \omega = f_\omega \partial \rho_0 + g \partial \omega'_0 + g_\omega \Sigma \omega_0 \partial \omega'_0. \quad (41)$$

The expression for f_ω may be simplified by means of (39). For

$$f_\omega = f \frac{w_0}{\rho_0} + \frac{\cos \beta}{r_0 r} \gamma^2 \left[\frac{\omega_0 r}{r_0} + \sqrt{2} \left(\omega'_0 r_0 \gamma + \sqrt{2} \omega_0 \right) \right] + \varphi_0 g_\omega,$$

or

$$f_\omega = f \frac{w_0}{\rho_0} + \frac{\cos \beta}{r_0^2} \omega_0 \gamma^2 + g_\omega \left[\varphi_0 + \sqrt{2} \frac{\cos \beta}{\gamma} \right]. \quad (42)$$

Equation (41) gives ∂x , ∂y , or ∂z in a closed form for any date, the auxiliary quantities $g_x, g_y, g_z, f_x, f_y, f_z$ being defined by (39) and (42). To obtain the values of ∂x , ∂y , or ∂z for a particular date we must introduce in (39), (41), and (42)

the values of f , γ , and r for that date. Thus we have, for instance, for the first date

$$g_{z_1} = \frac{\gamma_1^2}{r_0 r_1} \left[x_0' r_0 \gamma_1 + \sqrt{2} x_0 \right]; f_{z_1} = f_1 \frac{\xi_0}{\rho_0} + \frac{\cos \beta}{r_0^2} x_0 \gamma_1^2 + g_{z_1} \left[\varphi_0 + \frac{1}{2} \frac{\cos \beta}{\gamma_1} \right] \quad (43)$$

$$\partial x_1 = f_{z_1} \partial \rho_0 + [g_{z_1} + g_{z_1} x_0] \partial x_0' + g_{z_1} y_0 \partial y_0' + g_{z_1} z_0 \partial z_0'. \quad (44)$$

and analogous expressions for

$$g_{y_1}, g_{x_1}, f_{y_1}, f_{x_1}, g_{z_{11}}, g_{y_{11}}, g_{x_{11}}, f_{z_{11}}, f_{y_{11}}, f_{x_{11}}, \text{ and } \partial y_1, \partial z_1, \partial x_{11}, \partial y_{11}, \partial z_{11}.$$

If now equations (15), page 11, be applied to the first and third places, we obtain by (39), (41), and (42), for the first place

$$\left. \begin{aligned} \partial \alpha_1 &= A_{z_1} \partial \rho_0 - \sin \alpha_1 \left[\frac{g_{z_1}}{\rho_1} + x_0 \frac{g_{z_1}}{\rho_1} \right] \partial x_0' - \sin \alpha_1 y_0 \frac{g_{z_1}}{\rho_1} \partial y_0' - \sin \alpha_1 z_0 \frac{g_{z_1}}{\rho_1} \partial z_0' \\ &\quad + \cos \alpha_1 x_0 \frac{g_{z_1}}{\rho_1} \partial x_0' + \cos \alpha_1 \left[\frac{g_{z_1}}{\rho_1} + y_0 \frac{g_{z_1}}{\rho_1} \right] \partial y_0' + \cos \alpha_1 z_0 \frac{g_{z_1}}{\rho_1} \partial z_0' \\ \partial \delta_1 &= B_{z_1} \partial \rho_0 - \sin \delta_1 \cos \alpha_1 \left[\frac{g_{z_1}}{\rho_1} + x_0 \frac{g_{z_1}}{\rho_1} \right] \partial x_0' - \sin \delta_1 \cos \alpha_1 y_0 \frac{g_{z_1}}{\rho_1} \partial y_0' - \sin \delta_1 \cos \alpha_1 z_0 \frac{g_{z_1}}{\rho_1} \partial z_0' \\ &\quad - \sin \delta_1 \sin \alpha_1 x_0 \frac{g_{z_1}}{\rho_1} \partial x_0' - \sin \delta_1 \sin \alpha_1 \left[\frac{g_{z_1}}{\rho_1} + y_0 \frac{g_{z_1}}{\rho_1} \right] \partial y_0' - \sin \delta_1 \sin \alpha_1 z_0 \frac{g_{z_1}}{\rho_1} \partial z_0' \\ &\quad + \cos \delta_1 x_0 \frac{g_{z_1}}{\rho_1} \partial x_0' + \cos \delta_1 y_0 \frac{g_{z_1}}{\rho_1} \partial y_0' + \cos \delta_1 \left[\frac{g_{z_1}}{\rho_1} + z_0 \frac{g_{z_1}}{\rho_1} \right] \partial z_0'. \end{aligned} \right\} \quad (45)$$

Similarly, analogous expressions are obtained for $\partial \alpha_{11}$ and $\partial \delta_{11}$. The auxiliary quantities A_{z_1} , $A_{z_{11}}$, B_{z_1} , $B_{z_{11}}$ have the same definition as A_1 , A_{11} , B_1 , B_{11} in formulæ (23), page 12.

Equations (45) may be simplified by the introduction of further auxiliary quantities. Let A_{z_1} , $A_{z_{11}}$, B_{z_1} , $B_{z_{11}}$ be defined as functions of g_{z_1} , $g_{z_{11}}$ in the same manner as A_1 , A_{11} , B_1 , B_{11} or here A_{z_1} , $A_{z_{11}}$, B_{z_1} , $B_{z_{11}}$ are defined by (23), page 12, as functions of f_{z_1} and $f_{z_{11}}$, so that

$$\left. \begin{aligned} A_{z_1} &= \frac{1}{\rho_1} [\cos \alpha_1 g_{z_1} - \sin \alpha_1 g_{z_1}] \\ A_{z_{11}} &= \frac{1}{\rho_{11}} [\cos \alpha_{11} g_{z_{11}} - \sin \alpha_{11} g_{z_{11}}] \\ B_{z_1} &= -\frac{1}{\rho_1} [\sin \delta_1 (\sin \alpha_1 g_{z_1} + \cos \alpha_1 g_{z_1}) - \cos \delta_1 g_{z_1}] \\ B_{z_{11}} &= -\frac{1}{\rho_{11}} [\sin \delta_{11} (\sin \alpha_{11} g_{z_{11}} + \cos \alpha_{11} g_{z_{11}}) - \cos \delta_{11} g_{z_{11}}], \end{aligned} \right\} \quad (46)$$

and let C_1 and C_{11} be defined as in (23), page 12, then

$$\left. \begin{aligned} \partial \alpha_1 &= A_{z_1} \partial \rho_0 - [\sin \alpha_1 C_1 - x_0 A_{z_1}] \partial x_0' + [\cos \alpha_1 C_1 + y_0 A_{z_1}] \partial y_0' + z_0 A_{z_1} \partial z_0' \\ \partial \delta_1 &= B_{z_1} \partial \rho_0 - [\sin \delta_1 \cos \alpha_1 C_1 - x_0 B_{z_1}] \partial x_0' - [\sin \delta_1 \sin \alpha_1 C_1 - y_0 B_{z_1}] \partial y_0' \\ &\quad + [\cos \delta_1 C_1 + z_0 B_{z_1}] \partial z_0' \\ \partial \alpha_{11} &= A_{z_{11}} \partial \rho_0 - [\sin \alpha_{11} C_{11} - x_0 A_{z_{11}}] \partial x_0' + [\cos \alpha_{11} C_{11} + y_0 A_{z_{11}}] \partial y_0' + z_0 A_{z_{11}} \partial z_0' \\ \partial \delta_{11} &= B_{z_{11}} \partial \rho_0 - [\sin \delta_{11} \cos \alpha_{11} C_{11} - x_0 B_{z_{11}}] \partial x_0' - [\sin \delta_{11} \sin \alpha_{11} C_{11} - y_0 B_{z_{11}}] \partial y_0' \\ &\quad + [\cos \delta_{11} C_{11} + z_0 B_{z_{11}}] \partial z_0'. \end{aligned} \right\} \quad (47)$$

Equations (54) and (55) have the same form as equations (8), page 294, and together with (2), page 294, lead to the same equation (6), page 295, in the determination of a parabolic orbit. The same considerations as to the arrangement and the test of a parabolic solution apply here as in the case where the ∂f and ∂g were expressed in series (pages 294 to 296). In the foregoing arrangement, any remaining discrepancy will appear as a residual in the first declination.

It will be observed, however, that the form of ∂f and ∂g is the same for all classes of orbits only if they are expressed in series, but that the closed expressions (34) and (35) of ∂f and ∂g introduced here¹ for the parabola differ from KUEHNERT'S general expressions, (18) and (19), page 262, in that they are based on the condition that the orbit shall remain parabolic. Hence, if the application of the test referred to on page 296 reveals the infeasibility of a parabolic solution, and a general solution becomes necessary by solving equation (5), page 255, then this general solution still is in part conditioned by the parabolic relation $\partial G_0^2 = -\frac{2}{r_0^3} \partial r_0$ (cf. page 262), so that the resulting orbit will be as closely parabolic as the observations permit.

The effect of this process of passing from a parabolic to a general orbit is that the adoption of too short a period or of too small an eccentricity is practically excluded, which is of particular importance in the case of comet observations admitting of a large range of practical solutions.

¹The closed expressions of ∂f and ∂g for the parabola had been deduced independently before it was discovered that general expressions for ∂f and ∂g had been published by KUEHNERT.

positive and those of z^3 and z remain positive, then there will be no variation of sign in (47), and consequently no solution. Let

$$\left. \begin{aligned} k &= 2p'^2 + \frac{1}{a^2}, \text{ then these conditions are: } p' > 0, c < 0 \text{ and} \\ c &< -2p', \quad c < -\frac{\left(1 + \frac{1}{a^2}\right)p'}{k}, \quad c < -2p'a^2; \quad c > -\frac{1+2k}{8p'}, \quad c > -\frac{\frac{1}{a^2} + 2k}{8p'}a^2, \end{aligned} \right\} (5)$$

that is, there is no solution if c is smaller than the smallest of the expressions of the first group and larger than the largest of the expressions of the second group, which is readily ascertained by inspection of the auxiliary quantities.

Suppressing now the subscript zero for quantities referring to the epoch, we note that the condition $G^2 = \frac{1}{r}$ modifies the *general* solution only to the extent that the semi-major axis a shall be equal to the radius vector r at the middle date or epoch. Since three complete observations are used, the solution, therefore, will generally produce a *conditioned* ellipse with the planet at one of the extremities of the minor axis at the epoch. To test the possibility of a circle it is sufficient to ascertain whether r'^2 , which may be computed as soon as the heliocentric coördinates and velocities have been derived, is less than the numerical uncertainty of $G^2 = \frac{1}{r} = \frac{1}{a}$. For in $p = r^2 (G^2 - r'^2)$ we may let $r' = 0$, if r'^2 does not affect the definitely known portion of G^2 , so that $p = r = a$. The accuracy of r and, therefore, of G^2 is determined as on pages 270-276. But if $p = a(1 - e^2) = a$ within the accuracy attainable from the observations, then a *circular* orbit will represent the observations with the same degree of accuracy as the conditioned ellipse. If r' cannot be put equal to zero, the computer should proceed to the solution of a *general* orbit.

This *conditioned* solution prevents the computer from adopting a large eccentricity within the range of possible solutions. The elements Ω , i and u of the circle are the same as those of the conditioned ellipse while the radius is equal to $r (= a)$ of the latter. The residuals are computed most conveniently by means of the constants for the equator.

Similar considerations will sometimes enable the computer to change the *general* into a *circular* solution by letting $a = r$, if their difference is less than the uncertainty of r , and then ascertaining as above whether r' can be put equal to zero. The elements i , Ω , and u of the circle are then the same as those of the general solution, while the radius of the circle is equal to the computed radius vector at the middle date of the general solution.

Attention has already been called in the Introduction (pages 223 and 224) to the relation of the method outlined above to the usual methods of deriving circular orbits, so that this comparison may here be omitted.

general orbit. In the latter case $\partial\rho_0$ is computed by (5), page 255, and then $\partial x'_0$, $\partial y'_0$, and $\partial z'_0$ by (2) and (4), page 255. The remaining computation proceeds as usual by VI and VIII, Part I, except for such slight modifications as have been proposed above in various places. This solution is conditioned *in part* by $\partial r = \partial a$ and $\partial r' = 0$, so that it precludes the adoption of a large eccentricity, within the range of possible solutions, when a small one is possible.

If it be proposed to attempt a representation of the observations on the basis of an improved circular orbit, then the further procedure will be analogous to that adopted in the differential correction of the fundamental data in a parabolic orbit. The required formulæ are readily derived from those obtained for the parabola.

The corrected fundamental data must satisfy the conditions

$$(x'_0 + \partial x'_0)^2 + (y'_0 + \partial y'_0)^2 + (z'_0 + \partial z'_0)^2 = \frac{1}{r_0 + \partial r_0} \text{ and } r' = 0.$$

Allowing for the possibility that the initial orbit may not be strictly circular, we have for the same

$$x_0'^2 + y_0'^2 + z_0'^2 = \frac{2}{r_0} - \frac{1}{a_0} = \frac{1}{r_0} + \left(\frac{1}{r_0} - \frac{1}{a_0} \right) = \frac{1}{r_0} - \Delta.$$

Proceeding now exactly as in the case of the parabola on page 294 *et seq.*, we observe first of all that in equation (2), page 294, Δ takes the place of $\frac{1}{a_0}$ and that the coefficient 2 of ∂r and of its second and third powers drops out. Hence, when later ∂r is replaced by $\partial\rho$ and both sides of the equation are divided by 2, this number will appear in the denominators of all the terms containing $\partial\rho$, $(\partial\rho)^2$, and $(\partial\rho)^3$. In place of equations (5) and (6), we shall therefore obtain for circular orbits

$$\left. \begin{aligned} Q &= \frac{\cos \beta}{2 r_0^3} - [x'_0 Q_x + y'_0 Q_y + z'_0 Q_z]; & P &= \frac{\Delta}{2} - [x'_0 P_x + y'_0 P_y + z'_0 P_z] \\ Q \partial\rho &= P - \frac{1}{2} [(\partial x')^2 + (\partial y')^2 + (\partial z')^2] - \frac{1 - 3 \cos^2 \beta}{4 r_0^3} (\partial\rho)^2 + \frac{3 \cos \beta (1 - \frac{5}{2} \cos^2 \beta)}{4 r_0^4} (\partial\rho)^3, \end{aligned} \right\} \quad (8)$$

in which, on account of the usually large values of r_0 , the terms in $(\partial\rho)^2$ and $(\partial\rho)^3$ may practically always be omitted.

The further procedure is analogous to that in parabolic orbits.

As in the direct solution it remains to ascertain whether r' may be put equal to zero within the accuracy of the solution. After the differential correction the number of places to which r^2 is known with accuracy in general is not less than the number of places contained in the mean of the differences of α or δ , expressed in seconds of arc. The same criterion, however, together with the test whether $r = a$ within the accuracy of the solution, may be applied more conveniently to the corrected *general* orbit. In fact, the solution of equations (8) probably never will be of advantage, as the corrected orbit rarely will prove circular.

A collection of formulæ for the direct solution and for the correction of circular orbits is included in the *Synopsis of Formulae*, given at the end of this paper.

with similar expressions for $\frac{dy}{dt}$ and $\frac{dz}{dt}$. But

$$\frac{dv}{dt} = \frac{k\sqrt{p}}{r^2} = \frac{k(1+e\cos v)^2}{p^{3/2}}; \quad \frac{dr}{dt} = \frac{d}{dt} \left[\frac{p}{1+e\cos v} \right] = \frac{pe\sin v}{(1+e\cos v)^2} \frac{dv}{dt} = \frac{ke\sin v}{\sqrt{p}}. \quad (3)$$

Hence

$$\frac{dx}{dt} = \frac{k(1+e\cos v)}{\sqrt{p}} \sin a \cos (A' + v) + \frac{ke\sin v}{\sqrt{p}} \sin a \sin (A' + v),$$

where the unit of time is a mean solar day. Since $x' = \frac{1}{k} \frac{dx}{dt}$, etc., we finally have after a simple trigonometrical reduction

$$\left. \begin{aligned} x'_0 &= \frac{\sin a}{\sqrt{p}} [\cos (A' + v_0) + e \cos A'] \\ y'_0 &= \frac{\sin b}{\sqrt{p}} [\cos (B' + v_0) + e \cos B'] \\ z'_0 &= \frac{\sin c}{\sqrt{p}} [\cos (C' + v_0) + e \cos C'] \end{aligned} \right\} \quad (4)$$

For the parabola $p = 2q$ and $e = 1$. Hence equations (4) reduce to

$$\left. \begin{aligned} x'_0 &= \sqrt{\frac{2}{q}} \sin a \cos (A' + \frac{1}{2} v_0) \cos \frac{1}{2} v_0 \\ y'_0 &= \sqrt{\frac{2}{q}} \sin b \cos (B' + \frac{1}{2} v_0) \cos \frac{1}{2} v_0 \\ z'_0 &= \sqrt{\frac{2}{q}} \sin c \cos (C' + \frac{1}{2} v_0) \cos \frac{1}{2} v_0 \end{aligned} \right\} \quad (5)$$

For the circle $\frac{dr}{dt} = \mu = \frac{k}{a^{3/2}}$, $r = a$, and $\frac{dv}{dt} = 0$. Hence equations (2) become

$$x'_0 = \frac{1}{a^{1/2}} \sin a \cos (A' + v_0); \quad y'_0 = \frac{1}{a^{1/2}} \sin b \cos (B' + v_0); \quad z'_0 = \frac{1}{a^{1/2}} \sin c \cos (C' + v_0), \quad (6)$$

where $A' + v_0 = A + u_0$, etc.

Before the approximate values of the fundamental data, derived by the foregoing directions, may serve as the basis of the differential correction, they must be written out to the number of decimals to which the orbit is to be computed, if necessary by adding ciphers. Initial values of the heliocentric coördinates are then accurately computed from $x_0 = \rho_0 \cos \alpha_0 \cos \delta_0 - X_0$; etc., where α_0, δ_0 are the coördinates of the adopted zero or middle date.

From the heliocentric coördinates and velocities at the middle date, the residuals of the first and third places are next deduced with the aid of the expressions for f and g as in VI, page 16. The differential correction of the fundamental data may then be performed on the basis of these residuals in the usual manner. It is evident that closed expressions for f and g , ∂f and ∂g must be used in the computation, if the criteria given on pages 260 and 261 indicate the insufficiency of the series.

The particular formulæ to be used for the most convenient solution of the problem according to whether the initial orbit is circular, parabolic, or elliptic are outlined in the *Synopsis of Formulæ*, at the end of this paper.

The initial orbit which serves as the basis of the differential correction is an artificial orbit defined by the following six constants: The observed right ascension and declination at the middle date; the geocentric distance and the heliocentric velocities in rectangular coördinates corresponding to the preliminary orbit.

The preliminary orbit has, therefore, only the four elements ρ_0, x'_0, y'_0, z'_0 in common with the initial orbit, while the remaining two elements α_{m} and δ_{m} of the initial orbit are, in general, not satisfied by the preliminary orbit.

It will be observed that great accuracy is not required in computing ρ_0, x'_0, y'_0, z'_0 from the preliminary orbit and that these values may even receive slight arbitrary changes to suit the convenience of the computer. But after certain values have been adopted for these quantities they must be strictly retained to the required number of decimals in performing the comparison between theory and observation and the differential correction.

Concerning the elimination of the parallax and aberration the usual rules are to be observed. These corrections may therefore be applied in advance, if the artificial initial orbit may be considered a close approximation. It is evident that the initial orbit is more accurate than the preliminary orbit, for while the geocentric distance ρ_0 and the heliocentric velocities x'_0, y'_0, z'_0 are the same for both, the former also represents the observed $\alpha_{\text{m}}, \delta_{\text{m}}$, at the middle date. The geocentric distances ρ_i and ρ_{m} may be computed by (1) in the same manner as ρ_0 .

If the initial orbit can not be considered sufficiently accurate for deriving the parallax and aberration corrections in advance, then the parallax should be eliminated from the middle place alone by (4), page 234. For the first and third places the parallax is applied in the comparison between theory and observation with the values of ρ_i and ρ_{m} resulting in the course of this comparison as in VI, page 16. To allow for a possible change in the parallax in the course of the differential correction, the auxiliary quantities A and B should be corrected as in (41), page 244.

If the accuracy of the initial orbit be doubtful, the aberration may be treated as in the direct solution. The aberration of the fixed stars is included in the reduction of the observed places to the beginning of the year and the solar coördinates X, Y, Z are interpolated for the observed (uncorrected) dates $t_i, t_{\text{m}}, t_{\text{m}}'$. But in order to free the intervals at least in part from the planetary aberration in computing f and g either by series or closed expressions, $\rho_i, \rho_{\text{m}}, \rho_{\text{m}}'$ are also computed by (1) since true dates must be used in deriving the f and g . The true epoch is obtained by ultimately correcting the observed date t_{m} by means of the final value of ρ_0 . Theoretically, the value of ρ_0 for the uncorrected date should be included in the initial fundamental data together with the values of x'_0, y'_0, z'_0 for the true date t_{m} . But this latter refinement is superfluous, inasmuch as slight arbitrary changes of the fundamental data are permissible.

SYNOPSIS OF FORMULÆ.*

A. DIRECT SOLUTION OF ORBITS WITH OR WITHOUT ASSUMPTION REGARDING THE ECCENTRICITY.

-
- a. For parabolic orbits use Ia, IIa, IIIa, etc.
b. For circular orbits use Ib, IIb, IIIb, etc.
c. For general orbits use Ic, IIc, IIIc, etc.

GENERAL DIRECTIONS.

Ia; Ib; Ic.

It is assumed that the orbit is to be based on three places of short intervals.

Reduce the observed α and δ to the beginning of the year by including the aberration terms. Let the mean places be

$$\alpha_i, \delta_i; \alpha_u, \delta_u; \alpha_m, \delta_m.$$

From one of the Astronomical Ephemerides interpolate the solar coördinates for the observed dates $t_i, t_u = t_0, t_m$.

$$X_i, Y_i, Z_i; \quad X_u, Y_u, Z_u; \quad X_m, Y_m, Z_m.$$

At the same time obtain $X'_0 = X'_u, Y'_0 = Y'_u, Z'_0 = Z'_u$, by either of the following formulæ of numerical differentiation:

$$kw \frac{df(l)}{dl} = f'(a + iw) + n f''(a + iw) + N_1^3(n) f'''(a + iw) + \dots;$$

$$l = a + [i + n] w = t_0; \quad N_1^3(n) = \frac{3n^2 - 1}{6}; \quad N_1^4(n) = \frac{n(2n^2 - 1)}{12}; \text{ etc.}$$

$$kw \frac{df(l)}{dl} = f'(a + [i + \frac{1}{2}] w) + m f''(a + [i + \frac{1}{2}] w) + M_1^3(m) f'''(a + [i + \frac{1}{2}] w) + \dots;$$

$$l = a + [i + \frac{1}{2} + m] w = t_0; \quad M_1^3(m) = \frac{3m^2 - 1}{6}; \quad M_1^4(m) = \frac{m(2m^2 - 1)}{12}; \text{ etc.}$$

according as to whether t_0 lies nearer to a tabulated argument or to the mean of two such arguments. m and n are numerically < 0.25 . $N_1^3(n)$ or $M_1^3(m)$ may be taken from Table I or II, OPPOLZER, *Bahnbestimmung*, etc., Vol. II, page 515 or 523, but, in general, the terms in which they occur may be neglected. When the solar coördinates are tabulated for every 12 hours, $w = \frac{1}{2}$, $\log \frac{1}{kw} = 2.0654486$.

*All supplementary formulæ and terms printed in small type, especially those contained in the footnotes, may generally be omitted.

It is advisable to *partially*¹ eliminate the *geocentric* (ordinary) parallax from the middle place by applying the following corrections to the solar coördinates at the epoch $t_0 = t_u$:

Reduce the *geocentric* parallax factors to circular measure by multiplying the $(p_a\rho)'$ by $15 \sin 1''$ and the $(p_b\rho)''$ by $\sin 1''$. $\log (15 \sin 1'') = 5.8617 - 10$; $\log \sin 1'' = 4.6856 - 10$.

$$\begin{aligned}\Delta X_u &= (p_a\rho)_u \sin \alpha_u \cos \delta_u + (p_b\rho)_u \cos \alpha_u \sin \delta_u \\ \Delta Y_u &= -(p_a\rho)_u \cos \alpha_u \cos \delta_u + (p_b\rho)_u \sin \alpha_u \sin \delta_u \\ \Delta Z_u &= -(p_b\rho)_u \cos \delta_u.\end{aligned}$$

$$R \cos D \cos A = X_u; \quad R \cos D \sin A = Y_u; \quad S = R \cos D, \quad R \sin D = Z_u.$$

$$\theta_i = k(t_u - t_i), \quad \theta_{ii} = k(t_u - t_i); \quad \log k = 8.2355814 - 10.$$

¹But a *complete* elimination of the parallax should be performed, if the accuracy of the resulting orbit essentially depends on such elimination, (cf. pages 273 and 274). In brief, the parallax should be completely eliminated if the parallax factors for the three dates differ considerably, and if it be estimated that the percentage error of the observed motion due to neglecting the parallax corrections is large and in excess of the errors arising from the errors of observation and from the neglected third and higher derivatives, (cf. pages 270 to 278). For the *complete elimination* of the parallax compute for all three dates:

$$\begin{aligned}\Delta_1 X &= -d_1 \cos \delta_1 \cos \alpha_1; \quad \Delta_1 Y = -d_1 \cos \delta_1 \sin \alpha_1; \quad \Delta_1 Z = -d_1 \sin \delta_1. \\ p_a^m \rho_m &= \frac{d_1}{\sin 1''} \frac{\cos \delta_1}{\cos \delta} \sin(\alpha - \alpha_1); \quad p_b^m \rho_m = \frac{d_1}{\sin 1''} \left\{ -\sin \delta_1 \cos \delta + \cos \delta_1 \sin \delta \cos(\alpha - \alpha_1) \right\},\end{aligned}$$

where α_1, δ_1 represent the coördinates of the Moon and where

$$\log d_1 = 5.4943 - 10; \quad \frac{d_1}{\sin 1''} = 6''.4372.$$

Further, for all three dates:

$$p_a \rho = p_a^g \rho_g + p_a^m \rho_m; \quad p_b \rho = p_b^g \rho_g + p_b^m \rho_m,$$

where $p_a^g \rho_g, p_b^g \rho_g$ represent the geocentric parallax factors, usually designated by $p_a \rho$ or $p_a \Delta$, etc., and where all parallax factors are supposed to be expressed in seconds of arc. Further, for all three dates:

$$\begin{aligned}\Delta_2 X &= [(p_a\rho) \sin \alpha \cos \delta + (p_b\rho) \cos \alpha \sin \delta] \sin 1''; \quad \Delta_2 Y = [-(p_a\rho) \cos \alpha \cos \delta + (p_b\rho) \sin \alpha \sin \delta] \sin 1''; \\ \Delta_2 Z &= -(p_b\rho) \cos \delta \sin 1''.\end{aligned}$$

If the $\Delta_2 X, \Delta_2 Y, \Delta_2 Z$, be roughly proportional to the time, then the following auxiliary quantities d_x, d_y, d_z are equal to zero and need not be computed.

$$\begin{aligned}d_x &= \frac{\Delta_2 X_{ii}(t_u - t_i) + \Delta_2 X_u(t_{ii} - t_u)}{\Delta_2 X_{ii}(t_{ii} - t_i)} - 1; & d_y &= \frac{\Delta_2 Y_{ii}(t_u - t_i) + \Delta_2 Y_u(t_{ii} - t_u)}{\Delta_2 Y_{ii}(t_{ii} - t_i)} - 1 \\ d_z &= \frac{\Delta_2 Z_{ii}(t_u - t_i) + \Delta_2 Z_u(t_{ii} - t_u)}{\Delta_2 Z_{ii}(t_{ii} - t_i)} - 1.\end{aligned}$$

$$\Delta X = \Delta_1 X + \Delta_2 X, \quad \Delta Y = \Delta_1 Y + \Delta_2 Y, \quad \Delta Z = \Delta_1 Z + \Delta_2 Z; \quad (X) = X + \Delta X, \quad (Y) = Y + \Delta Y, \quad (Z) = Z + \Delta Z.$$

Compute further, but only for the middle date (epoch):

$$R \cos D \cos A = (X)_u = X_u + \Delta X_u, \quad R \cos D \sin A = (Y)_u = Y_u + \Delta Y_u, \quad R \sin D = (Z)_u = Z_u + \Delta Z_u.$$

Also for general orbits, Ic:

$$\Delta_3 R = \frac{X_u \Delta_2 X_{ii} + Y_u \Delta_2 Y_{ii} + Z_u \Delta_2 Z_{ii}}{R_u},$$

where we may use the corrected or uncorrected values for X, Y, Z .

$$j \cos a = \left[\frac{1}{R^2} + \frac{2d_x}{\theta_i \theta_{ii}} \right] \Delta_2 X_{ii}; \quad j \sin a = \left[\frac{1}{R^2} + \frac{2d_y}{\theta_i \theta_{ii}} \right] \Delta_2 Y_{ii}; \quad j \tan d = \left[\frac{1}{R^2} + \frac{2d_z}{\theta_i \theta_{ii}} \right] \Delta_2 Z_{ii}.$$

Compute $(\Delta X)'_0; (\Delta Y)'_0; (\Delta Z)'_0$ from $\Delta X_i, \Delta X_{ii}, \Delta X_u; \Delta Y_i, \Delta Y_{ii}, \Delta Y_u; \Delta Z_i, \Delta Z_{ii}, \Delta Z_u$ in the same manner as $\alpha'_0; \delta'_0$ are computed from $\alpha_i, \alpha_{ii}, \alpha_u; \delta_i, \delta_{ii}, \delta_u$; but omitting the factors $\sin 1''$ in formulæ II which here has already been taken account of. Or, if the $\Delta X, \Delta Y, \Delta Z$ be roughly proportional to the time, then simply

$$\log \frac{1}{k} = 1.7644; \quad (\Delta X)'_0 = \frac{1}{k} \frac{\Delta X_{ii} - \Delta X_i}{t_{ii} - t_i}; \quad (\Delta Y)'_0 = \frac{1}{k} \frac{\Delta Y_{ii} - \Delta Y_i}{t_{ii} - t_i}; \quad (\Delta Z)'_0 = \frac{1}{k} \frac{\Delta Z_{ii} - \Delta Z_i}{t_{ii} - t_i}.$$

IIa; IIb; IIc.¹

It is most convenient to express the differences in the coördinates in seconds of arc. Then, with the mean solar day as unit of time:

$$\alpha'_{iii} = \frac{\alpha_{iii} - \alpha_i}{t_{iii} - t_i}; \quad \alpha'_i = \frac{\alpha_{iii} - \alpha_{ii}}{t_{iii} - t_{ii}}; \quad \delta'_{iii} = \frac{\delta_{iii} - \delta_i}{t_{iii} - t_i}; \quad \delta'_i = \frac{\delta_{iii} - \delta_{ii}}{t_{iii} - t_{ii}}.$$

Or expressed in circular measure and in $\frac{1}{k}$ mean solar days as unit of time:

$$\alpha'_0 = \frac{\sin 1''}{k} \frac{(t_{iii} - t_{ii}) \alpha'_{iii} + (t_{ii} - t_i) \alpha'_i}{t_{iii} - t_i}; \quad \delta'_0 = \frac{\sin 1''}{k} \frac{(t_{iii} - t_{ii}) \delta'_{iii} + (t_{ii} - t_i) \delta'_i}{t_{iii} - t_i}.$$

$$\alpha''_0 = \frac{2}{k^2} \sin 1'' \frac{\alpha'_i - \alpha'_{iii}}{t_{iii} - t_i}; \quad \delta''_0 = \frac{2}{k^2} \sin 1'' \frac{\delta'_i - \delta'_{iii}}{t_{iii} - t_i}.$$

$$\log \frac{\sin 1''}{k} = 6.4499934 - 10; \quad \log \frac{2}{k^2} \sin 1'' = 8.5154420 - 10.$$

$$(\tan \delta)_0' = \sec^2 \delta_{ii} \delta'_0; \quad (\tan \delta)_0'' = \sec^2 \delta_{ii} [2 \tan \delta_{ii} (\delta'_0)^2 + \delta''_0].$$

¹In *extremely rare* cases the geocentric motion may be so irregular as to necessitate laborious subsequent computation by requiring successive differential corrections to the fundamental data ρ_0, x'_0, y'_0, z'_0 based on the velocities and accelerations in α and δ determined from these formulæ. Complications may then be avoided by using shorter intervals. Difficulties may also arise, if the computer is obliged to base the direct solution on long intervals. In such cases more accurate values of $\alpha'_0, \alpha''_0, \delta'_0, \delta''_0$ may be derived by plotting the available coördinates with the dates as abscissæ; then drawing a smooth curve for the right ascensions as well as for the declinations; then forming a table of right ascensions and declinations by taking these coördinates from their respective curves for equidistant dates; and finally determining $\alpha'_0, \alpha''_0, \delta'_0, \delta''_0$ from the tabulated values by numerical differentiation in a manner similar to that used in Ia, Ib, Ic in deriving the solar velocities X'_0 , etc. Three observations may suffice for this purpose, but more should be used if available.

Or more accurate values $\alpha', \alpha'', \delta', \delta''$ for the middle date may be determined by means of TAYLOR'S Theorem by the following method of solution, based on five observations. The scheme proposed here is readily extended to more than five observations.

Let the five right ascensions be $\alpha_1, \alpha_2, \alpha_3 = \alpha_0, \alpha_4, \alpha_5$; the dates $t_1, t_2, t_3 = t_0, t_4, t_5$, with a similar notation for the declinations.

$$i = 1, 2, 4, 5; \quad a_i = \frac{(t_i - t_0)^3}{4!}; \quad b_i = \frac{(t_i - t_0)^2}{3!}; \quad c_i = \frac{(t_i - t_0)}{2!}.$$

$$p_i = b_i, c_i, 1; \quad i = 2, 4, 5; \quad [p_i \cdot 1] = p_i - \frac{a_i}{a_1} p_1.$$

$$p_i = c_i, 1; \quad i = 2, 4; \quad [p_i \cdot 2] = [p_i \cdot 1] \frac{[b_i \cdot 1]}{[b_2 \cdot 1]} [p_2 \cdot 1].$$

$$i = 2, 4; \quad n_{i,a} = \frac{\alpha_i - [\alpha_0 + 2c_i \alpha'_0 + 3b_i \alpha''_0]}{2c_i},$$

where α'_0, α''_0 are derived as usual by the formulæ of this group, except for the factors $\frac{1}{k}$ and $\frac{1}{k^2}$, which are omitted for the present, the provisional unit of time being a mean solar day.

Compute $n_{2,\delta}, n_{4,\delta}$ in a manner analogous to the computation of $n_{2,a}, n_{4,a}$.

$$y_a = \frac{n_{2,a} [1_4 \cdot 2] - n_{4,a} [1_2 \cdot 2]}{[c_2 \cdot 2] [1_4 \cdot 2] - [c_4 \cdot 2] [1_2 \cdot 2]}; \quad z_a = \frac{n_{4,a} [c_2 \cdot 2] - n_{2,a} [c_4 \cdot 2]}{[c_2 \cdot 2] [1_4 \cdot 2] - [c_4 \cdot 2] [1_2 \cdot 2]}.$$

Compute y_δ and z_δ from the same formulæ after replacing $n_{2,a}$ by $n_{2,\delta}$ and $n_{4,a}$ by $n_{4,\delta}$. Then

$$\alpha' = \alpha'_0 + z_a; \quad \alpha'' = \alpha''_0 + y_a; \quad \delta' = \delta'_0 + z_\delta; \quad \delta'' = \delta''_0 + y_\delta.$$

To express these velocities in units of $\frac{1}{k}$ mean solar days, multiply α', δ' by $\frac{1}{k}$ and α'', δ'' by $\left(\frac{1}{k}\right)^2$; $\log \frac{1}{k} = 1.7644186$; $\log \left(\frac{1}{k}\right)^2 = 3.5288372$.

Let \mathfrak{D}_0 be a value of \mathfrak{D} corresponding to $y' = 0$. There will be either one such value \mathfrak{D}_0 or three, according to the number of positive roots. Then for each root

$$z = s \tan \mathfrak{D}_0 + c.$$

First find the region of each possible \mathfrak{D}_0 with the aid of a table of natural tangents and cosines and of a multiplication table, such as CRELLE'S *Rechentafeln*, or with the aid of a mechanism, such as a slide rule, by noting approximate values of $z' = \mathfrak{Z} + c'$ and $y = \eta - q'^2$ for a few values of \mathfrak{D} from \mathfrak{D}_m down to $\psi - 90$ until the value or values of \mathfrak{D} are passed for which $y' = z'^2 - y$ changes sign. This is conveniently accomplished, for instance, with a table like CRELLE'S, by copying from the page headed s , the products $\mathfrak{Z} = s \tan \mathfrak{D}$ increased by c' , c' being written on the edge of a card and held successively over the products in the table. In this manner a small table of z' is formed with \mathfrak{D} as argument. As only rough values are needed, most computers will write down the z'^2 as readily as the z' . In the next column write down the values of $y = \frac{h}{s} \cos \mathfrak{D} - q'^2$ in a similar manner from the page headed $\frac{h}{s}$ using a card on the edge of which the value of q'^2 is written. Then inspection of the z'^2 and y columns readily reveals the approximate value or values \mathfrak{D}_0 for which y' changes sign.¹

Then compute y' for at least one value of \mathfrak{D} greater and less, respectively, than the approximate \mathfrak{D}_0 to the degree of accuracy to which it is desired to find \mathfrak{D}_0 and, therefore, z from the plot.

Finally plot the short arc of the curve $y' = f(\mathfrak{D})$ for each possible root and read off the accurate values of \mathfrak{D}_0 for which $y' = 0$.

2). *By tabulating $y' = f(\mathfrak{D})$.* Proceed as under (1), but instead of plotting $y' = f(\mathfrak{D})$, construct a small table of y' with argument \mathfrak{D} and derive \mathfrak{D}_0 by interpolation.

3). *Graphically.* The roots are given by the values of z' at the intersections of the parabola and curve

$$y = z'^2 \text{ and } y = \eta - q'^2, \quad z' = \mathfrak{Z} + c',$$

where η and \mathfrak{Z} are defined as functions of \mathfrak{D} as under (1).

The parabola is the same for all orbits and may be plotted once for all.

To plot the short arc or arcs of the curve, first find an approximate value of \mathfrak{D}_0 as in (1). Then compute y and z' for at least one value \mathfrak{D} greater and less, respectively, than the approximate \mathfrak{D}_0 to the degree of accuracy to which it is desired to find z' from the plot. Finally plot the short arc or arcs of the curve and read off the values of z' at the intersections of the curve and the parabola. Then

$$z = z' + p'.$$

¹To test the feasibility of a parabolic solution cf. page 290, (a). In case of three roots the two fictitious roots may be rejected by the application of (b), page 291.

4). From the Tables at the end of this Paper, (Part 7).

This method gives the solutions without hypothesis regarding the eccentricity.¹

$$m = -\frac{\kappa}{R^4 \cos \delta_{\kappa}}; \quad (m) = m \left(1 + \frac{3 \Delta_2 R}{R} \right) + \frac{\Delta \kappa}{R \cos \delta_{\kappa}}.$$

With $\frac{1}{m}$ or $\frac{1}{(m)}$ and ψ as arguments interpolate z from the Tables. As the range of practical solutions is often large for preliminary orbits from short arcs, the value of z obtained from the Tables may differ considerably from the value of z corresponding to a parabola. This method of deriving an initial value z_i of z may, therefore, involve several subsequent corrections of z_i , before a value of z is obtained which satisfies the equation of z for the parabola.

Let z_1 be the value of z obtained by any one of the foregoing methods.

$$\mu_1 = (z_1 - c)^2 + s^2; \quad M_1 = (z_1 - p')^2 + q'^2 - \frac{h}{\mu_1^{1/2}}; \quad z_2 = z_1 - \frac{M_1}{2(z_1 - p') + \frac{h}{\mu_1^{1/2}} - \frac{z_1 - c}{\mu_1}}.$$

Continue these approximations until $M=0$. Denote the final value by z , without subscript.

IVb. CIRCULAR ORBITS.

A circular orbit is impossible, if both c and p' are negative; or if the inequalities given on page 306 are fulfilled. The solution must then be performed without assumption regarding the eccentricity by IVc, etc. No criteria regarding the number of solutions, in general, are required.

$$h = \frac{1}{a^2 R^3}; \quad \cos \vartheta_m = a^2 q'^2 s.$$

In all other respects formulæ IVb are the same as formulæ IVa. The result of the solution so far is a *conditioned* ellipse so that the criterion $r'_0 = 0$ must be applied at the end of Vb. (cf. page 306.)

Further, a circular solution may be derived even more easily directly from the general solution IVc by applying in Vc the criteria $r_0 = a$ and $r'_0 = 0$ (cf. page 306).

IVc. GENERAL ORBITS.²

$$m = -\frac{\kappa}{R^4 \cos \delta_{\kappa}}; \quad (m) = m \left(1 + \frac{3 \Delta_2 R}{R} \right) + \frac{\Delta \kappa}{R \cos \delta_{\kappa}}.$$

With $\frac{1}{m}$ or $\frac{1}{(m)}$ and ψ as arguments take $z = z_1$ from the Tables. The several solutions, which can not exceed two in number, are obtained from the Tables.

$$\mu_1 = (z_1 - c)^2 + s^2; \quad (v)_1 = z_1 - (m); \quad \mu_1^2 (v)_1^2 - m^2 = M_1;$$

For check: $M_1 = 0$; or for correction of z_1 , if necessary:

$$z_2 = z_1 - \frac{M_1}{2 \mu_1^2 (v)_1 [\mu_1 + 3 (v)_1 (z_1 - c)]}.$$

Let z , without subscript, be the final value.

¹Attention is called to the last paragraph on page 291 and the second paragraph on page 292.

²It should be remembered that (m) is to be computed only in case of *complete elimination* of the parallax. Otherwise $(m) = m$ and $(v) = v$.

Va, Vb, Vc.

$$\rho_0 = R z; \quad \sigma_0 = \rho_0 \cos \delta_{\alpha}$$

$$\sigma_0'' = \sigma_0 \left[\frac{C_3}{\kappa} - \frac{1}{r_0^2} + (\alpha_0')^2 \right] + c_3 - C_3 \frac{\Delta \kappa}{\kappa},$$

where r_0 is given below.

$$\rho_i = \frac{\sigma_0}{\cos \delta_i} \left[1 - \frac{\lambda}{\kappa} \theta_{\alpha} \right]^1 + \frac{\theta_{\alpha}^2 \sigma_0''}{2 \cos \delta_i}; \quad \rho_{\alpha} = \frac{\sigma_0}{\cos \delta_{\alpha}} \left[1 + \frac{\lambda}{\kappa} \theta_i \right]^1 + \frac{\theta_i^2 \sigma_0''}{2 \cos \delta_{\alpha}}.$$

With ρ_i , ρ_0 , ρ_{α} correct t_i , t_0 , t_{α} for planetary aberration; $\log \alpha = 7.76128$; corrected $t = t - \alpha \rho$. Recompute from the corrected dates

$$\theta_i = k(t_{\alpha} - t_i); \quad \theta_{\alpha} = k(t_i - t_0); \quad \log k = 8.2355814.$$

The computation of σ_0'' and of the terms in ρ_i and ρ_{α} depending on it are practically always superfluous. It may become essential only if the body should be changing the direction of motion in the line of sight, that is, if $\lambda = 0$, and if σ_0'' is large. These conditions may be estimated by inspection.

$$\begin{cases} x_0 = \sigma_0 \cos \alpha_{\alpha} - X_{\alpha}; & y_0 = \sigma_0 \sin \alpha_{\alpha} - Y_{\alpha}; & z_0 = \sigma_0 \tan \delta_{\alpha} - Z_{\alpha}; & r_0^2 = x_0^2 + y_0^2 + z_0^2 \\ \quad = \xi_0 - X_{\alpha}; & \quad = \eta_0 - Y_{\alpha}; & \quad = \zeta_0 - Z_{\alpha} \\ x'_0 = a_x \sigma_0 - X'_0; & y'_0 = a_y \sigma_0 - Y'_0; & z'_0 = a_z \sigma_0 - Z'_0; & r_0 r'_0 = x_0 x'_0 + y_0 y'_0 + z_0 z'_0. \end{cases}$$

Although not required at this point, the elements which determine the character of the orbit may now be computed from

$$G_0^2 = x_0'^2 + y_0'^2 + z_0'^2; \quad \frac{1}{a} = \frac{2}{r_0} - G_0^2; \quad p = r_0^2 [G_0^2 - r_0'^2]; \quad e^2 = \frac{a - p}{a}.$$

For a parabolic orbit $p = 2q = 2r_0 - (r_0 r'_0)^2$, and the computation is partially checked by $G_0^2 = \frac{2}{r_0}$. For the *conditioned* ellipse the check is $a = r_0$, but the criterion $r'_0 = 0$ must also be applied. Both criteria, $a = r_0$ and $r'_0 = 0$, must be applied if the *circular* is to be based on the *general* solution (cf. page 306).

VIa, VIc. PARABOLIC AND GENERAL ORBITS.

$$f_2 = -\frac{1}{2r_0^3}; \quad f_3 = \frac{r'_0}{2r_0^4}; \quad f_4 = \frac{1}{6r_0^6} \left[1 - \left(\frac{15}{4} \frac{(r_0 r'_0)^2}{r_0} + \frac{3}{4} \frac{r_0}{a} \right) \right]$$

$$f_5 = -\frac{r'_0}{2r_0^5} \left[1 - \left(\frac{7}{4} \frac{(r_0 r'_0)^2}{r_0} + \frac{3}{4} \frac{r_0}{a} \right) \right] \dots$$

$$g_3 = -\frac{1}{6r_0^3}; \quad g_4 = \frac{r'_0}{4r_0^4}; \quad g_5 = \frac{1}{12r_0^6} \left[1 - \left(\frac{9}{2} \frac{(r_0 r'_0)^2}{r_0} + \frac{9}{10} \frac{r_0}{a} \right) \right]$$

$$g_6 = -\frac{7}{24} \frac{r'_0}{r_0^5} \left[1 - \left(2 \frac{(r_0 r'_0)^2}{r_0} + \frac{6}{7} \frac{r_0}{a} \right) \right] \dots$$

$$f_i = 1 + \theta_{\alpha}^2 f_2 - \theta_{\alpha}^3 f_3 + \theta_{\alpha}^4 f_4 - \theta_{\alpha}^5 f_5 + \dots; \quad g_i = -\theta_{\alpha} - \theta_{\alpha}^3 g_3 + \theta_{\alpha}^4 g_4 - \theta_{\alpha}^5 g_5 + \theta_{\alpha}^6 g_6 \dots$$

$$f_{\alpha} = 1 + \theta_i^2 f_2 + \theta_i^3 f_3 + \theta_i^4 f_4 + \theta_i^5 f_5 + \dots; \quad g_{\alpha} = \theta_i + \theta_i^3 g_3 + \theta_i^4 g_4 + \theta_i^5 g_5 + \theta_i^6 g_6 \dots$$

¹In case of complete elimination of parallax replace $\frac{\lambda}{\kappa}$ by $\frac{\sigma_0'}{\sigma_0} = \frac{\lambda}{\kappa} - \frac{\beta}{\sigma_0}$.

²In the complete elimination of the parallax use $(X)_{\alpha}$ for X_{α} , etc., and $[X]_0'$ for X_0' , etc.

For parabolic orbits the terms containing a disappear. In general only f_2, f_3 , and g_3 are required. The necessity of including the higher terms $\theta_4 f_4, \theta_4 g_4$, etc., is readily determined by estimating their numerical values.

For unusually long intervals or excessively small values of r_0 the f and g series, as given, may prove insufficient. Then the f and g should be computed from the closed expressions given in B II, page 330, etc.

VIIb. CIRCULAR ORBITS.

The criterion $r' = 0$ in the *conditioned* or the criteria $a = r$ and $r' = 0$ in the *general* ellipse having been verified, the computation of the orbit may be concluded by first deriving the elements $(\Omega), (i), (u)$ referred to the *equator* from the values of $x, y, z; x', y', z'$ for the *conditioned* or the *general* ellipse, as the case may be, and then computing the constants for the equator and the elements referred to the *ecliptic* in one of the usual ways. The formulæ for the elements, therefore, are given here, instead of later, under VIII.

Equator.

$$\sqrt{p} \cos(i) = x_0 y'_0 - y_0 x'_0; \quad r_0 \sin(u_0) = \frac{z_0}{\sin(i)}$$

$$\sqrt{p} \sin(i) \sin(\Omega) = y_0 z'_0 - z_0 y'_0; \quad r_0 \cos(u_0) = x \cos(\Omega) + y \sin(\Omega)$$

$$\sqrt{p} \sin(i) \cos(\Omega) = x_0 z'_0 - z_0 x'_0; \quad r_0^2 = x_0^2 + y_0^2 + z_0^2, \text{ check.}$$

p is to be discarded and, therefore, need not be computed. Let radius $a = r_0$.

$$\sin a \sin(A) = \cos(\Omega); \quad \sin b \sin(B) = \sin(\Omega); \quad (C) = 0$$

$$\sin a \cos(A) = -\sin(\Omega) \cos(i); \quad \sin b \cos(B) = \cos(\Omega) \cos(i); \quad \sin c = \sin(i).$$

$$A' = (A) + (u_0), \quad B' = (B) + (u_0), \quad C' = (C) + (u_0); \quad \alpha = r_0 \sin a; \quad \beta = r_0 \sin b; \quad \gamma = r_0 \sin c.$$

$$\mu = k'' a^{-3/2}; \quad \log k'' = 3.550007; \quad \text{Epoch of } (u_0) = \text{true } t_0. \quad \text{Let } \Delta t = (t - t_0).$$

$$x = \alpha \sin(A' + \mu \Delta t); \quad y = \beta \sin(B' + \mu \Delta t); \quad z = \gamma \sin(C' + \mu \Delta t).$$

Ecliptic.

$$\sin m \sin M = \sin(i) \cos(\Omega);$$

$$\sin i \sin \Omega = \sin(i) \sin(\Omega)$$

$$\sin m \cos M = \cos(i);$$

$$\sin i \cos \Omega = \sin m \sin(M - \epsilon)$$

$$\sin n \sin N = \sin(i);$$

$$\sin i \sin \sigma = \sin \epsilon \sin(\Omega)$$

$$\sin n \cos M = \cos(i) \cos(\Omega);$$

$$\sin i \cos \sigma = \sin n \sin(N - \epsilon)$$

$$\cos i = \sin m \cos(M - \epsilon);$$

$$u_0 = (u_0) - \sigma.$$

$\epsilon =$ obliquity of the ecliptic at the beginning of the year.

VIa, VIb, VIc—Continued.

$$\rho, \cos \delta, \cos \alpha = X_i + f_i x_0 + g_i x'_0 = \mathcal{E}_i;$$

$$\rho_{iii} \cos \delta_{iii} \cos \alpha_{iii} = X_{iii} + f_{iii} x_0 + g_{iii} x'_0 = \mathcal{E}_{iii}$$

$$\rho, \cos \delta, \sin \alpha = Y_i + f_i y_0 + g_i y'_0 = \eta_i;$$

$$\rho_{iii} \cos \delta_{iii} \sin \alpha_{iii} = Y_{iii} + f_{iii} y_0 + g_{iii} y'_0 = \eta_{iii}$$

$$\rho, \sin \delta = Z_i + f_i z_0 + g_i z'_0 = \mathcal{Z}_i;$$

$$\rho_{iii} \sin \delta_{iii} = Z_{iii} + f_{iii} z_0 + g_{iii} z'_0 = \mathcal{Z}_{iii}.$$

For VIb use $x = \alpha \sin(A' + \mu \Delta t)$, etc., in place of $f x_0 + g x'_0$, etc.

¹If the orbit is to be made the basis of a differential correction the necessary values of x'_0, y'_0, z'_0 are obtained from $x'_0 = a^{-1/2} \sin a \sin A'$; $y'_0 = a^{-1/2} \sin b \sin B'$; $z'_0 = a^{-1/2} \sin c \sin C'$.

Use here the geocentric solar coördinates and correct the first and third observed places for parallax on the basis of the resulting values of ρ , and ρ_{III} ¹.

Form the residuals, $(O - C)$, $\partial\alpha = \partial\alpha \cos \delta$, $\partial\delta$, $\partial\alpha_{\text{III}} = \partial\alpha_{\text{III}} \cos \delta_{\text{III}}$, $\partial\delta_{\text{III}}$. If these residuals are sufficiently small for the purpose in hand, the elements and an ephemeris may be computed at once by VIII. If it be intended, however, to represent the observations more accurately the necessary corrections to ρ_0 , x'_0 , y'_0 , z'_0 may be determined by means of [VII].

[VIIa], [VIIb], [VIIc].

For the correction of an initial parabolic or general orbit the differential formulæ of this group are based on the ∂f and ∂g series. These are usually sufficient for the removal of any residuals that may remain after the direct solution I—VI. They include terms of the order θ^3 multiplied by a variation. The differential formulæ based on *closed expressions* for ∂f and ∂g are given in *B*. But for the correction of an initial *circular* orbit the *closed* expressions for ∂f and ∂g are adopted in this group as well as in *B*.

For the correction of circular orbits the procedure is, therefore, the same in all cases [VIIb].

For the correction of an initial parabolic or general orbit apply either the formulæ of this group—with or without the supplementary terms—or the closed formulæ *B*, page 327, in accordance with the following criterion:

If in the computation of the f and g series by VIa, or VIc, as the case may be, terms containing powers of θ higher than θ^4 are negligible, apply [VIIa] or [VIIc]. Otherwise apply *B*, page 327.

Before adopting any values of ρ_0 , x'_0 , y'_0 , z'_0 as final, compute and test the residuals by VI. As a further check the observations may later also be reproduced from the constants for the equator.

A second application of the differential formulæ will be required only if linear relations do not prevail, that is, for unusually large initial residuals. This condition will be revealed by the residuals remaining after the first differential correction. In the second differential correction it is generally sufficient to recompute only those auxiliary quantities which depend on the new residuals.

As soon as $\partial\rho_0$ (or in case of improvement of a *conditioned* elliptic or parabolic orbit a first approximation of $\partial\rho_0$) has been found, test the applicability of the differential relations to the correction of the fundamental quantities ρ_0 , x'_0 , y'_0 , z'_0 . If

$$\partial\rho_0 > \frac{1}{\sqrt{2}} \frac{r_0}{\sin(\beta - 45^\circ)},$$

then the differential formulæ will not converge and must be abandoned. In such

¹If the parallax has been eliminated *completely*, so that $(X)_i$; $(X)_{\text{III}}$, etc., are available, these corrected solar coördinates may be used here, but then the observations require no further correction in the derivation of the residuals.

cases, which, however, are not to be expected in practice, if proper precautions have been taken in the derivation of the initial velocities and accelerations, recourse must be had to the method of arbitrary variation.

If $\partial\rho_0$ be less than the foregoing limiting value but comparable in magnitude to the same, then the convergence of the differential formulæ will be slow. In that case compute the auxiliary terms given in footnote (1) on page 323 as soon as $\partial\rho_0$, $\partial x'_0$, $\partial y'_0$, $\partial z'_0$, $\partial r'_0$ have been found. In *no other case* are these auxiliary terms to be calculated.

[VIIb]. FROM CIRCULAR ORBITS.

$$\cos \beta = \frac{\rho_0 - R_0 \cos \psi}{r_0}.$$

$$d = \frac{3 \cos \beta}{2a}; \quad c = -\frac{\theta_{00} g_0}{a^2}; \quad c' = \theta_{00} f_0 + g_0; \quad c_{00} = \frac{\theta_0 g_{00}}{a^3}; \quad c'_{00} = -(\theta_0 f_{00} - g_{00}).$$

$$f_{x_0} = f_0 \frac{\xi_0}{\rho_0} + d [c_0 x_0 + c'_0 x'_0]; \quad f_{x_{00}} = f_{00} \frac{\xi_0}{\rho_0} + d [c_{00} x_0 + c'_{00} x'_0]$$

$$f_{y_0} = f_0 \frac{\eta_0}{\rho_0} + d [c_0 y_0 + c'_0 y'_0]; \quad f_{y_{00}} = f_{00} \frac{\eta_0}{\rho_0} + d [c_{00} y_0 + c'_{00} y'_0]$$

$$f_{z_0} = f_0 \frac{z_0}{\rho_0} + d [c_0 z_0 + c'_0 z'_0]; \quad f_{z_{00}} = f_{00} \frac{z_0}{\rho_0} + d [c_{00} z_0 + c'_{00} z'_0].$$

[VIIa], [VIIc]. FROM PARABOLIC OR ELLIPTIC ORBITS.

$$\cos \beta = \frac{\rho_0 - R_0 \cos \psi}{r_0};$$

$$f_{x_0} = f_0 \frac{\xi_0}{\rho_0} + \frac{\cos \beta}{2r_0^2} (3\theta_{00}^2 x_0 - \theta_{00}^3 [x'_0 - \frac{4r'_0 x_0}{r_0}]); \quad f_{x_{00}} = f_{00} \frac{\xi_0}{\rho_0} + \frac{\cos \beta}{2r_0^2} (3\theta_{00}^2 x_0 + \theta_{00}^3 [x'_0 - \frac{4r'_0 x_0}{r_0}])$$

$$f_{y_0} = f_0 \frac{\eta_0}{\rho_0} + \frac{\cos \beta}{2r_0^2} (3\theta_{00}^2 y_0 - \theta_{00}^3 [y'_0 - \frac{4r'_0 y_0}{r_0}]); \quad f_{y_{00}} = f_{00} \frac{\eta_0}{\rho_0} + \frac{\cos \beta}{2r_0^2} (3\theta_{00}^2 y_0 + \theta_{00}^3 [y'_0 - \frac{4r'_0 y_0}{r_0}])$$

$$f_{z_0} = f_0 \frac{z_0}{\rho_0} + \frac{\cos \beta}{2r_0^2} (3\theta_{00}^2 z_0 - \theta_{00}^3 [z'_0 - \frac{4r'_0 z_0}{r_0}]); \quad f_{z_{00}} = f_{00} \frac{z_0}{\rho_0} + \frac{\cos \beta}{2r_0^2} (3\theta_{00}^2 z_0 + \theta_{00}^3 [z'_0 - \frac{4r'_0 z_0}{r_0}]).$$

[VIIa], [VIIb], [VIIc]—Continued.

$$A_0 = -\frac{1}{\rho_0} [\cos \alpha_0 f_{y_0} - \sin \alpha_0 f_{x_0}] + \cos \delta_0 \frac{(p_0 \rho_0)_0}{\rho_0^2};$$

$$B_0 = -\frac{1}{\rho_0} [\sin \delta_0 (\sin \alpha_0 f_{y_0} + \cos \alpha_0 f_{x_0}) - \cos \delta_0 f_{z_0}] + \frac{(p_0 \rho_0)_0}{\rho_0^2}; \quad C_0 = \frac{g_0}{\rho_0};$$

$$A_{00} = -\frac{1}{\rho_{00}} [\cos \alpha_{00} f_{y_{00}} - \sin \alpha_{00} f_{x_{00}}] + \cos \delta_{00} \frac{(p_0 \rho_0)_{00}}{\rho_{00}^2};$$

$$B_{00} = -\frac{1}{\rho_{00}} [\sin \delta_{00} (\sin \alpha_{00} f_{y_{00}} + \cos \alpha_{00} f_{x_{00}}) - \cos \delta_{00} f_{z_{00}}] + \frac{(p_0 \rho_0)_{00}}{\rho_{00}^2}; \quad C_{00} = \frac{g_{00}}{\rho_{00}}.$$

[VIIa], [VIIb], [VIIc]—Continued.

$$\begin{aligned}
P_x &= \frac{C_{xxx} \cos \alpha_{xxx} \partial_i \alpha_i - C_i \cos \alpha_i \partial_i \alpha_{xxx}}{C_i C_{xxx} \sin (\alpha_{xxx} - \alpha_i)} ; & Q_x &= \frac{A_i C_{xxx} \cos \alpha_{xxx} - A_{xxx} C_i \cos \alpha_i}{C_i C_{xxx} \sin (\alpha_{xxx} - \alpha_i)} \\
P_y &= \frac{C_{xxx} \sin \alpha_{xxx} \partial_i \alpha_i - C_i \sin \alpha_i \partial_i \alpha_{xxx}}{C_i C_{xxx} \sin (\alpha_{xxx} - \alpha_i)} ; & Q_y &= \frac{A_i C_{xxx} \sin \alpha_{xxx} - A_{xxx} C_i \sin \alpha_i}{C_i C_{xxx} \sin (\alpha_{xxx} - \alpha_i)} \\
P_{z_i} &= \frac{\partial \delta_i + C_i \sin \delta_i (\cos \alpha_i P_x + \sin \alpha_i P_y)}{C_i \cos \delta_i} ; & Q_{z_i} &= \frac{B_i + C_i \sin \delta_i (\cos \alpha_i Q_x + \sin \alpha_i Q_y)}{C_i \cos \delta_i} \\
P_{z_{xxx}} &= \frac{\partial \delta_{xxx} + C_{xxx} \sin \delta_{xxx} (\cos \alpha_{xxx} P_x + \sin \alpha_{xxx} P_y)}{C_{xxx} \cos \delta_{xxx}} ; & Q_{z_{xxx}} &= \frac{B_{xxx} + C_{xxx} \sin \delta_{xxx} (\cos \alpha_{xxx} Q_x + \sin \alpha_{xxx} Q_y)}{C_{xxx} \cos \delta_{xxx}} .
\end{aligned}$$

A parabolic, circular, or general orbit, may now be attempted irrespective of the character of the initial orbit.

For the computation of a parabola or conditioned ellipse a single value must be adopted for the P_x as well as for the Q_x . If any possible subsequent discrepancy in the parabolic or circular orbit is to be concentrated in the residual of the declination of the first place, then let $P_x = P_{z_{xxx}}$; $Q_x = Q_{z_{xxx}}$. If it be decided, however, that the final residuals be distributed between the first and third declinations, then this may be accomplished by forming mean values P_x and Q_x from P_{z_i} , $P_{z_{xxx}}$ and Q_{z_i} , $Q_{z_{xxx}}$ by assigning to the latter such arbitrary weights as may seem most expedient. In order to obtain equal residuals for the first and third declinations, the weights must be based on the intervals and the rate of variation of $\tan \delta_i$ and $\tan \delta_{xxx}$. In general, the ephemeris will hold better by concentrating any discrepancy in the first declination.

[VIIa]. FOR THE DERIVATION OF PARABOLIC ORBITS.

If not computed in Va, obtain

$$a_0^2 = (x_0'^2 + y_0'^2 + z_0'^2) ; \quad \frac{1}{a_0} = \frac{2}{r_0} - a_0^2 .$$

If the direct solution or the initial orbit was parabolic, then $\frac{1}{a_0}$ will be a small quantity comparable to the inaccuracy of the computation.

$$\begin{aligned}
Q &= \frac{\cos \beta}{r_0^2} - [x_0' Q_x + y_0' Q_y + z_0' Q_z] ; & P &= \frac{1}{2 a_0} - [x_0' P_x + y_0' P_y + z_0' P_z] \\
Q \partial \rho_0 &= P - \frac{1}{2} [(\partial x_0')^2 + (\partial y_0')^2 + (\partial z_0')^2] - \frac{1}{2} \frac{3 \cos^2 \beta}{r_0^3} (\partial \rho_0)^2 + \frac{3 \cos \beta (1 - \frac{1}{2} \cos^2 \beta)}{2 r_0^4} (\partial \rho_0)^3 \dots
\end{aligned}$$

As a first approximation: $\partial \rho_0 = \frac{P}{Q}$;

$$\partial x_0' = P_x - Q_x \partial \rho_0 ; \quad \partial y_0' = P_y - Q_y \partial \rho_0 ; \quad \partial z_0' = P_z - Q_z \partial \rho_0 .$$

Substitute these approximate values of $\partial \rho_0$, $\partial x_0'$, $\partial y_0'$, $\partial z_0'$ in the right-hand member

[VIIa]. TEST OF THE POSSIBILITY OF A PARABOLIC ORBIT.

With the final value of $\partial\rho_0$ compute

$$\partial z'_0 = P_z - Q_z \partial\rho_0; \quad \partial z'_0 = P_{z_{\infty}} - Q_{z_{\infty}} \partial\rho_0.$$

If the two values of $\partial z'_0$ agree within the limits of accuracy of the numerical computation, then the observations will be represented by a parabolic orbit.

If they disagree, the resulting residuals may be obtained at once by subtracting the residuals computed from

$$(\partial\delta)_c = B_z \partial\rho_0 - C_z \sin\delta_z \cos\alpha_z \partial x'_0 - C_z \sin\delta_z \sin\alpha_z \partial y'_0 + C_z \cos\delta_z \partial z'_0$$

$$(\partial\delta_{\infty})_c = B_{z_{\infty}} \partial\rho_0 - C_{z_{\infty}} \sin\delta_{z_{\infty}} \cos\alpha_{z_{\infty}} \partial x'_0 - C_{z_{\infty}} \sin\delta_{z_{\infty}} \sin\alpha_{z_{\infty}} \partial y'_0 + C_{z_{\infty}} \cos\delta_{z_{\infty}} \partial z'_0$$

from the initial residuals $\partial\delta_z$ and $\partial\delta_{z_{\infty}}$. The value of $\partial z'_0$ to be used in these formulæ for $\partial\delta_z$ and $\partial\delta_{z_{\infty}}$ is the value derived from $\partial z'_0 = P_z - Q_z \partial\rho_0$. If P_z , Q_z were assumed equal to $P_{z_{\infty}}$, $Q_{z_{\infty}}$, then $\partial\delta_{z_{\infty}} - (\partial\delta_{z_{\infty}})_c$ will be equal to zero, provided that no error has been committed in the computation of the differential corrections. If, then, $\partial\delta_z - (\partial\delta_z)_c$ is sufficiently small the corrections to ρ_0 , x'_0 , y'_0 , z'_0 , may be adopted as final.

If P_z and Q_z represent weighted means then neither $\partial\delta_z - (\partial\delta_z)_c$ nor $\partial\delta_{z_{\infty}} - (\partial\delta_{z_{\infty}})_c$ will be equal to zero, but if these residuals are sufficiently small, then the corrections to ρ_0 , x'_0 , y'_0 , z'_0 may be adopted as final.

If the representation should be unsatisfactory, then the attempt at a parabolic orbit in general should be abandoned and instead the corrections should be determined without regard to the eccentricity, as in [VIIc], etc.

The parabola, however, should not as yet be abandoned, if the residuals after the differential correction appear to be due to the insufficiency of the *linear* differential relations adopted in the differential formulæ.

[VIIc]. FOR THE DERIVATION OF ORBITS WITHOUT ASSUMPTION REGARDING THE ECCENTRICITY.¹

$$\partial\rho_0 = \frac{P_{z_{\infty}} - P_z}{Q_{z_{\infty}} - Q_z}; \quad \partial x'_0 = P_x - Q_x \partial\rho_0; \quad \partial y'_0 = P_y - Q_y \partial\rho_0; \quad \partial z'_0 = P_{z_{\infty}} - Q_{z_{\infty}} \partial\rho_0 = P_z - Q_z \partial\rho_0.$$

[VIIa], [VIIb], [VIIc].

The corrected heliocentric coördinates and velocities are

$$x_0 + \varepsilon_0 \frac{\partial\rho_0}{\rho_0}; \quad y_0 + \eta_0 \frac{\partial\rho_0}{\rho_0}; \quad z_0 + \zeta_0 \frac{\partial\rho_0}{\rho_0}; \quad x'_0 + \partial x'_0; \quad y'_0 + \partial y'_0; \quad z'_0 + \partial z'_0.$$

With these compute r_0 and r'_0 as in V and then the new residuals by VI. If these prove satisfactory, then compute the elements by VIII.

¹See footnote on preceding page.

2). Or compute first the elements defining the position of the orbit plane with reference to the *equator* and from them the constants for the *equator*.

Equator.

$$\begin{aligned}\sqrt{p} \cos(i) &= xy' - yx'; & r \sin(u) &= \frac{z}{\sin(i)} \\ \sqrt{p} \sin(i) \sin(\Omega) &= yz' - zy'; & r \cos(u) &= x \cos(\Omega) + y \sin(\Omega) \\ \sqrt{p} \sin(i) \cos(\Omega) &= xz' - zx'; & r^2 &= x^2 + y^2 + z^2, \text{ check.}\end{aligned}$$

ϕ should check with its previously computed value.

$$\begin{aligned}\sin a \sin(A) &= \cos(\Omega); & \sin b \sin(B) &= \sin(\Omega); & (C) &= 0 \\ \sin a \cos(A) &= -\sin(\Omega) \cos(i); & \sin b \cos(B) &= \cos(\Omega) \cos(i); & \sin c &= \sin(i): \\ (\omega) &= (u) - r; & A' &= (A) + (\omega), & B' &= (B) + (\omega), & C' &= (\omega).\end{aligned}$$

$$\begin{aligned}\sin \frac{1}{2} i \sin \frac{1}{2} [\Omega + \sigma] &= \sin \frac{1}{2} [(i) + \epsilon] \sin \frac{1}{2} (\Omega) \\ \sin \frac{1}{2} i \cos \frac{1}{2} [\Omega + \sigma] &= \sin \frac{1}{2} [(i) - \epsilon] \cos \frac{1}{2} (\Omega) \\ \cos \frac{1}{2} i \sin \frac{1}{2} [\Omega - \sigma] &= \cos \frac{1}{2} [(i) + \epsilon] \sin \frac{1}{2} (\Omega) \\ \cos \frac{1}{2} i \cos \frac{1}{2} [\Omega - \sigma] &= \cos \frac{1}{2} [(i) - \epsilon] \cos \frac{1}{2} (\Omega),\end{aligned}$$

where e = obliquity of the ecliptic. Another set of formulæ for the transformation of the elements from the equator to the ecliptic is given under [VII b]: $m \sin M$, etc.

$$\omega = (\omega) - \sigma; \quad \pi = \Omega + \omega,$$

where i , Ω , ω , π are referred to the *ecliptic*.

<i>Ellipse.</i>	<i>Parabola.</i>	<i>Hyperbola.</i>
$\tan \frac{1}{2} E = \sqrt{\frac{1-e}{1+e}} \tan \frac{1}{2} v$ $M = E - \frac{e}{\sin 1''} \sin E$ $\mu = k'' a^{-\frac{3}{2}}$ $\log k'' = 3.550\,006\,6$ $\text{Epoch} = \text{true } t_{\mu}.$	<p>With r as argument, take M_{μ} from OP- POLZER, vol. I, table IV.</p> $T = \text{true } t_{\mu} - M_{\mu} q^{\frac{3}{2}}.$	$\tan \frac{1}{2} F = \sqrt{\frac{e-1}{e+1}} \tan \frac{1}{2} v$ $T = \text{true } t_{\mu} - \frac{(-a)^{\frac{3}{2}}}{k} \left[e \tan F - \frac{\log \tan (45^{\circ} + \frac{1}{2} F)}{\text{Mod.}} \right].$

An ephemeris may be computed in the usual manner from the constants for the equator or by means of the series for f and g as in VI, or by means of the closed expressions for f and g as in B.

B. THE DERIVATION OF AN ORBIT FROM THREE OBSERVATIONS ON THE BASIS OF A PREVIOUS APPROXIMATION.

I.

The derivation of an orbit from three observations on the basis of a previous approximation (preliminary orbit) is accomplished either by formulæ A [VII] or by the formulæ of this group.

Formulæ A [VII] are the more convenient of the two groups. They are based on the series for ∂f and ∂g and are applicable to moderate intervals and if the heliocentric distance r_0 at the middle date is not excessively small, that is, in general if in the f and g series A VI, the terms containing higher powers of θ than θ^4 are negligible.

Formulæ A [VII], therefore, in general have a wider range of applicability for minor planets than for comets.

The formulæ of this group are based on the closed expressions for ∂f and ∂g and are applicable to intervals of any length as well as to small values of r_0 .

Both groups of formulæ are applicable to the derivation of an orbit with or without assumption regarding the eccentricity on the basis of a preliminary orbit of any eccentricity and readily permit of the transition from one class of orbit to another for the purpose of improving the representation of the observations.

While the possibility of obtaining a hyperbolic orbit from a preliminary parabolic or nearly parabolic orbit is directly included, no special formulæ are given for basing the computation on an *initial* hyperbolic orbit. Since, however, the eccentricity of a hyperbolic orbit rarely differs considerably from unity, the differential formulæ for the correction of an initial parabolic orbit will prove sufficient for the correction of a possible hyperbolic orbit.

The method to be chosen for applying the corrections for parallax and aberration will depend on the accuracy of the geocentric distances which the preliminary orbit may yield for the dates of the new orbit. This may be judged by a rough comparison of the preliminary orbit with the observations.

If it be deemed possible to derive from the preliminary orbit values of the geocentric distances that are close enough to yield the *final* values of the corrections for parallax and planetary aberration, these corrections should be applied in advance. The observations are then to be reduced to the beginning of the year by the usual reduction formulæ, *exclusive* of the aberration terms, and the solar coördinates X , Y , Z are to be interpolated, also for the beginning of the year, from an astronomical ephemeris for the dates corrected for planetary aberration, that is, for the *true* dates.

In case of doubt concerning the accuracy of the geocentric distances of the

preliminary orbit, the *geocentric* parallax and aberration should be *eliminated* by correcting the solar coördinates for the middle date by

$$\begin{aligned}\Delta X_u &= (p_a \rho)_u \sin \alpha_u \cos \delta_u + (p_s \rho)_u \cos \alpha_u \sin \delta_u; \quad \Delta Y_u = -(p_a \rho)_u \cos \alpha_u \cos \delta_u + (p_s \rho) \sin \alpha_u \sin \delta_u, \\ \Delta Z_u &= -(p_s \rho)_u \cos \delta_u,\end{aligned}$$

and by using the solar coördinates at the beginning of the year interpolated for the *uncorrected* dates together with the observed places reduced to the beginning of the year by the usual reduction formulæ, *inclusive* of the aberration terms.

Let the data resulting from either of the foregoing methods of reduction be

$$\alpha_i, \delta_i; \quad \alpha_u, \delta_u; \quad \alpha_m, \delta_m; \quad X_i, Y_i, Z_i; \quad X_u, Y_u, Z_u; \quad X_m, Y_m, Z_m.$$

$$R \cos D \cos A = X_u; \quad R \cos D \sin A = Y_u; \quad R \sin D = Z_u; \quad \cos \psi = \sin D \sin \delta_u + \cos D \cos \delta_u \cos (A - \alpha_u).$$

Derive initial values of the fundamental quantities ρ_0, x'_0, y'_0, z'_0 from the preliminary orbit:

The number of decimals to which these quantities should be computed should correspond to the number of decimals to which they may be considered to agree with their physical values. But as only approximate values are required, slight arbitrary changes may be made in them to suit the convenience of the computer. Hence, although theoretically the initial fundamental data x'_0, y'_0, z'_0 should always correspond to the true date t_0 , while ρ_0 should correspond to the true date only if the observations have been corrected in advance for parallax and aberration, this distinction, if inconvenient, may be disregarded.

In deriving the fundamental quantities any data that may be available from the numerical work on the preliminary orbit in addition to the elements and the constants for the equator should be used for the simplification of the computation.

Thus ρ_0 for the apparent or the true middle date, as the case may be, is frequently directly given by an ephemeris.

If the numerical work on the ephemeris be available, the velocities x'_0, y'_0, z'_0 for the true date may be derived from the heliocentric coördinates of the ephemeris computation by means of the formulæ for numerical differentiation given in A I.

If the constants for the equator are not available, but if the middle date is covered by an ephemeris giving α, δ, ρ , then the velocities of $\alpha'_0, \delta'_0, \rho'_0$ may be obtained by numerical differentiation and then

$$\begin{aligned}(\tan \delta)_0'' &= \sec^2 \delta_u \delta'_0; \quad \sigma'_0 = \cos \delta_u \rho'_0 - \rho_0 \sin \delta_u \delta'_0; \quad \sigma_0 = \rho_0 \cos \delta_u, \\ x'_0 &= \cos \alpha_u \sigma'_0 - \sin \alpha_u \sigma_0 \alpha'_0 - X'_0; \quad y'_0 = \sin \alpha_u \sigma'_0 + \cos \alpha_u \sigma_0 \alpha'_0 - Y'_0; \quad z'_0 = \tan \delta_u \sigma'_0 + \sigma_0 (\tan \delta)_0' - Z'_0,\end{aligned}$$

where X'_0, Y'_0, Z'_0 are to be obtained by numerical differentiation as in A I.

In the majority of cases, only the elements and the constants for the equator are available.

Then derive v_u from the elements (or u_u from a circular orbit) in the usual

manner according to the eccentricity of the preliminary orbit. (If the preliminary orbit is *very nearly* parabolic, let $e = 1$, $q = \frac{p}{2}$ and proceed as in case of a parabolic orbit.)

Let t_0 be the epoch of the preliminary orbit, T the date of perihelion passage.

<i>Ellipse</i>	<i>Parabola</i>	<i>Circle</i>
$E'' = \frac{e}{\sin 1''} \sin E'' = M_0 + (t'' - t_0) \mu$	$M'' = \frac{t'' - T}{q^{3/2}}$	$u'' = (t'' - t_0) \mu$
$\tan \frac{1}{2} v'' = \sqrt{\frac{1+e}{1-e}} \tan \frac{1}{2} E''$	with M'' as argument take v'' from OPPOLZER, Vol. I, Table IV.	$z'' = a \sin e \sin (C' + u'')$
$r'' = a (1 - e \cos E'')$	$r'' = q \sec^2 \frac{1}{2} v''$	
$z'' = r'' \sin e \sin (C' + v'')$	$z'' = r'' \sin e \sin (C' + v'')$	
	$\rho_0 = \frac{z'' + Z''}{\sin \delta''}$	
<i>Ellipse</i>	<i>Parabola</i>	<i>Circle</i>
$x'_0 = \frac{\sin a}{\sqrt{p}} [\cos (A' + v'') + e \cos A']$	$x'_0 = \sqrt{\frac{2}{q}} \sin a \cos (A' + \frac{1}{2} v'') \cos \frac{1}{2} v''$	$x'_0 = \frac{\sin a \cos (A' + v'')}{a^{1/2}}$
$y'_0 = \frac{\sin b}{\sqrt{p}} [\cos (B' + v'') + e \cos B']$	$y'_0 = \sqrt{\frac{2}{q}} \sin b \cos (B' + \frac{1}{2} v'') \cos \frac{1}{2} v''$	$y'_0 = \frac{\sin b \cos (B' + v'')}{a^{1/2}}$
$z'_0 = \frac{\sin c}{\sqrt{p}} [\cos (C' + v'') + e \cos C']$	$z'_0 = \sqrt{\frac{2}{q}} \sin c \cos (C' + \frac{1}{2} v'') \cos \frac{1}{2} v''$	$z'_0 = \frac{\sin c \cos (C' + v'')}{a^{1/2}}$

Compute ρ_i and ρ_{iii} in the same manner as ρ_0 to free the intervals from aberration before computing the f and g . These values of ρ_i and ρ_{iii} will also serve for correcting the observations in advance for parallax and aberration, if the preliminary orbit is sufficiently accurate for that purpose.

The adopted values of ρ_0, x'_0, y'_0, z'_0 must now be written out to the number of decimals to which the orbit is to be computed, by adding ciphers if necessary.

$$\begin{aligned} \sigma_0 &= \rho_0 \cos \delta''; & x_0 &= \sigma_0 \cos \alpha'' - X''; & y_0 &= \sigma_0 \sin \alpha'' - Y''; & z_0 &= \sigma_0 \tan \delta'' - Z'' \\ & & &= X_0 - X''; & &= Y_0 - Y''; & &= Z_0 - Z''. \end{aligned}$$

$$r_0^2 = x_0^2 + y_0^2 + z_0^2; \quad G_0^2 = x_0'^2 + y_0'^2 + z_0'^2; \quad \frac{1}{a} = \frac{2}{r_0} - G_0^2.$$

$x_0, y_0, z_0; x'_0, y'_0, z'_0$ are the constants (or elements) of an artificial *initial* orbit which has ρ_0, x'_0, y'_0, z'_0 in common with the preliminary orbit, but accurately represents α'' and δ'' .

The eccentricity of the *initial* orbit, therefore, is not necessarily the same as that of the preliminary orbit. Nevertheless, if the latter was parabolic or circular

¹With ρ_0 thus obtained from the *observed* time we may correct t'' for planetary aberration and with the reduced time $t'' = t'' - \alpha \rho_0$ ($\log \alpha = 7.76128$) we may compute a second approximation of v'' and ρ_0 . Then if the parallax and aberration have been eliminated, we may choose as the fundamental data the first approximation of ρ_0 together with the values of x', y', z' corresponding to the second approximation of v'' , but if the corrections for parallax and aberration have been applied in advance, then all four fundamental data should correspond to the second approximation of v'' . But, as stated above, these distinctions are not important. In any case x', y', z' are to be computed but once.

the *differential formulæ* for the correction of an initial parabolic or circular orbit will serve for the correction of the adopted fundamental data.

But in such cases one or more (for the circle) of the heliocentric velocities may first be changed arbitrarily so as to furnish a value of G_0 which will make $\frac{1}{a} = 0$, or $a = r_0$ and $r'_0 = 0$, so that the initial orbit is changed into a strictly parabolic or a strictly circular orbit, as the case may be. To produce an *initial* parabola it is most convenient to change z'_0 so as to satisfy the equation $x_0'^2 + y_0'^2 + z_0'^2 = \frac{2}{r_0}$. To produce an *initial* circle let $a = r_0$; $r'_0 = 0$ and compute x'_0, y'_0, z'_0 as in footnote 1, page 319. The practical gain arising from this arbitrary variation is that the computation of the f and g by their closed expressions is simplified since the application of the more complicated expressions for very nearly parabolic or for hyperbolic orbits is avoided. Nevertheless, to provide for the case that an arbitrary variation of the heliocentric velocities has not been made or that the required arbitrary change would materially alter the form of the initial orbit, the closed expressions for f and g in nearly parabolic and in hyperbolic orbits are included below.

II.

Compute f, g, f'', g'' in strict accordance with the *adopted initial* fundamental data or orbit.

If these may be computed from the f and g series, apply the proper expressions given in A VI.

If it be necessary to derive the f and g from closed expressions, compute, except for circular orbits,

$$\begin{aligned} \text{Epoch} &= t_u; & r_0 r'_0 &= x_0 x'_0 + y_0 y'_0 + z_0 z'_0; & e \sin r_0 &= r'_0 \sqrt{p} \\ \log k'' &= 3.550\,006\,6; & p &= r_0^2 (r_0^2 - r_0'^2); & e \cos r_0 &= \frac{p}{r_0} - 1 \\ \mu &= k'' a^{-\frac{3}{2}}; & e^2 &= 1 - \frac{p}{a}, \text{ check,} \end{aligned}$$

and then in the usual manner, except where special formulæ are given, according to the character of the *initial* orbit, where for the sake of uniformity $f, f''; g, g''$ are computed from the corresponding γ, γ'' for all classes of orbits,¹ except for the circle:

$$\text{Circle: } 2\bar{\gamma}_i = -\frac{\theta'''}{a^{3/2}}, 2\bar{\gamma}_{iii} = \frac{\theta'''}{a^{3/2}}; f_i = \cos 2\bar{\gamma}_i, f_{iii} = \cos 2\bar{\gamma}_{iii}; \gamma_i = a^{3/2} \sin 2\bar{\gamma}_i, g_{iii} = a^{3/2} \sin 2\bar{\gamma}_{iii}.$$

$$\text{Parabola: } r_0, q, T, r_i, r_{iii}; \quad r_i, r_{iii}; \quad \gamma_i = \sqrt{r_i - q} - \sqrt{r_0 - q}; \quad \gamma_{iii} = \sqrt{r_{iii} - q} - \sqrt{r_0 - q},$$

where the algebraical sign of each square root is the same as that of the corresponding $\tan \frac{1}{2} v$.

¹But $f, f''; g, g''$ may also be computed directly from formulæ (8) and (9), page 248, except for parabolic or nearly parabolic orbits for which g, g'' may be computed directly from formula (17), page 249. Careful attention should be paid to the quadrants of the angles involved, particularly, if the observations are of different oppositions.

$$\text{Ellipse: } r_0, E_0, M_0; \quad \mu, t_0, E_1, E_{III}; \quad r_1, r_{III}; \quad 2g_1 = E_1 - E_0, \quad 2g_{III} = E_{III} - E_0 \\ \gamma_1 = \sqrt{2a \sin \bar{g}_1}, \quad \gamma_{III} = \sqrt{2a \sin \bar{g}_{III}}.$$

$$\text{Hyperbola: } F_0, T; F_1, F_{III}; \quad r_1, r_{III}; \quad r_1, r_{III};$$

$$\gamma_1 = \sqrt{\frac{2r_1 r_0}{p} \sin \frac{1}{2}(v_1 - v_0)}, \quad \gamma_{III} = \sqrt{\frac{2r_{III} r_0}{p} \sin \frac{1}{2}(v_{III} - v_0)}.$$

$$\text{Very nearly parabolic orbits: } \theta_0, P_1, P_3, T; \quad \varepsilon, \alpha, \beta; \quad M_1, x_1, u_1, r_1, \theta_1, v_1; \quad M_{III}, x_{III}, u_{III}, r_{III}, \theta_{III}, v_{III}.$$

$$\gamma_1 = \sqrt{\frac{2r_1 r_0}{p} \sin \frac{1}{2}(v_1 - v_0)}, \quad \gamma_{III} = \sqrt{\frac{2r_{III} r_0}{p} \sin \frac{1}{2}(v_{III} - v_0)}.$$

The usual formulæ for computing the true anomalies appear in A VIII, except for nearly parabolic orbits for which *cf.* OPPOLZER, Vol. I, pages 73, 75. The *radii vectors* may be computed from the polar equation $r = \frac{p}{1 + e \cos v}$, or for the ellipse from $r = a(1 - e \cos E)$, for the parabola from $r = q \sec^2 \frac{1}{2} v$, etc.

For all, except circular orbits:

$$f_1 = 1 - \frac{\gamma_1^2}{r_0}; \quad f_{III} = 1 - \frac{\gamma_{III}^2}{r_0}; \quad g_1^2 = [2r_1 r_0 - p \gamma_1^2] \gamma_1^2; \quad g_{III}^2 = [2r_{III} r_0 - p \gamma_{III}^2] \gamma_{III}^2.$$

g_i in general is *positive* for dates *after* and *negative* for dates *before* $t_{II} = t_0$, but the algebraical sign should be carefully checked by reference to the quadrants of the angles involved in formulæ (8), (9), (17), page 248 and 249.¹

With $x_0, y_0, z_0; x'_0, y'_0, z'_0$ and with $f_1, f_{III}; g_1, g_{III}$ as obtained either by series or closed expressions, compute the residuals for the first and third dates as in A VI, the corrections for parallax being applied on the basis of the values of ρ , and ρ_{III} resulting from the computation of the places from the initial orbit.

III.

By the application of the proper differential formulæ determine such corrections to ρ_0, x'_0, y'_0, z'_0 as will remove the residuals.

Choose the formulæ of group A [VII] or those given below in accordance with the criteria contained above, page 327, second paragraph.

For the improvement of an initial circular orbit the differential formulæ based on the closed expressions are given in A [VII b] and are not repeated below.

If the differential formulæ based on the closed expressions for the ∂f and ∂g be chosen, apply the formulæ designated 1) *from circular orbits* to the improvement of *initial circular* or *very nearly circular orbits*; those designated by 2) *from parabolic orbits* to the improvement of *initial parabolic, very nearly parabolic, or hyperbolic orbits*; those designated by 3) *from elliptic orbits* to the improvement of *initial elliptic orbits* in general.

1). *From initial circular orbits.*

Apply the formulæ of A [VII b].

¹See footnote 1, page 330.

2). From initial parabolic orbits.

$$\cos \beta = \frac{\rho_0 - R_0 \cos \psi}{r_0}; \quad \varphi_0 = \frac{x'_0 \xi_0 + y'_0 \eta_0 + z'_0 \zeta_0}{\rho_0}$$

$$\Phi_i = \varphi_0 + \frac{1}{\gamma_i} \frac{\cos \beta}{r_0}; \quad \Phi_{iii} = \varphi_0 + \frac{1}{\gamma_{iii}} \frac{\cos \beta}{r_0}.$$

$$g_{x_i} = \frac{\gamma_i^3}{r_0 r_i} [x'_0 r_0 \gamma_i + 1/2 x_0]; \quad f_{x_i} = f_i \frac{\xi_0}{\rho_0} + \frac{\cos \beta \gamma_i^2}{r_0^2} x_0 + g_{x_i} \Phi_i$$

$$g_{y_i} = \frac{\gamma_i^3}{r_0 r_i} [y'_0 r_0 \gamma_i + 1/2 y_0]; \quad f_{y_i} = f_i \frac{\eta_0}{\rho_0} + \frac{\cos \beta \gamma_i^2}{r_0^2} y_0 + g_{y_i} \Phi_i$$

$$g_{z_i} = \frac{\gamma_i^3}{r_0 r_i} [z'_0 r_0 \gamma_i + 1/2 z_0]; \quad f_{z_i} = f_i \frac{\zeta_0}{\rho_0} + \frac{\cos \beta \gamma_i^2}{r_0^2} z_0 + g_{z_i} \Phi_i$$

$$g_{x_{iii}} = \frac{\gamma_{iii}^3}{r_0 r_{iii}} [x'_0 r_0 \gamma_{iii} + 1/2 x_0]; \quad f_{x_{iii}} = f_{iii} \frac{\xi_0}{\rho_0} + \frac{\cos \beta \gamma_{iii}^2}{r_0^2} x_0 + g_{x_{iii}} \Phi_{iii}$$

$$g_{y_{iii}} = \frac{\gamma_{iii}^3}{r_0 r_{iii}} [y'_0 r_0 \gamma_{iii} + 1/2 y_0]; \quad f_{y_{iii}} = f_{iii} \frac{\eta_0}{\rho_0} + \frac{\cos \beta \gamma_{iii}^2}{r_0^2} y_0 + g_{y_{iii}} \Phi_{iii}$$

$$g_{z_{iii}} = \frac{\gamma_{iii}^3}{r_0 r_{iii}} [z'_0 r_0 \gamma_{iii} + 1/2 z_0]; \quad f_{z_{iii}} = f_{iii} \frac{\zeta_0}{\rho_0} + \frac{\cos \beta \gamma_{iii}^2}{r_0^2} z_0 + g_{z_{iii}} \Phi_{iii}$$

$$A_{f_i} = \frac{1}{\rho_i} [\cos \alpha_i f_{y_i} - \sin \alpha_i f_{x_i}] + \cos \delta_i \frac{(\rho_a \rho)_i}{\rho_i^2}$$

$$B_{f_i} = -\frac{1}{\rho_i} [\sin \delta_i (\sin \alpha_i f_{y_i} + \cos \alpha_i f_{x_i}) - \cos \delta_i f_{z_i}] + \frac{(\rho_b \rho)_i}{\rho_i^2}$$

$$A_{g_i} = \frac{1}{\rho_i} [\cos \alpha_i g_{y_i} - \sin \alpha_i g_{x_i}]$$

$$B_{g_i} = -\frac{1}{\rho_i} [\sin \delta_i (\sin \alpha_i g_{y_i} + \cos \alpha_i g_{x_i}) - \cos \delta_i g_{z_i}]; \quad C_i = \frac{g_i}{\rho_i}$$

$$A_{f_{iii}} = \frac{1}{\rho_{iii}} [\cos \alpha_{iii} f_{y_{iii}} - \sin \alpha_{iii} f_{x_{iii}}] + \cos \delta_{iii} \frac{(\rho_a \rho)_{iii}}{\rho_{iii}^2}$$

$$B_{f_{iii}} = -\frac{1}{\rho_{iii}} [\sin \delta_{iii} (\sin \alpha_{iii} f_{y_{iii}} + \cos \alpha_{iii} f_{x_{iii}}) - \cos \delta_{iii} f_{z_{iii}}] + \frac{(\rho_b \rho)_{iii}}{\rho_{iii}^2}$$

$$A_{g_{iii}} = \frac{1}{\rho_{iii}} [\cos \alpha_{iii} g_{y_{iii}} - \sin \alpha_{iii} g_{x_{iii}}]$$

$$B_{g_{iii}} = -\frac{1}{\rho_{iii}} [\sin \delta_{iii} (\sin \alpha_{iii} g_{y_{iii}} + \cos \alpha_{iii} g_{x_{iii}}) - \cos \delta_{iii} g_{z_{iii}}]; \quad C_{iii} = \frac{g_{iii}}{\rho_{iii}}$$

$$A = A_{f_i} A_{g_{iii}} - A_{f_{iii}} A_{g_i}$$

$$a_1 = [\sin \alpha_i C_i A_{g_{iii}} - \sin \alpha_{iii} C_{iii} A_{g_i}]; \quad b_1 = -[\cos \alpha_i C_i A_{g_{iii}} - \cos \alpha_{iii} C_{iii} A_{g_i}]$$

$$a_2 = [\sin \alpha_i C_i A_{f_{iii}} - \sin \alpha_{iii} C_{iii} A_{f_i} + A x_0]; \quad b_2 = -[\cos \alpha_i C_i A_{f_{iii}} - \cos \alpha_{iii} C_{iii} A_{f_i} - A y_0]$$

$$a_3 = [\sin \delta_i \cos \alpha_i C_i - x_0 B_{g_i}]; \quad b_3 = [\sin \delta_i \sin \alpha_i C_i - y_0 B_{g_i}]$$

$$a_4 = [\sin \delta_{iii} \cos \alpha_{iii} C_{iii} - x_0 B_{g_{iii}}]; \quad b_4 = [\sin \delta_{iii} \sin \alpha_{iii} C_{iii} - y_0 B_{g_{iii}}]$$

$$c_2 = A z_0; \quad d_1 = A; \quad e_1 = A_{g_i} \partial_i \alpha_{iii} - A_{g_{iii}} \partial_i \alpha_i$$

$$c_3 = -[\cos \delta_i C_i + z_0 B_{g_i}]; \quad d_3 = B_{f_i}; \quad e_2 = A_{f_i} \partial_i \alpha_{iii} - A_{f_{iii}} \partial_i \alpha_i$$

$$c_4 = -[\cos \delta_{iii} C_{iii} + z_0 B_{g_{iii}}]; \quad d_4 = B_{f_{iii}}; \quad e_3 = -\partial \delta_i$$

$$e_4 = -\partial \delta_{iii}$$

$$\begin{aligned}
A_{f_i} &= \frac{1}{\rho_i} [\cos \alpha_i f_{y_i} - \sin \alpha_i f_{x_i}] + \cos \delta_i \frac{(\rho_i \rho)_i}{\rho_i^2}; & A_{f_{i'''}} &= \frac{1}{\rho_{i'''}} [\cos \alpha_{i'''} f_{y_{i'''}} - \sin \alpha_{i'''} f_{x_{i'''}}] + \cos \delta_{i'''} \frac{(\rho_{i'''} \rho)_{i'''}}{\rho_{i'''}} \\
A_{g_i} &= \frac{1}{\rho_i} [\cos \alpha_i g_{y_i} - \sin \alpha_i g_{x_i}]; & A_{g_{i'''}} &= \frac{1}{\rho_{i'''}} [\cos \alpha_{i'''} g_{y_{i'''}} - \sin \alpha_{i'''} g_{x_{i'''}}] \\
A_{m_i} &= \frac{1}{\rho_i} [\cos \alpha_i m_{y_i} - \sin \alpha_i m_{x_i}]; & A_{m_{i'''}} &= \frac{1}{\rho_{i'''}} [\cos \alpha_{i'''} m_{y_{i'''}} - \sin \alpha_{i'''} m_{x_{i'''}}]. \\
B_{f_i} &= -\frac{1}{\rho_i} [\sin \delta_i (\sin \alpha_i f_{y_i} + \cos \alpha_i f_{x_i}) - \cos \delta_i f_{z_i}] + \frac{(\rho_i \rho)_i}{\rho_i^2} \\
B_{g_i} &= -\frac{1}{\rho_i} [\sin \delta_i (\sin \alpha_i g_{y_i} + \cos \alpha_i g_{x_i}) - \cos \delta_i g_{z_i}] \\
B_{m_i} &= -\frac{1}{\rho_i} [\sin \delta_i (\sin \alpha_i m_{y_i} + \cos \alpha_i m_{x_i}) - \cos \delta_i m_{z_i}] \\
B_{f_{i'''}} &= -\frac{1}{\rho_{i'''}} [\sin \delta_{i'''} (\sin \alpha_{i'''} f_{y_{i'''}} + \cos \alpha_{i'''} f_{x_{i'''}}) - \cos \delta_{i'''} f_{z_{i'''}}] + \frac{(\rho_{i'''} \rho)_{i'''}}{\rho_{i'''}} \\
B_{g_{i'''}} &= -\frac{1}{\rho_{i'''}} [\sin \delta_{i'''} (\sin \alpha_{i'''} g_{y_{i'''}} + \cos \alpha_{i'''} g_{x_{i'''}}) - \cos \delta_{i'''} g_{z_{i'''}}] \\
B_{m_{i'''}} &= -\frac{1}{\rho_{i'''}} [\sin \delta_{i'''} (\sin \alpha_{i'''} m_{y_{i'''}} + \cos \alpha_{i'''} m_{x_{i'''}}) - \cos \delta_{i'''} m_{z_{i'''}}] \\
C_i &= \frac{g_i}{\rho_i}; & C_{i'''} &= \frac{g_{i'''}}{\rho_{i'''}}.
\end{aligned}$$

Then $\partial \rho_0, \partial x'_0, \partial y'_0, \partial z'_0$ are obtained by solving four equations of the form

$$\alpha_i \partial \rho_0 + \beta_i \partial x'_0 + \gamma_i \partial y'_0 + \delta_i \partial z'_0 = \nu_i, \quad i = 1, 2, 3, 4,$$

where

$$\begin{aligned}
\nu_1 &= \partial \alpha_i; & \alpha_1 &= A_{f_i}; & \beta_1 &= -[\sin \alpha_i C_i - x_0 A_{g_i} - 2 x'_0 A_{m_i}] \\
\nu_2 &= \partial \delta_i; & \alpha_2 &= B_{f_i}; & \beta_2 &= -[\sin \delta_i \cos \alpha_i C_i - x_0 B_{g_i} - 2 x'_0 B_{m_i}] \\
\nu_3 &= \partial \alpha_{i'''}; & \alpha_3 &= A_{f_{i'''}}; & \beta_3 &= -[\sin \alpha_{i'''} C_{i'''} - x_0 A_{g_{i'''}} - 2 x'_0 A_{m_{i'''}}] \\
\nu_4 &= \partial \delta_{i'''}; & \alpha_4 &= B_{f_{i'''}}; & \beta_4 &= -[\sin \delta_{i'''} \cos \alpha_{i'''} C_{i'''} - x_0 B_{g_{i'''}} - 2 x'_0 B_{m_{i'''}}] \\
\delta_1 &= [z_0 A_{g_i} + 2 z'_0 A_{m_i}]; & \gamma_1 &= [\cos \alpha_i C_i + y_0 A_{g_i} + 2 y'_0 A_{m_i}] \\
\delta_2 &= [\cos \delta_i C_i + z_0 B_{g_i} + 2 z'_0 B_{m_i}]; & \gamma_2 &= -[\sin \delta_i \sin \alpha_i C_i - y_0 B_{g_i} - 2 y'_0 B_{m_i}] \\
\delta_3 &= [z_0 A_{g_{i'''}} + 2 z'_0 A_{m_{i'''}}]; & \gamma_3 &= [\cos \alpha_{i'''} C_{i'''} + y_0 A_{g_{i'''}} + 2 y'_0 A_{m_{i'''}}] \\
\delta_4 &= [\cos \delta_{i'''} C_{i'''} + z_0 B_{g_{i'''}} + 2 z'_0 B_{m_{i'''}}]; & \gamma_4 &= -[\sin \delta_{i'''} \sin \alpha_{i'''} C_{i'''} - y_0 B_{g_{i'''}} - 2 y'_0 B_{m_{i'''}}].
\end{aligned}$$

Of the available methods of solution the computer will choose the one which admits of the most convenient determination of the unknowns on the basis of the numerical values of the coefficients. Then

$$x = x_0 + \xi_0 \frac{\partial \rho_0}{\rho_0}, \quad y = y_0 + \eta_0 \frac{\partial \rho_0}{\rho_0}, \quad z = z_0 + \zeta_0 \frac{\partial \rho_0}{\rho_0}; \quad x' = x'_0 + \partial x'_0, \quad y' = y'_0 + \partial y'_0, \quad z' = z'_0 + \partial z'_0.$$

IV.

$x, y, z; x', y', z'$ are the new values of the heliocentric coördinates and velocities from which the residuals are computed as a check by A VI with new values of f and g , and then the elements by A VIII.

TABLES FROM WHICH $z = \frac{\rho}{R}$ MAY BE INTERPOLATED WITH
THE ARGUMENTS ψ AND $\frac{1}{m}$ IN THE DETERMINATION
OF PRELIMINARY ORBITS WITHOUT ASSUMP-
TION REGARDING THE ECCENTRICITY.

IN Part I of this volume I have pointed out that the geocentric distance which is defined by the positive real roots of the equation of the seventh degree, to which the orbit problem reduces if no assumption is made regarding the eccentricity, may be obtained directly from an extension of V. OPPOLZER'S Table XIIIa, *Bahnbestimmung*, Vol. I. Such a preliminary extension has been made in the form of the Tables which follow. They serve for the interpolation of $z = \frac{\rho}{R}$ with the arguments ψ and $\frac{1}{m}$ in general orbits; for determining the feasibility of a solution with hypothesis regarding the eccentricity (*cf.* pages 290, 291); and for determining the accuracy of the adopted solution (*cf.* pages 270–278).

The tabular values were computed under my direction by students of the University, Miss SOPHIA H. LEVY, a graduate student, performing by far the major portion of the work. Miss LEVY has not only completed the computations commenced by a number of students, but has also checked most of the work and read the proofs.

The equation from which the Tables were constructed is (*cf.* equation (27), page 240, where $m = (m)$, if parallax be neglected, and where the absolute term m^2 is the square of $-m$)

$$(z^2 - 2z \cos \psi + 1)^2 (z - m)^2 - m^2 = 0.$$

For the computation of the tabular values of $\frac{1}{m}$ this equation was written in the form

$$a = [z(z - 2 \cos \psi) + 1]^{-\frac{3}{2}}; \quad \frac{1}{m} = \frac{1 - a}{z}.$$

The values of $\frac{1}{m}$ were computed directly from $z = 0.01$ to $z = 4.00$ for every 10 degrees of ψ . This computation was performed to seven figures and cut down to six figures in the Tables.

It is evident that for a particular value of ψ the second differences of $z^2 - 2z \cos \psi + 1$ are constant, so that this function may be built up by differences for equidistant values of z and then checked at intervals by direct computation.

For the intermediate values $\psi = 5^\circ, 15^\circ$, etc., the tabular values of $\frac{1}{m}$ were either computed directly or obtained by interpolation from the values for $\psi = 10^\circ, 20^\circ$, etc., taken to six figures if the interpolation could be performed with sufficient accuracy. The last figure of the tabulated values of $\frac{1}{m}$, therefore, is not absolutely

reliable throughout the Tables and the computer is advised to check the interpolated value of z in each case by substitution in the equation and to correct it, if necessary, by the differential formula (*cf. Synopsis of Formulae*, A IVc).

To interpolate z (of which there can be at most two values corresponding to given arguments ψ and $\frac{1}{m}$) it is most convenient to first enter the ψ column with its given value and to determine by inspection (along the horizontal lines) the page and region of the given value of $\frac{1}{m}$. Two such regions will be found in case of a double solution. Then the values of $\frac{1}{m}$ corresponding to the given ψ should be accurately interpolated from two or more of the vertical columns, according to the run of the differences.

The interpolated values of $\frac{1}{m}$ furnish a small table with z as argument, from which the required value of z may be interpolated with the given value of $\frac{1}{m}$. In the greater part of the Tables, interpolation to second differences will prove sufficient to obtain the definitive value of z and, therefore, of the geocentric distance. The laborious trials hitherto necessary are thus avoided.

If it is found by inspection that the required value (or values) of z falls within a part of the Table where the interpolation requires the use of more than the second differences, it is more convenient to perform the interpolation approximately and to correct the approximate value of z by the differential formula. A single correction is practically always sufficient.

The author hopes that ultimately means may be placed at his disposal to enable him to further extend the Table so that interpolation by proportional parts will prove sufficient in all parts of the Tables.

The Tables are applicable to all orbit methods in which the solution of the problem has been reduced to the solution of the equation on which the Tables are based. The importance of the Tables as a direct means of obtaining the geocentric distance, therefore, can not be overestimated.

Examples of the use of the Tables for the determination of the geocentric distance, for testing the accuracy of the solution (*cf. page 270 et seq.*), and for testing the feasibility of a solution with hypothesis regarding the eccentricity (*cf. pages 290, 291*) will be found in Part 8 of this volume.

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$

ψ	$\frac{1}{m}$	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.11	z
0°	—	3.061 01	3.124 12	3.189 42	3.257 02	3.327 02	3.399 54	3.474 70	3.552 64	3.633 50	3.717 42	3.804 57	0°
5	—	3.049 00	3.111 50	3.176 14	3.243 02	3.312 28	3.384 00	3.458 30	3.535 33	3.615 22	3.698 11	3.784 14	5
10	—	3.013 11	3.073 73	3.136 41	3.201 22	3.268 24	3.337 60	3.409 39	3.483 74	3.560 76	3.640 59	3.723 38	10
15	—	2.953 59	3.011 24	3.070 70	3.132 11	3.195 52	3.261 05	3.328 76	3.398 78	3.471 19	3.546 11	3.623 64	15
20	—	2.871 06	2.924 60	2.979 73	3.036 58	3.095 14	3.155 52	3.217 77	3.282 00	3.348 25	3.416 63	3.487 22	20
25	—	2.766 15	2.814 67	2.864 53	2.915 76	2.968 42	3.022 54	3.078 20	3.135 42	3.194 28	3.254 82	3.317 10	25
30	—	2.639 81	2.682 56	2.726 32	2.771 14	2.817 06	2.864 08	2.912 24	2.961 57	3.012 10	3.063 86	3.116 88	30
35	—	2.493 11	2.529 52	2.566 62	2.604 45	2.643 00	2.682 34	2.722 43	2.763 29	2.804 93	2.847 36	2.890 59	35
40	—	2.327 36	2.357 00	2.387 07	2.417 58	2.448 48	2.479 81	2.511 56	2.543 70	2.576 25	2.609 20	2.642 54	40
45	—	2.143 92	2.166 66	2.189 57	2.212 63	2.235 84	2.259 15	2.282 60	2.306 12	2.329 72	2.353 39	2.377 12	45
50	—	1.944 34	1.960 26	1.976 11	1.991 88	2.007 56	2.023 14	2.038 59	2.053 88	2.069 01	2.083 95	2.098 68	50
55	—	1.730 29	1.739 66	1.748 78	1.757 65	1.766 27	1.774 61	1.782 65	1.790 37	1.797 76	1.804 78	1.811 45	55
60	—	1.503 57	1.506 86	1.509 78	1.512 35	1.514 56	1.516 39	1.517 82	1.518 84	1.519 61	1.519 61	1.519 33	60
65	—	1.266 06	1.263 89	1.261 33	1.258 37	1.255 01	1.251 25	1.247 04	1.242 43	1.237 37	1.231 87	1.225 92	65
70	—	1.019 65	1.012 86	1.005 68	0.998 12	0.990 20	0.981 87	0.973 16	0.964 05	0.954 56	0.944 68	0.934 41	70
75	—	0.766 31	0.755 86	0.745 06	0.733 94	0.722 54	0.710 80	0.698 75	0.686 41	0.673 77	0.660 83	0.647 62	75
80	—	0.508 06	0.494 99	0.481 67	0.468 12	0.454 38	0.440 42	0.426 26	0.411 94	0.397 42	0.382 75	0.367 92	80
85	—	0.246 97	0.232 36	0.217 64	0.202 82	0.187 93	0.172 95	0.157 91	0.142 82	0.127 68	0.112 51	0.097 33	85
90	+	0.015 02	0.029 99	0.044 95	0.059 88	0.074 77	0.089 60	0.104 36	0.119 04	0.133 65	0.148 15	0.162 54	90
95	+	0.275 85	0.290 05	0.304 14	0.318 05	0.331 81	0.345 39	0.358 78	0.371 99	0.385 00	0.397 80	0.410 39	95
100	+	0.533 58	0.545 92	0.558 05	0.569 89	0.581 48	0.592 80	0.603 84	0.614 60	0.625 09	0.635 30	0.645 21	100
105	+	0.786 27	0.795 75	0.804 91	0.813 71	0.822 22	0.830 38	0.838 21	0.845 71	0.852 89	0.859 75	0.866 28	105
110	+	1.032 09	1.037 78	1.043 09	1.048 00	1.052 62	1.056 87	1.060 77	1.064 34	1.067 57	1.070 49	1.073 08	110
115	+	1.269 28	1.270 34	1.271 05	1.271 39	1.271 45	1.271 17	1.270 58	1.269 70	1.268 49	1.267 02	1.265 28	115
120	+	1.496 11	1.491 89	1.487 40	1.482 64	1.477 62	1.472 34	1.466 84	1.461 11	1.455 17	1.449 03	1.442 70	120
125	+	1.710 95	1.701 00	1.690 87	1.680 60	1.670 16	1.659 60	1.648 92	1.638 14	1.627 25	1.616 29	1.605 26	125
130	+	1.912 34	1.896 34	1.880 30	1.864 28	1.848 25	1.832 26	1.816 30	1.800 38	1.784 52	1.768 72	1.752 98	130
135	+	2.098 90	2.076 70	2.054 66	2.032 82	2.011 18	1.989 77	1.968 55	1.947 57	1.926 81	1.906 27	1.885 96	135
140	+	2.269 34	2.240 97	2.213 01	2.185 47	2.158 36	2.131 66	2.105 37	2.079 51	2.054 05	2.028 99	2.004 33	140
145	+	2.422 48	2.388 17	2.354 55	2.321 59	2.289 26	2.257 57	2.226 51	2.196 06	2.166 20	2.136 94	2.108 24	145
150	+	2.557 32	2.517 48	2.478 59	2.440 60	2.403 48	2.367 21	2.331 77	2.297 14	2.263 29	2.230 20	2.197 85	150
155	+	2.672 96	2.628 16	2.584 54	2.542 05	2.500 65	2.460 31	2.420 98	2.382 68	2.345 32	2.308 89	2.273 35	155
160	+	2.768 62	2.719 56	2.671 89	2.625 55	2.580 51	2.536 72	2.494 12	2.452 68	2.412 35	2.373 10	2.334 88	160
165	+	2.843 68	2.791 18	2.740 24	2.690 80	2.642 84	2.596 28	2.551 04	2.507 11	2.464 43	2.422 93	2.382 59	165
170	+	2.897 61	2.842 58	2.789 26	2.737 58	2.687 48	2.638 88	2.591 74	2.546 00	2.501 53	2.458 47	2.416 58	170
175	+	2.930 10	2.873 52	2.818 76	2.765 71	2.714 31	2.664 48	2.616 17	2.569 34	2.523 87	2.479 78	2.436 94	175
180	+	2.940 96	2.883 87	2.828 61	2.775 09	2.723 25	2.673 01	2.624 32	2.577 10	2.531 29	2.486 85	2.443 71	180
ψ	z	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.11	$\frac{1}{m}$

1
TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ	z	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.20	0.21	z	ψ
0°	0°	3.804 57	3.895 10	3.989 20	4.087 06	4.188 89	4.294 89	4.405 31	4.520 39	4.640 40	4.765 62	4.896 37	0°	0°
5	5	3.784 14	3.873 50	3.966 33	4.062 84	4.163 21	4.267 66	4.376 40	4.489 68	4.607 75	4.730 89	4.859 38	5	5
10	10	3.723 38	3.809 24	3.898 35	3.990 87	4.086 97	4.186 84	4.290 68	4.398 70	4.511 11	4.628 17	4.750 12	10	10
15	15	3.623 64	3.703 92	3.787 06	3.873 21	3.962 51	4.055 10	4.151 15	4.250 82	4.354 30	4.461 77	4.573 43	15	15
20	20	3.487 22	3.560 11	3.635 38	3.713 16	3.793 54	3.876 62	3.962 52	4.051 37	4.143 29	4.238 41	4.336 87	20	20
25	25	3.317 10	3.381 18	3.447 13	3.515 01	3.584 88	3.656 81	3.730 87	3.807 15	3.885 70	3.966 61	4.049 96	25	25
30	30	3.116 88	3.171 19	3.226 83	3.283 82	3.342 19	3.401 99	3.463 23	3.525 95	3.590 19	3.655 97	3.723 32	30	30
35	35	2.890 59	2.934 63	2.979 48	3.025 15	3.071 63	3.118 95	3.167 09	3.216 07	3.265 86	3.316 48	3.367 92	35	35
40	40	2.642 54	2.676 24	2.710 32	2.744 75	2.779 52	2.814 61	2.850 01	2.885 70	2.921 66	2.957 87	2.994 30	40	40
45	45	2.377 12	2.400 86	2.424 62	2.448 36	2.472 07	2.495 70	2.519 25	2.542 70	2.565 99	2.589 11	2.612 03	45	45
50	50	2.098 68	2.113 19	2.127 44	2.141 42	2.155 10	2.168 45	2.181 46	2.194 09	2.206 31	2.218 10	2.229 44	50	50
55	55	1.811 45	1.817 73	1.823 59	1.829 03	1.834 00	1.838 52	1.842 54	1.846 05	1.849 05	1.851 48	1.853 35	55	55
60	60	1.519 33	1.518 59	1.517 37	1.515 67	1.513 47	1.510 76	1.507 52	1.503 76	1.499 44	1.494 58	1.489 15	60	60
65	65	1.225 92	1.219 82	1.212 65	1.205 33	1.197 53	1.189 26	1.180 52	1.171 31	1.161 62	1.151 46	1.140 82	65	65
70	70	0.934 41	0.923 76	0.912 72	0.901 30	0.889 51	0.877 34	0.864 81	0.851 91	0.838 65	0.825 05	0.811 11	70	70
75	75	0.647 62	0.634 13	0.620 37	0.606 35	0.592 08	0.577 57	0.562 85	0.547 87	0.532 69	0.517 31	0.501 74	75	75
80	80	0.367 92	0.352 95	0.337 85	0.322 62	0.307 28	0.291 85	0.276 33	0.260 73	0.245 08	0.229 37	0.213 62	80	80
85	85	0.097 33	0.082 14	0.066 95	0.051 78	0.036 63	0.021 53	0.006 48	0.008 52	0.023 44	0.038 29	0.053 04	85	85
90	90	0.162 54	0.176 81	0.190 96	0.204 97	0.218 83	0.232 54	0.246 09	0.259 46	0.272 66	0.285 67	0.298 49	90	90
95	95	0.410 39	0.422 75	0.434 89	0.446 80	0.458 48	0.469 91	0.481 08	0.492 04	0.502 73	0.513 16	0.523 34	95	95
100	100	0.645 21	0.654 84	0.664 17	0.673 22	0.681 97	0.690 44	0.698 61	0.706 49	0.714 08	0.721 39	0.728 40	100	100
105	105	0.866 28	0.872 49	0.878 39	0.883 98	0.889 25	0.894 23	0.898 90	0.903 28	0.907 36	0.911 16	0.914 69	105	105
110	110	1.073 08	1.075 37	1.077 35	1.079 04	1.080 44	1.081 56	1.082 40	1.082 98	1.083 31	1.083 38	1.083 21	110	110
115	115	1.265 28	1.263 28	1.261 03	1.258 53	1.255 80	1.252 85	1.249 68	1.246 31	1.242 73	1.238 98	1.235 05	115	115
120	120	1.442 70	1.436 19	1.429 51	1.422 68	1.415 70	1.408 58	1.401 34	1.393 97	1.386 50	1.378 93	1.371 26	120	120
125	125	1.605 26	1.594 16	1.583 00	1.571 80	1.560 56	1.549 30	1.538 02	1.526 72	1.515 42	1.504 12	1.492 84	125	125
130	130	1.752 98	1.737 32	1.721 74	1.706 25	1.690 85	1.675 56	1.660 36	1.645 27	1.630 30	1.615 44	1.600 70	130	130
135	135	1.885 96	1.865 88	1.846 04	1.826 42	1.807 06	1.787 90	1.768 99	1.750 33	1.731 90	1.713 70	1.695 74	135	135
140	140	2.004 33	1.980 06	1.956 19	1.932 71	1.909 60	1.886 87	1.864 52	1.842 53	1.820 90	1.799 63	1.778 71	140	140
145	145	2.108 24	2.080 10	2.052 52	2.025 48	1.998 96	1.972 96	1.947 46	1.922 46	1.897 95	1.873 90	1.850 32	145	145
150	150	2.197 85	2.166 24	2.135 32	2.105 09	2.075 53	2.046 61	2.018 33	1.990 66	1.963 59	1.937 10	1.911 18	150	150
155	155	2.273 35	2.238 69	2.204 88	2.171 88	2.139 67	2.108 23	2.077 55	2.047 58	2.018 32	1.989 73	1.961 81	155	155
160	160	2.334 88	2.297 67	2.261 43	2.226 12	2.191 71	2.158 18	2.125 49	2.093 62	2.062 54	2.032 22	2.002 64	160	160
165	165	2.382 59	2.343 57	2.305 19	2.268 06	2.231 91	2.196 73	2.162 46	2.129 10	2.096 64	2.065 11	2.034 03	165	165
170	170	2.416 58	2.375 87	2.336 32	2.297 86	2.260 47	2.224 09	2.188 70	2.154 25	2.120 72	2.088 07	2.056 27	170	170
175	175	2.436 94	2.395 35	2.354 95	2.315 69	2.277 55	2.240 45	2.204 45	2.169 28	2.135 12	2.101 89	2.069 53	175	175
180	180	2.443 71	2.401 83	2.361 15	2.321 63	2.283 23	2.245 89	2.209 58	2.174 27	2.139 92	2.106 48	2.073 93	180	180
ψ	z	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.20	0.21	ψ	z

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{p}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ	z	0.21	0.22	0.23	0.24	0.25	0.26	0.27	0.28	0.29	0.30	0.31	z	ψ
0°	0°	4.896 37	5.032 96	5.175 75	5.325 12	5.481 48	5.645 27	5.816 97	5.997 09	6.186 17	6.384 83	6.593 73	0°	5
5	5	4.859 38	4.993 55	5.133 73	5.280 27	5.433 58	5.594 06	5.762 16	5.938 39	6.123 25	6.317 31	6.521 19	5	10
10	10	4.750 12	4.877 24	5.009 82	5.148 17	5.292 62	5.443 53	5.601 28	5.766 28	5.938 98	6.119 84	6.309 38	10	15
15	15	4.573 43	4.689 50	4.810 20	4.935 78	5.066 47	5.202 56	5.344 33	5.492 08	5.646 16	5.806 87	5.974 61	15	20
20	20	4.336 87	4.438 82	4.544 41	4.653 80	4.767 15	4.884 65	5.006 46	5.132 80	5.263 85	5.399 84	5.540 96	20	25
25	25	4.049 96	4.135 82	4.224 29	4.315 44	4.409 38	4.506 18	4.605 93	4.708 73	4.814 68	4.923 87	5.036 39	25	30
30	30	3.723 32	3.792 28	3.862 86	3.935 10	4.009 02	4.084 65	4.162 01	4.241 11	4.321 98	4.404 63	4.489 07	30	35
35	35	3.367 92	3.420 18	3.473 23	3.527 07	3.581 69	3.637 07	3.693 18	3.750 00	3.807 52	3.865 68	3.924 46	35	40
40	40	2.994 30	3.030 92	3.067 70	3.104 62	3.141 63	3.178 70	3.215 80	3.252 88	3.289 89	3.326 79	3.363 52	40	45
45	45	2.612 03	2.634 72	2.657 13	2.679 24	2.701 01	2.722 38	2.743 34	2.763 81	2.783 81	2.803 24	2.822 07	45	50
50	50	2.229 44	2.240 28	2.250 61	2.260 39	2.269 59	2.278 19	2.286 14	2.293 42	2.300 01	2.305 86	2.310 94	50	55
55	55	1.853 35	1.854 63	1.855 31	1.855 36	1.854 76	1.853 50	1.851 56	1.848 94	1.845 59	1.841 52	1.836 71	55	60
60	60	1.489 15	1.483 15	1.476 58	1.469 42	1.461 66	1.453 31	1.444 36	1.434 81	1.424 64	1.413 87	1.402 49	60	65
65	65	1.140 82	1.129 70	1.118 11	1.106 06	1.093 54	1.080 56	1.067 12	1.053 23	1.038 90	1.024 14	1.008 96	65	70
70	70	0.811 109	0.796 834	0.782 233	0.767 316	0.752 096	0.736 583	0.720 786	0.704 720	0.688 396	0.671 827	0.655 026	70	75
75	75	0.501 741	0.486 014	0.470 111	0.454 059	0.437 868	0.421 551	0.405 121	0.388 593	0.371 978	0.355 291	0.338 546	75	80
80	80	0.213 624	0.197 853	0.182 070	0.166 285	0.150 509	0.134 761	0.119 044	0.103 375	0.087 764	0.072 224	0.056 764	80	85
85	85	0.053 037	0.067 677	0.082 216	0.096 627	0.110 907	0.125 045	0.139 030	0.152 863	0.166 530	0.180 022	0.193 332	85	90
90	90	0.298 488	0.311 104	0.323 515	0.335 716	0.347 699	0.359 460	0.370 994	0.382 299	0.393 369	0.404 201	0.414 792	90	95
95	95	0.523 345	0.533 243	0.542 893	0.552 280	0.561 401	0.570 258	0.578 846	0.587 170	0.595 228	0.603 025	0.610 549	95	100
100	100	0.728 404	0.735 135	0.741 583	0.747 732	0.753 644	0.759 260	0.764 605	0.769 682	0.774 495	0.779 050	0.783 347	100	105
105	105	0.914 690	0.917 923	0.920 899	0.923 610	0.926 062	0.928 261	0.930 214	0.931 929	0.933 404	0.934 641	0.935 681	105	110
110	110	1.083 209	1.082 802	1.082 168	1.081 314	1.080 248	1.078 976	1.077 506	1.075 846	1.074 004	1.071 964	1.069 798	110	115
115	115	1.235 048	1.230 949	1.226 684	1.222 266	1.217 703	1.213 000	1.208 170	1.203 216	1.198 145	1.192 965	1.187 688	115	120
120	120	1.371 259	1.363 507	1.355 679	1.347 780	1.339 820	1.331 803	1.323 738	1.315 629	1.307 482	1.299 307	1.291 104	120	125
125	125	1.492 839	1.481 553	1.470 298	1.459 068	1.447 869	1.436 704	1.425 583	1.414 505	1.403 474	1.392 500	1.381 575	125	130
130	130	1.600 701	1.586 084	1.571 592	1.557 229	1.542 996	1.528 894	1.514 927	1.501 096	1.487 401	1.473 843	1.460 425	130	135
135	135	1.695 736	1.678 006	1.660 512	1.643 240	1.626 219	1.609 418	1.592 844	1.576 498	1.560 374	1.544 473	1.528 793	135	140
140	140	1.778 710	1.758 136	1.737 903	1.718 006	1.698 437	1.679 194	1.660 268	1.641 658	1.623 354	1.605 356	1.587 654	140	145
145	145	1.850 324	1.827 195	1.804 510	1.782 259	1.760 430	1.739 016	1.718 005	1.697 391	1.677 169	1.657 316	1.637 836	145	150
150	150	1.911 182	1.885 812	1.860 978	1.836 670	1.812 869	1.789 564	1.766 742	1.744 390	1.722 496	1.701 050	1.680 036	150	155
155	155	1.961 808	1.934 523	1.907 857	1.881 797	1.856 320	1.831 411	1.807 054	1.783 232	1.759 931	1.737 138	1.714 823	155	160
160	160	2.002 640	1.973 777	1.945 604	1.918 104	1.891 252	1.865 029	1.839 417	1.814 393	1.789 943	1.766 048	1.742 691	160	165
165	165	2.034 032	2.003 935	1.974 587	1.945 963	1.918 039	1.890 795	1.864 206	1.838 249	1.812 906	1.788 160	1.763 988	165	170
170	170	2.056 266	2.025 283	1.995 091	1.965 664	1.936 974	1.908 997	1.881 710	1.855 088	1.829 109	1.803 755	1.779 002	170	175
175	175	2.069 528	2.038 012	2.007 313	1.977 403	1.948 254	1.919 837	1.892 131	1.865 111	1.838 750	1.813 034	1.787 932	175	180
180	180	2.073 933	2.042 242	2.011 373	1.981 303	1.952 000	1.923 438	1.895 593	1.868 438	1.841 952	1.816 112	1.790 896	180	
ψ	z	0.21	0.22	0.23	0.24	0.25	0.26	0.27	0.28	0.29	0.30	0.31	z	ψ

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ	z	0.31	0.32	0.33	0.34	0.35	0.36	0.37	0.38	0.39	0.40	0.41	z	ψ
0°	0°	6.593 73	6.813 56	7.045 09	7.289 15	7.546 66	7.818 61	8.106 08	8.410 26	8.732 45	9.074 07	9.436 70	0°	0°
5	5	6.521 19	6.735 55	6.961 12	7.198 66	7.449 04	7.713 16	7.992 06	8.286 81	8.598 62	8.928 80	9.278 79	5	5
10	10	6.309 38	6.508 13	6.716 69	6.935 71	7.165 86	7.407 88	7.662 58	7.930 82	8.213 54	8.511 76	8.826 58	10	10
15	15	5.974 61	6.149 76	6.332 74	6.523 98	6.723 97	6.933 19	7.152 20	7.381 55	7.621 86	7.873 78	8.138 00	15	15
20	20	5.540 96	5.687 47	5.839 60	5.997 59	6.161 72	6.332 25	6.509 46	6.693 66	6.885 14	7.084 23	7.291 25	20	20
25	25	5.036 39	5.152 35	5.271 84	5.394 96	5.521 80	5.652 46	5.787 04	5.925 63	6.068 31	6.215 16	6.366 26	25	25
30	30	4.489 07	4.575 31	4.663 35	4.753 19	4.844 82	4.938 22	5.033 39	5.130 29	5.228 88	5.329 14	5.431 00	30	30
35	35	3.924 46	3.983 82	4.043 71	4.104 09	4.164 88	4.226 05	4.287 52	4.349 22	4.411 06	4.472 97	4.534 86	35	35
40	40	3.363 52	3.400 03	3.436 28	3.472 18	3.507 68	3.542 72	3.577 23	3.611 13	3.644 35	3.676 80	3.708 41	40	40
45	45	2.822 07	2.840 25	2.857 74	2.874 48	2.890 44	2.905 55	2.919 76	2.933 03	2.945 30	2.956 52	2.966 63	45	45
50	50	2.310 94	2.315 24	2.318 71	2.321 32	2.323 05	2.323 88	2.323 76	2.322 68	2.320 60	2.317 52	2.313 39	50	50
55	55	1.836 71	1.831 14	1.824 80	1.817 69	1.809 80	1.801 10	1.791 61	1.781 31	1.770 20	1.758 28	1.745 54	55	55
60	60	1.402 49	1.390 51	1.377 92	1.364 73	1.350 94	1.336 56	1.321 60	1.306 05	1.289 95	1.273 28	1.256 08	60	60
65	65	1.008 96	0.993 358	0.977 356	0.960 963	0.944 189	0.927 050	0.909 558	0.891 727	0.873 574	0.855 114	0.836 363	65	65
70	70	0.655 526	0.638 005	0.620 782	0.603 369	0.585 781	0.568 035	0.550 142	0.532 123	0.513 990	0.495 761	0.477 450	70	70
75	75	0.338 546	0.321 755	0.304 933	0.288 095	0.271 250	0.254 417	0.237 605	0.220 830	0.204 103	0.187 437	0.170 846	75	75
80	80	0.056 764	0.041 395	0.026 131	0.010 978	0.004 051	0.018 947	0.033 701	0.048 303	0.062 747	0.077 022	0.091 121	80	80
85	85	0.193 332	0.206 456	0.219 385	0.232 114	0.244 635	0.256 946	0.269 041	0.280 915	0.292 563	0.303 982	0.315 170	85	85
90	90	0.414 792	0.425 140	0.435 242	0.445 098	0.454 703	0.464 059	0.473 165	0.482 018	0.490 622	0.498 973	0.507 073	90	90
95	95	0.610 549	0.617 815	0.624 820	0.631 566	0.638 053	0.644 235	0.650 266	0.655 997	0.661 480	0.666 720	0.671 720	95	95
100	100	0.783 347	0.787 392	0.791 191	0.794 746	0.798 063	0.801 146	0.804 001	0.806 632	0.809 044	0.811 243	0.813 233	100	100
105	105	0.935 681	0.936 489	0.937 090	0.937 486	0.937 683	0.937 689	0.937 509	0.937 148	0.936 614	0.935 912	0.935 047	105	105
110	110	1.069 798	1.067 444	1.064 938	1.062 282	1.059 483	1.056 548	1.053 483	1.050 293	1.046 986	1.043 566	1.040 039	110	110
115	115	1.187 688	1.182 310	1.176 845	1.171 266	1.165 670	1.159 973	1.154 209	1.148 385	1.142 505	1.136 575	1.130 600	115	115
120	120	1.291 104	1.282 882	1.274 644	1.266 395	1.258 140	1.249 885	1.241 630	1.233 383	1.225 145	1.216 922	1.208 717	120	120
125	125	1.381 575	1.370 714	1.359 914	1.349 177	1.338 510	1.327 911	1.317 385	1.306 933	1.296 556	1.286 260	1.276 043	125	125
130	130	1.460 425	1.447 147	1.434 009	1.421 009	1.408 152	1.395 433	1.382 857	1.370 420	1.358 123	1.345 969	1.333 953	130	130
135	135	1.528 793	1.513 328	1.498 084	1.483 050	1.468 229	1.453 614	1.439 207	1.425 003	1.411 001	1.397 198	1.383 590	135	135
140	140	1.587 654	1.570 240	1.553 123	1.536 284	1.519 722	1.503 432	1.487 410	1.471 650	1.456 149	1.440 899	1.425 898	140	140
145	145	1.637 836	1.618 717	1.599 555	1.581 536	1.563 455	1.545 706	1.528 278	1.511 166	1.494 367	1.477 862	1.461 655	145	145
150	150	1.680 036	1.659 448	1.639 270	1.619 495	1.600 112	1.581 111	1.562 481	1.544 216	1.526 302	1.508 734	1.491 502	150	150
155	155	1.714 823	1.692 976	1.671 630	1.650 727	1.630 253	1.610 205	1.590 569	1.571 331	1.552 501	1.534 044	1.515 958	155	155
160	160	1.742 691	1.719 855	1.697 527	1.675 691	1.654 331	1.633 434	1.612 987	1.592 977	1.573 390	1.554 216	1.535 442	160	160
165	165	1.763 988	1.740 366	1.717 297	1.694 748	1.672 704	1.651 153	1.630 079	1.609 468	1.589 308	1.569 581	1.550 278	165	165
170	170	1.779 002	1.754 833	1.731 228	1.708 169	1.685 640	1.663 625	1.642 106	1.621 068	1.600 498	1.580 381	1.560 702	170	170
175	175	1.787 932	1.763 437	1.739 511	1.716 148	1.693 328	1.671 036	1.649 252	1.627 961	1.607 145	1.586 796	1.566 892	175	175
180	180	1.790 896	1.766 285	1.742 258	1.718 792	1.695 880	1.673 494	1.651 620	1.630 247	1.609 350	1.588 922	1.568 945	180	180
ψ	z	0.31	0.32	0.33	0.34	0.35	0.36	0.37	0.38	0.39	0.40	0.41	z	ψ

$\frac{1}{m}$ TABLE FOR INTERPOLATING z ρ r WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ	z	0.41	0.42	0.43	0.44	0.45	0.46	0.47	0.48	0.49	0.50	0.51	z	ψ
0°	0°	9.436 70	9.822 05	10.232 0	10.668 7	11.134 5	11.631 9	12.163 7	12.733 3	13.344 0	14.000 0	14.705 6	0°	5
5	5	9.278 79	9.650 16	10.044 7	10.464 2	10.910 8	11.386 9	11.895 0	12.438 0	13.019 0	13.641 5	14.309 5	5	10
10	10	8.826 58	9.159 20	9.510 93	9.883 18	10.277 5	10.695 6	11.139 3	11.610 6	12.111 7	12.645 0	13.213 2	10	15
15	15	8.138 00	8.415 26	8.706 35	9.012 11	9.333 44	9.671 29	10.026 7	10.400 7	10.794 5	11.209 4	11.646 5	15	20
20	20	7.291 25	7.506 54	7.730 45	7.963 35	8.205 58	8.457 54	8.719 59	8.992 14	9.275 54	9.570 22	9.876 54	20	25
25	25	6.366 26	6.521 70	6.681 52	6.845 77	7.014 52	7.187 78	7.365 58	7.547 90	7.734 73	7.926 02	8.121 73	25	30
30	30	5.431 00	5.534 40	5.639 27	5.745 52	5.853 07	5.961 79	6.071 57	6.182 26	6.293 71	6.405 74	6.518 18	30	35
35	35	4.534 86	4.596 62	4.658 15	4.719 33	4.780 06	4.840 19	4.899 60	4.958 18	5.015 67	5.072 03	5.127 05	35	40
40	40	3.708 41	3.739 09	3.768 76	3.797 33	3.824 71	3.850 81	3.875 53	3.898 78	3.920 46	3.940 48	3.958 75	40	45
45	45	2.966 63	2.975 59	2.983 35	2.989 85	2.995 05	2.998 89	3.001 34	3.002 34	3.001 87	2.999 87	2.996 30	45	50
50	50	2.313 39	2.308 21	2.301 96	2.294 61	2.286 15	2.276 57	2.265 86	2.254 01	2.241 01	2.226 87	2.211 58	50	55
55	55	1.745 54	1.731 99	1.717 63	1.702 46	1.686 50	1.669 74	1.652 20	1.633 89	1.614 83	1.595 02	1.574 50	55	60
60	60	1.256 08	1.238 34	1.220 08	1.201 32	1.182 08	1.162 36	1.142 20	1.121 60	1.100 60	1.079 20	1.057 41	60	65
65	65	0.836 363	0.817 339	0.798 061	0.778 544	0.758 804	0.738 867	0.718 747	0.698 465	0.678 038	0.657 486	0.636 832	65	70
70	70	0.477 450	0.459 074	0.440 649	0.422 190	0.403 714	0.385 238	0.366 774	0.348 338	0.329 947	0.311 614	0.293 355	70	75
75	75	0.170 846	0.154 341	0.137 934	0.121 636	0.105 459	0.089 414	0.073 511	0.057 759	0.042 170	0.026 752	0.011 513	75	80
80	80	0.091 121	0.105 036	0.118 761	0.132 289	0.145 611	0.158 726	0.171 626	0.184 306	0.196 761	0.208 987	0.220 982	80	85
85	85	0.315 170	0.326 123	0.336 838	0.347 313	0.357 546	0.367 537	0.377 283	0.386 783	0.396 039	0.405 048	0.413 812	85	90
90	90	0.507 073	0.514 924	0.522 526	0.529 881	0.536 950	0.543 854	0.550 477	0.556 860	0.563 006	0.568 915	0.574 596	90	95
95	95	0.671 720	0.676 489	0.681 013	0.685 308	0.689 391	0.693 246	0.696 881	0.700 365	0.703 518	0.706 528	0.709 337	95	100
100	100	0.813 233	0.815 029	0.816 607	0.818 002	0.819 208	0.820 231	0.821 077	0.821 748	0.822 253	0.822 595	0.822 779	100	105
105	105	0.935 047	0.934 030	0.932 853	0.931 535	0.930 112	0.928 484	0.926 752	0.924 912	0.922 949	0.920 870	0.918 679	105	110
110	110	1.040 039	1.036 411	1.032 687	1.028 872	1.024 972	1.020 990	1.016 933	1.012 803	1.008 608	1.004 350	1.000 031	110	115
115	115	1.130 400	1.124 579	1.118 526	1.112 438	1.106 322	1.100 181	1.094 019	1.087 838	1.081 646	1.075 442	1.069 228	115	120
120	120	1.208 717	1.200 528	1.192 364	1.184 226	1.176 117	1.168 039	1.159 994	1.151 984	1.144 013	1.136 081	1.128 190	120	125
125	125	1.276 043	1.265 904	1.255 848	1.245 876	1.235 988	1.226 185	1.216 467	1.206 836	1.197 291	1.187 835	1.178 467	125	130
130	130	1.333 953	1.322 076	1.310 337	1.298 736	1.287 272	1.275 944	1.264 750	1.253 693	1.242 765	1.231 971	1.221 309	130	135
135	135	1.383 590	1.370 176	1.356 954	1.343 919	1.331 070	1.318 406	1.305 919	1.293 613	1.281 480	1.269 519	1.257 731	135	140
140	140	1.425 898	1.411 140	1.396 622	1.382 338	1.368 284	1.354 456	1.340 849	1.327 460	1.314 286	1.301 317	1.288 555	140	145
145	145	1.461 655	1.445 737	1.430 102	1.414 742	1.399 651	1.384 825	1.370 257	1.355 940	1.341 872	1.328 044	1.314 449	145	150
150	150	1.491 502	1.474 598	1.458 014	1.441 742	1.425 774	1.410 103	1.394 722	1.379 622	1.364 800	1.350 248	1.335 955	150	155
155	155	1.515 958	1.498 236	1.480 863	1.463 833	1.447 138	1.430 767	1.414 711	1.398 976	1.383 520	1.368 369	1.353 500	155	160
160	160	1.535 442	1.517 058	1.499 051	1.481 411	1.464 131	1.447 198	1.430 603	1.414 337	1.398 391	1.382 758	1.367 429	160	165
165	165	1.550 278	1.531 386	1.512 892	1.494 784	1.477 055	1.459 691	1.442 681	1.426 018	1.409 691	1.393 687	1.378 006	165	170
170	170	1.560 702	1.541 450	1.522 612	1.504 175	1.486 127	1.468 458	1.451 157	1.434 212	1.417 615	1.401 355	1.385 422	170	175
175	175	1.566 802	1.547 424	1.528 383	1.509 750	1.491 512	1.473 662	1.456 187	1.439 074	1.422 317	1.405 905	1.389 822	175	180
180	180	1.568 945	1.549 407	1.530 295	1.511 595	1.493 296	1.475 386	1.457 852	1.440 684	1.423 873	1.407 408	1.391 277	180	

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ	z	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59	0.60	0.61	z	ψ
0°	0°	14.705 6	15.465 8	16.286 4	17.173 5	18.134 4	19.177 3	20.311 4	21.547 4	22.897 2	24.375 0	25.996 7	0°	0°
5	5	14.309 5	15.027 2	15.799 7	16.632 4	17.531 4	18.503 6	19.557 0	20.700 4	21.943 8	23.298 6	24.778 1	5	5
10	10	13.213 2	13.819 1	14.466 0	15.157 3	15.896 9	16.689 0	17.538 3	18.449 9	19.429 5	20.483 4	21.618 6	10	10
15	15	11.646 5	12.107 4	12.593 5	13.106 3	13.647 5	14.218 9	14.822 2	15.459 5	16.132 6	16.843 7	17.594 9	15	15
20	20	9.876 54	10.194 9	10.525 6	10.869 0	11.225 5	11.595 4	11.978 8	12.376 1	12.787 3	13.212 6	13.652 0	20	20
25	25	8.121 73	8.321 73	8.525 92	8.734 15	8.946 21	9.161 87	9.380 86	9.602 86	9.827 49	10.054 3	10.282 9	25	25
30	30	6.518 18	6.630 80	6.743 39	6.855 70	6.967 46	7.078 38	7.188 17	7.296 49	7.403 00	7.507 34	7.609 13	30	30
35	35	5.127 05	5.180 58	5.232 43	5.282 42	5.330 37	5.376 10	5.419 41	5.460 11	5.498 01	5.532 90	5.564 61	35	35
40	40	3.958 75	3.975 17	3.989 64	4.002 08	4.012 40	4.020 50	4.026 32	4.030 77	4.034 20	4.036 27	4.037 20	40	40
45	45	2.996 30	2.991 15	2.984 37	2.975 94	2.965 83	2.954 03	2.940 52	2.925 30	2.908 35	2.889 67	2.869 28	45	45
50	50	2.211 58	2.195 14	2.177 56	2.158 85	2.139 01	2.118 07	2.096 03	2.072 93	2.048 78	2.023 60	1.997 43	50	50
55	55	1.574 50	1.553 26	1.531 35	1.508 76	1.485 55	1.461 71	1.437 29	1.412 31	1.386 80	1.360 79	1.334 31	55	55
60	60	1.057 44	1.035 33	1.012 89	0.990 161	0.967 154	0.943 895	0.920 412	0.896 727	0.872 867	0.848 856	0.824 721	60	60
65	65	0.636 832	0.616 092	0.595 286	0.574 436	0.553 557	0.532 670	0.511 796	0.490 951	0.470 153	0.449 421	0.428 771	65	65
70	70	0.293 355	0.275 183	0.257 112	0.239 155	0.221 325	0.203 634	0.186 096	0.168 720	0.151 519	0.134 500	0.117 678	70	70
75	75	0.011 513	0.003 537	0.018 392	0.033 043	0.047 484	0.061 707	0.075 711	0.089 485	0.103 025	0.116 328	0.129 389	75	75
80	80	0.220 982	0.232 742	0.244 262	0.255 542	0.266 579	0.277 372	0.287 915	0.298 214	0.308 264	0.318 065	0.327 618	80	80
85	85	0.413 812	0.422 331	0.430 606	0.438 637	0.446 426	0.453 974	0.461 285	0.468 357	0.475 196	0.481 802	0.488 178	85	85
90	90	0.574 596	0.580 047	0.585 272	0.590 273	0.595 057	0.599 625	0.603 982	0.608 130	0.612 074	0.615 818	0.619 365	90	90
95	95	0.709 337	0.711 950	0.714 373	0.716 608	0.718 661	0.720 535	0.722 237	0.723 773	0.725 138	0.726 346	0.727 399	95	95
100	100	0.822 779	0.822 809	0.822 692	0.822 433	0.822 034	0.821 502	0.820 840	0.820 054	0.819 148	0.818 126	0.816 993	100	100
105	105	0.918 679	0.916 384	0.913 990	0.911 500	0.908 919	0.906 251	0.903 500	0.900 671	0.897 767	0.894 792	0.891 750	105	105
110	110	1.000 031	0.995 659	0.991 236	0.986 766	0.982 253	0.977 700	0.973 111	0.968 488	0.963 835	0.959 156	0.954 451	110	110
115	115	1.069 228	1.063 011	1.056 792	1.050 573	1.044 358	1.038 148	1.031 947	1.025 755	1.019 576	1.013 412	1.007 263	115	115
120	120	1.128 190	1.120 343	1.112 540	1.104 783	1.097 074	1.089 414	1.081 803	1.074 243	1.066 736	1.059 281	1.051 879	120	120
125	125	1.178 467	1.169 186	1.159 994	1.150 891	1.141 876	1.132 950	1.124 112	1.115 362	1.106 701	1.098 127	1.089 641	125	125
130	130	1.221 309	1.210 775	1.200 370	1.190 093	1.179 942	1.169 915	1.160 013	1.150 232	1.140 573	1.131 032	1.121 611	130	130
135	135	1.257 731	1.246 108	1.234 651	1.223 358	1.212 223	1.201 246	1.190 424	1.179 755	1.169 237	1.158 864	1.148 639	135	135
140	140	1.288 555	1.275 995	1.263 632	1.251 464	1.239 484	1.227 692	1.216 082	1.204 652	1.193 398	1.182 316	1.171 403	140	140
145	145	1.314 449	1.301 090	1.287 955	1.275 042	1.262 343	1.249 858	1.237 589	1.225 503	1.213 616	1.201 935	1.190 436	145	145
150	150	1.335 955	1.321 921	1.308 137	1.294 598	1.281 297	1.268 230	1.255 391	1.242 773	1.230 375	1.218 189	1.206 209	150	150
155	155	1.353 500	1.338 910	1.324 591	1.310 538	1.296 741	1.283 196	1.269 895	1.256 833	1.244 008	1.231 408	1.219 031	155	155
160	160	1.367 429	1.352 394	1.337 647	1.323 181	1.308 988	1.295 061	1.281 393	1.267 978	1.254 809	1.241 881	1.229 186	160	160
165	165	1.378 006	1.362 632	1.347 558	1.332 778	1.318 281	1.304 064	1.290 114	1.276 432	1.263 001	1.249 822	1.236 885	165	165
170	170	1.385 422	1.369 808	1.354 504	1.339 501	1.324 790	1.310 369	1.296 223	1.282 348	1.268 736	1.255 380	1.242 273	170	170
175	175	1.389 822	1.374 065	1.358 624	1.343 490	1.328 651	1.314 108	1.299 846	1.285 855	1.272 137	1.258 676	1.245 468	175	175
180	180	1.391 277	1.375 474	1.359 988	1.344 810	1.329 932	1.315 346	1.301 044	1.287 018	1.273 261	1.259 766	1.246 525	180	180

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$

ψ	z	0.71	0.72	0.73	0.74	0.75	0.76	0.77	0.78	0.79	0.80	0.81	z	ψ
0°	0°	—	—	—	—	—	—	—	—	—	—	—	0°	0°
5	5	56.341 0	61.880 4	68.226 4	75.534 8	84.000 0	93.865 7	105.441	119.121	135.417	155.000	178.758	5	5
10	10	51.191 0	55.783 0	60.966 2	66.838 2	73.515 6	81.139 0	89.878 0	99.938 7	111.572	125.084	140.851	10	10
15	15	39.592 9	42.350 1	45.354 2	48.628 6	52.198 0	56.088 2	60.326 6	64.940 6	69.957 2	75.401 4	81.295 7	15	15
20	20	27.798 7	29.130 2	30.522 4	31.974 5	33.484 9	35.050 9	36.668 4	38.331 9	40.034 3	41.766 3	43.516 9	20	20
25	25	18.732 4	19.287 9	19.845 3	20.401 7	20.954 5	21.500 2	22.035 4	22.556 3	23.058 8	23.538 5	23.991 0	25	25
30	30	12.511 0	12.708 2	12.896 3	13.074 2	13.240 4	13.393 8	13.533 1	13.657 1	13.764 5	13.854 2	13.925 1	30	30
35	35	8.393 19	8.438 47	8.476 00	8.505 40	8.526 33	8.538 46	8.541 51	8.535 24	8.519 46	8.494 02	8.458 84	35	35
40	40	5.667 64	5.653 37	5.634 24	5.610 20	5.581 22	5.547 32	5.508 51	5.464 84	5.416 36	5.363 18	5.305 40	40	40
45	45	3.834 86	3.800 68	3.763 83	3.724 36	3.682 36	3.637 87	3.591 00	3.541 82	3.490 45	3.436 97	3.381 51	45	45
50	50	2.575 15	2.537 52	2.498 61	2.458 48	2.417 18	2.374 81	2.331 42	2.287 10	2.241 91	2.195 94	2.149 28	50	50
55	55	1.688 85	1.654 24	1.619 14	1.583 59	1.547 64	1.511 36	1.474 78	1.437 96	1.400 96	1.363 80	1.326 56	55	55
60	60	1.050 93	1.021 49	0.991 984	0.962 448	0.932 915	0.903 415	0.873 979	0.844 639	0.815 419	0.786 356	0.757 469	60	60
65	65	0.581 897	0.557 980	0.534 215	0.510 623	0.487 221	0.464 026	0.441 056	0.418 326	0.395 851	0.373 645	0.351 721	65	65
70	70	0.230 259	0.211 506	0.193 001	0.174 752	0.156 766	0.139 052	0.121 618	0.104 471	0.087 616	0.071 060	0.054 805	70	70
75	75	0.038 061	0.052 248	0.066 158	0.079 795	0.093 154	0.106 239	0.119 038	0.131 558	0.143 793	0.155 746	0.167 416	75	75
80	80	0.246 084	0.256 330	0.266 317	0.276 042	0.285 515	0.294 729	0.303 699	0.312 393	0.320 837	0.329 056	0.337 018	80	80
85	85	0.409 672	0.416 566	0.423 231	0.429 671	0.435 887	0.441 884	0.447 664	0.453 231	0.458 588	0.463 740	0.468 688	85	85
90	90	0.539 965	0.544 019	0.547 885	0.551 565	0.555 063	0.558 383	0.561 529	0.564 504	0.567 314	0.569 961	0.572 449	90	90
95	95	0.644 920	0.646 574	0.648 080	0.649 443	0.650 667	0.651 754	0.652 710	0.653 538	0.654 242	0.654 826	0.655 293	95	95
100	100	0.730 317	0.729 937	0.729 454	0.728 868	0.728 183	0.727 403	0.726 531	0.725 571	0.724 526	0.723 399	0.722 194	100	100
105	105	0.800 412	0.798 311	0.796 144	0.793 913	0.791 621	0.789 272	0.786 868	0.784 413	0.781 908	0.779 356	0.776 761	105	105
110	110	0.858 386	0.854 822	0.851 227	0.847 605	0.843 956	0.840 285	0.836 592	0.832 880	0.829 150	0.825 406	0.821 647	110	110
115	115	0.906 642	0.901 832	0.897 025	0.892 222	0.887 425	0.882 635	0.877 853	0.873 081	0.868 321	0.863 575	0.858 838	115	115
120	120	0.947 022	0.941 150	0.935 310	0.929 504	0.923 731	0.917 992	0.912 388	0.906 620	0.900 989	0.895 395	0.889 835	120	120
125	125	0.980 941	0.974 163	0.967 443	0.960 782	0.954 179	0.947 634	0.941 147	0.934 719	0.928 349	0.922 037	0.915 782	125	125
130	130	1.009 510	1.001 959	0.994 488	0.987 098	0.979 788	0.972 558	0.965 407	0.958 333	0.951 337	0.944 416	0.937 573	130	130
135	135	1.033 594	1.025 384	1.017 273	1.009 263	1.001 352	0.993 539	0.985 823	0.978 202	0.970 674	0.963 239	0.955 896	135	135
140	140	1.053 881	1.045 110	1.036 455	1.027 919	1.019 498	1.011 192	1.002 995	0.994 911	0.986 934	0.979 063	0.971 297	140	140
145	145	1.070 915	1.061 667	1.052 553	1.043 572	1.034 720	1.025 997	1.017 395	1.008 920	1.000 564	0.992 326	0.984 204	145	145
150	150	1.085 128	1.075 481	1.065 981	1.056 625	1.047 412	1.038 338	1.029 399	1.020 595	1.011 921	1.003 376	0.994 956	150	150
155	155	1.096 864	1.086 885	1.077 064	1.067 398	1.057 885	1.048 520	1.039 302	1.030 225	1.021 288	1.012 488	1.003 822	155	155
160	160	1.106 393	1.096 143	1.086 061	1.076 142	1.066 383	1.056 783	1.047 336	1.038 038	1.028 887	1.019 880	1.011 012	160	160
165	165	1.113 929	1.103 464	1.093 174	1.083 055	1.073 103	1.063 315	1.053 686	1.044 213	1.034 893	1.025 721	1.016 696	165	165
170	170	1.119 636	1.109 007	1.098 560	1.088 289	1.078 190	1.068 262	1.058 491	1.048 887	1.039 440	1.030 143	1.020 997	170	170
175	175	1.123 626	1.112 883	1.102 325	1.091 947	1.081 744	1.071 714	1.061 851	1.052 152	1.042 613	1.033 230	1.024 000	175	175
180	180	1.125 992	1.115 181	1.104 557	1.094 115	1.083 853	1.073 760	1.063 842	1.054 087	1.044 494	1.035 060	1.025 780	180	180
		1.126 772	1.115 940	1.105 294	1.094 832	1.084 548	1.074 439	1.064 500	1.054 727	1.045 117	1.035 665	1.026 368		
ψ	z	0.71	0.72	0.73	0.74	0.75	0.76	0.77	0.78	0.79	0.80	0.81	z	ψ

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ	z	0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99	1.00	1.01	z	ψ
0°	5	-1506.31	-2121.88	-3133.82	-4924.07	-8420.00	-16275.0	-38181.5	-127550	-1010100	-	-990098	0°	5
10	10	-161.476	-169.623	-177.020	-183.386	-188.449	-191.962	-193.733	-193.639	-191.646	-187.813	-182.288	10	10
15	15	-58.090 1	-58.758 1	-59.194 0	-59.383 7	-59.317 9	-58.992 4	-58.408 9	-57.574 7	-56.502 6	-55.210 4	-53.719 9	15	15
20	20	-26.060 0	-25.946 5	-25.768 2	-25.526 1	-25.221 6	-24.857 2	-24.436 0	-23.961 9	-23.439 1	-22.872 6	-22.267 1	20	20
25	25	-13.457 8	-13.290 6	-13.103 2	-12.896 6	-12.672 0	-12.430 9	-12.174 5	-11.904 2	-11.621 6	-11.328 2	-11.025 7	25	25
30	30	-7.591 28	-7.458 88	-7.319 88	-7.174 83	-7.024 28	-6.868 83	-6.709 06	-6.545 56	-6.378 93	-6.209 77	-6.038 64	30	30
35	35	-4.512 14	-4.416 11	-4.318 06	-4.218 24	-4.116 92	-4.014 39	-3.910 89	-3.806 68	-3.702 01	-3.597 11	-3.492 22	35	35
40	40	-2.745 91	-2.677 23	-2.608 17	-2.538 85	-2.469 39	-2.399 94	-2.330 59	-2.261 46	-2.192 67	-2.124 32	-2.056 49	40	40
45	45	-1.661 44	-1.612 22	-1.563 19	-1.514 43	-1.465 98	-1.417 92	-1.370 28	-1.323 13	-1.276 50	-1.230 44	-1.184 99	45	45
50	50	-0.958 863	-0.923 446	-0.888 413	-0.853 791	-0.819 606	-0.785 883	-0.752 643	-0.719 908	-0.687 693	-0.656 015	-0.624 897	50	50
55	55	-0.483 534	-0.458 054	-0.432 982	-0.408 329	-0.384 105	-0.360 318	-0.336 976	-0.314 085	-0.291 651	-0.269 676	-0.248 168	55	55
60	60	-0.150 275	-0.132 073	-0.114 244	-0.096 791	-0.079 714	-0.063 014	-0.046 694	-0.030 752	-0.015 188	0.000 000	0.014 813	60	60
65	65	+0.090 450	+0.103 221	+0.115 675	+0.127 812	+0.139 636	+0.151 148	+0.162 352	+0.173 249	+0.183 843	+0.194 138	+0.204 138	65	65
70	70	+0.268 780	+0.277 428	+0.285 817	+0.293 949	+0.301 829	+0.309 460	+0.316 846	+0.323 992	+0.330 900	+0.337 576	+0.344 023	70	70
75	75	+0.403 772	+0.409 238	+0.414 500	+0.419 558	+0.424 423	+0.429 093	+0.433 574	+0.437 871	+0.441 988	+0.445 925	+0.449 691	75	75
80	80	+0.507 877	+0.510 852	+0.513 670	+0.516 335	+0.518 852	+0.521 224	+0.523 455	+0.525 549	+0.527 509	+0.529 340	+0.531 044	80	80
85	85	+0.580 467	+0.590 464	+0.591 347	+0.592 120	+0.592 785	+0.593 347	+0.593 809	+0.594 173	+0.594 443	+0.594 622	+0.594 713	85	85
90	90	+0.654 312	+0.653 722	+0.653 054	+0.652 311	+0.651 498	+0.650 615	+0.649 666	+0.648 653	+0.647 579	+0.646 447	+0.645 258	90	90
95	95	+0.706 480	+0.704 602	+0.702 677	+0.700 709	+0.698 699	+0.696 650	+0.694 563	+0.692 441	+0.690 286	+0.688 099	+0.685 881	95	95
100	100	+0.748 892	+0.745 959	+0.743 007	+0.740 038	+0.737 052	+0.734 052	+0.731 038	+0.728 013	+0.724 978	+0.721 934	+0.718 882	100	100
105	105	+0.783 686	+0.779 883	+0.776 083	+0.772 288	+0.768 499	+0.764 717	+0.760 942	+0.757 176	+0.753 419	+0.749 672	+0.745 935	105	105
110	110	+0.812 446	+0.807 920	+0.803 417	+0.798 936	+0.794 483	+0.790 053	+0.785 648	+0.781 269	+0.776 914	+0.772 586	+0.768 284	110	110
115	115	+0.836 369	+0.831 239	+0.826 149	+0.821 098	+0.816 089	+0.811 118	+0.806 189	+0.801 298	+0.796 448	+0.791 636	+0.786 864	115	115
120	120	+0.856 360	+0.850 724	+0.845 141	+0.839 613	+0.834 138	+0.828 716	+0.823 347	+0.818 029	+0.812 764	+0.807 549	+0.802 386	120	120
125	125	+0.873 124	+0.867 062	+0.861 065	+0.855 135	+0.849 270	+0.843 468	+0.837 730	+0.832 055	+0.826 442	+0.820 888	+0.815 396	125	125
130	130	+0.887 203	+0.880 781	+0.874 436	+0.868 168	+0.861 974	+0.855 854	+0.849 806	+0.843 830	+0.837 924	+0.832 087	+0.826 319	130	130
135	135	+0.899 023	+0.892 300	+0.885 662	+0.879 109	+0.872 639	+0.866 251	+0.859 945	+0.853 714	+0.847 562	+0.841 507	+0.835 500	135	135
140	140	+0.908 920	+0.901 944	+0.895 060	+0.888 268	+0.881 567	+0.874 954	+0.868 429	+0.861 987	+0.855 630	+0.849 355	+0.843 161	140	140
145	145	+0.917 158	+0.909 971	+0.902 883	+0.895 892	+0.888 998	+0.882 197	+0.875 490	+0.868 873	+0.862 345	+0.855 904	+0.849 550	145	145
150	150	+0.923 947	+0.916 585	+0.909 328	+0.902 174	+0.895 120	+0.888 165	+0.881 308	+0.874 546	+0.867 877	+0.861 299	+0.854 811	150	150
155	155	+0.929 450	+0.921 946	+0.914 552	+0.907 265	+0.900 083	+0.893 002	+0.886 024	+0.879 144	+0.872 360	+0.865 672	+0.859 076	155	155
160	160	+0.933 796	+0.926 181	+0.918 679	+0.911 286	+0.904 002	+0.896 824	+0.889 749	+0.882 775	+0.875 902	+0.869 125	+0.862 445	160	160
165	165	+0.937 084	+0.929 386	+0.921 801	+0.914 327	+0.906 967	+0.899 714	+0.892 568	+0.885 522	+0.878 579	+0.871 737	+0.864 994	165	165
170	170	+0.939 381	+0.931 622	+0.923 980	+0.916 452	+0.909 037	+0.901 732	+0.894 533	+0.887 440	+0.880 451	+0.873 562	+0.866 772	170	170
175	175	+0.940 743	+0.932 947	+0.925 272	+0.917 712	+0.910 264	+0.902 928	+0.895 698	+0.888 577	+0.881 561	+0.874 644	+0.867 827	175	175
180	180	+0.941 191	+0.933 386	+0.925 698	+0.918 127	+0.910 669	+0.903 323	+0.896 084	+0.888 953	+0.881 925	+0.875 000	+0.868 174	180	180

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ϕ AND $\frac{1}{m}$.

ϕ	z	1.11	1.12	1.13	1.14	1.15	1.16	1.17	1.18	1.19	1.20	1.21	z	ϕ
0°	0°	-675.959	-515.807	-401.917	-318.800	-256.779	-209.604	-173.113	-144.464	-121.675	-103.333	-88.413 0	5	0°
5	5	-304.962	-256.352	-216.440	-183.609	-156.521	-134.086	-115.424	-99.823 1	-86.732 5	-75.684 2	-66.318 7	10	5
10	10	-90.933 5	-82.881 1	-75.425 7	-68.568 0	-62.293 4	-56.576 0	-51.383 4	-46.678 9	-42.424 3	-38.581 6	-35.113 5	15	10
15	15	-33.760 5	-31.779 7	-29.866 8	-28.029 5	-26.273 2	-24.601 5	-23.016 1	-21.517 5	-20.104 7	-18.776 0	-17.529 3	20	15
20	20	-15.251 1	-14.555 1	-13.874 0	-13.210 1	-12.565 1	-11.940 4	-11.337 1	-10.755 9	-10.197 4	-9.661 73	-9.149 03	25	20
25	25	-7.823 92	-7.513 80	-7.209 36	-6.911 26	-6.620 02	-6.336 13	-6.059 93	-5.791 75	-5.531 77	-5.280 18	-5.037 05	30	25
30	30	-4.330 72	-4.169 40	-4.011 06	-3.855 91	-3.704 12	-3.555 84	-3.411 21	-3.270 33	-3.133 27	-3.000 08	-2.870 81	35	30
35	35	-2.486 25	-2.393 28	-2.302 13	-2.212 89	-2.125 61	-2.040 33	-1.957 10	-1.875 96	-1.796 91	-1.719 97	-1.645 16	40	35
40	40	-1.424 22	-1.366 85	-1.310 69	-1.255 78	-1.202 13	-1.149 74	-1.098 64	-1.048 82	-1.000 29	-0.953 043	-0.907 074	45	40
45	45	-0.770 468	-0.733 450	-0.697 282	-0.661 968	-0.627 510	-0.593 909	-0.561 158	-0.529 260	-0.498 206	-0.467 989	-0.438 602	50	45
50	50	-0.346 175	-0.321 652	-0.297 739	-0.274 433	-0.251 731	-0.229 626	-0.208 115	-0.187 190	-0.166 844	-0.147 071	-0.127 862	55	50
55	55	-0.058 747	-0.042 334	-0.026 367	-0.010 840	0.004 253	0.018 918	0.033 160	0.046 987	0.060 405	0.073 421	0.086 042	60	55
60	60	0.142 970	0.153 877	0.164 457	0.174 715	0.184 656	0.194 287	0.203 613	0.212 640	0.221 373	0.229 820	0.237 987	65	60
65	65	0.288 788	0.295 819	0.302 609	0.309 164	0.315 487	0.321 584	0.327 461	0.333 121	0.338 571	0.343 815	0.348 857	70	65
70	70	0.396 873	0.401 092	0.405 136	0.409 007	0.412 711	0.416 252	0.419 633	0.422 859	0.425 934	0.428 862	0.431 647	75	70
75	75	0.478 730	0.480 857	0.482 857	0.484 733	0.486 492	0.488 133	0.489 663	0.491 083	0.492 398	0.493 607	0.494 718	80	75
80	80	0.541 882	0.542 417	0.542 864	0.543 226	0.543 506	0.543 707	0.543 831	0.543 881	0.543 859	0.543 768	0.543 611	85	80
85	85	0.591 391	0.590 693	0.589 937	0.589 127	0.588 264	0.587 351	0.586 389	0.585 379	0.584 326	0.583 229	0.582 092	90	85
90	90	0.630 750	0.629 081	0.627 380	0.625 647	0.623 886	0.622 097	0.620 281	0.618 441	0.616 577	0.614 691	0.612 784	95	90
95	95	0.662 421	0.659 977	0.657 522	0.655 054	0.652 575	0.650 089	0.647 594	0.645 092	0.642 583	0.640 068	0.637 549	100	95
100	100	0.688 174	0.685 106	0.682 041	0.678 980	0.675 925	0.672 875	0.669 832	0.666 796	0.663 767	0.660 746	0.657 734	105	100
105	105	0.709 304	0.705 726	0.702 165	0.698 621	0.695 094	0.691 587	0.688 097	0.684 626	0.681 174	0.677 741	0.674 327	110	105
110	110	0.726 773	0.722 774	0.718 805	0.714 864	0.710 952	0.707 067	0.703 207	0.699 382	0.695 583	0.691 811	0.688 068	115	110
115	115	0.741 303	0.736 958	0.732 651	0.728 381	0.724 149	0.719 953	0.715 795	0.711 670	0.707 583	0.703 532	0.699 516	120	115
120	120	0.753 450	0.748 816	0.744 227	0.739 683	0.735 183	0.730 728	0.726 316	0.721 948	0.717 621	0.713 337	0.709 094	125	120
125	125	0.763 638	0.758 760	0.753 936	0.749 163	0.744 441	0.739 769	0.735 146	0.730 572	0.726 046	0.721 568	0.717 136	130	125
130	130	0.772 189	0.767 111	0.762 090	0.757 126	0.752 218	0.747 365	0.742 566	0.737 821	0.733 128	0.728 487	0.723 897	135	130
135	135	0.779 371	0.774 124	0.768 939	0.763 815	0.758 750	0.753 746	0.748 799	0.743 910	0.739 077	0.734 300	0.729 577	140	135
140	140	0.785 386	0.779 997	0.774 674	0.769 415	0.764 220	0.759 090	0.754 020	0.749 010	0.744 061	0.739 170	0.734 337	145	140
145	145	0.790 392	0.784 886	0.779 448	0.774 079	0.768 776	0.763 540	0.758 367	0.753 259	0.748 212	0.743 228	0.738 303	150	145
150	150	0.794 518	0.788 915	0.783 383	0.777 922	0.772 530	0.767 207	0.761 951	0.756 760	0.751 634	0.746 571	0.741 571	155	150
155	155	0.797 861	0.792 180	0.786 572	0.781 037	0.775 573	0.770 180	0.764 855	0.759 598	0.754 407	0.749 282	0.744 221	160	155
160	160	0.800 503	0.794 760	0.789 092	0.783 499	0.777 978	0.772 529	0.767 150	0.761 841	0.756 600	0.751 425	0.746 315	165	160
165	165	0.802 502	0.796 712	0.790 998	0.785 360	0.779 796	0.774 306	0.768 888	0.763 540	0.758 260	0.753 049	0.747 903	170	165
170	170	0.803 896	0.798 075	0.792 330	0.786 662	0.781 068	0.775 548	0.770 100	0.764 723	0.759 417	0.754 179	0.749 007	175	170
175	175	0.804 725	0.798 883	0.793 119	0.787 432	0.781 821	0.776 284	0.770 822	0.765 431	0.760 110	0.754 857	0.749 669	180	175
180	180	0.804 998	0.799 150	0.793 380	0.787 686	0.782 069	0.776 527	0.771 057	0.765 658	0.760 330	0.755 071	0.749 879		180

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{E}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ	z	1.31	1.32	1.33	1.34	1.35	1.36	1.37	1.38	1.39	1.40	1.41	z	ψ
0°	0°	24.860 4	22.361 8	20.170 3	18.240 8	16.536 0	15.024 6	13.680 4	12.481 3	11.408 6	10.446 4	9.581 11	0°	5
5	5	21.334 0	19.333 8	17.558 9	15.979 4	14.570 0	13.309 0	12.178 0	11.161 3	10.245 1	9.417 68	8.668 91	5	10
10	10	14.473 1	13.324 7	12.280 1	11.329 0	10.461 8	9.670 21	8.946 76	8.284 79	7.678 32	7.122 05	6.611 23	10	15
15	15	8.800 96	8.222 25	7.683 64	7.182 30	6.715 58	6.280 96	5.876 13	5.498 93	5.147 33	4.819 48	4.513 64	15	20
20	20	5.106 12	4.904 05	4.627 73	4.366 42	4.119 36	3.885 82	3.665 11	3.456 53	3.259 45	3.073 23	2.897 27	20	25
25	25	3.060 50	2.905 06	2.756 47	2.614 49	2.478 88	2.349 41	2.225 83	2.107 92	1.995 44	1.888 16	1.785 86	25	30
30	30	1.789 94	1.701 92	1.617 24	1.535 82	1.457 56	1.382 37	1.310 15	1.240 81	1.174 25	1.110 38	1.049 11	30	35
35	35	1.010 86	0.958 220	0.907 400	0.858 352	0.811 036	0.765 402	0.721 408	0.679 005	0.638 155	0.598 805	0.560 912	35	40
40	40	0.515 142	0.482 331	0.450 592	0.419 902	0.390 236	0.361 570	0.333 878	0.307 134	0.281 314	0.256 392	0.232 344	40	45
45	45	0.188 014	0.166 993	0.146 649	0.126 965	0.107 927	0.089 518	0.071 723	0.054 525	0.037 911	0.021 863	0.006 368	45	50
50	50	0.035 225	0.048 852	0.062 034	0.074 780	0.087 102	0.099 010	0.110 514	0.121 625	0.132 352	0.142 708	0.152 700	50	55
55	55	0.192 195	0.200 974	0.209 449	0.217 629	0.225 521	0.233 133	0.240 471	0.247 543	0.254 356	0.260 915	0.267 234	55	60
60	60	0.305 529	0.311 002	0.316 265	0.321 324	0.326 185	0.330 852	0.335 331	0.339 627	0.343 746	0.347 692	0.351 470	60	65
65	65	0.389 283	0.392 427	0.395 426	0.398 281	0.401 000	0.403 586	0.406 040	0.408 369	0.410 576	0.412 664	0.414 638	65	70
70	70	0.452 460	0.453 915	0.455 270	0.456 527	0.457 691	0.458 762	0.459 745	0.460 642	0.461 457	0.462 190	0.462 847	70	75
75	75	0.500 983	0.501 184	0.501 318	0.501 382	0.501 393	0.501 341	0.501 227	0.501 056	0.500 832	0.500 555	0.500 228	75	80
80	80	0.538 848	0.538 098	0.537 305	0.536 472	0.535 599	0.534 689	0.533 743	0.532 763	0.531 749	0.530 704	0.529 629	80	85
85	85	0.568 815	0.567 329	0.565 821	0.564 290	0.562 742	0.561 167	0.559 576	0.557 968	0.556 343	0.554 703	0.553 048	85	90
90	90	0.592 825	0.590 763	0.588 692	0.586 614	0.584 528	0.582 437	0.580 341	0.578 239	0.576 134	0.574 026	0.571 914	90	95
95	95	0.612 273	0.609 752	0.607 233	0.604 710	0.602 209	0.599 704	0.597 205	0.594 710	0.592 222	0.589 741	0.587 266	95	100
100	100	0.628 176	0.625 284	0.622 406	0.619 540	0.616 688	0.613 850	0.611 025	0.608 215	0.605 418	0.602 636	0.599 868	100	105
105	105	0.641 285	0.638 093	0.634 921	0.631 770	0.628 640	0.625 530	0.622 441	0.619 373	0.616 325	0.613 298	0.610 291	105	110
110	110	0.652 167	0.648 727	0.645 315	0.641 929	0.638 570	0.635 239	0.631 932	0.628 652	0.625 398	0.622 170	0.618 967	110	115
115	115	0.661 250	0.657 607	0.653 996	0.650 417	0.646 870	0.643 353	0.639 867	0.636 412	0.632 987	0.629 593	0.626 228	115	120
120	120	0.668 865	0.665 052	0.661 276	0.657 536	0.653 831	0.650 161	0.646 526	0.642 926	0.639 360	0.635 827	0.632 328	120	125
125	125	0.675 268	0.671 313	0.667 398	0.663 523	0.659 687	0.655 890	0.652 129	0.648 409	0.644 724	0.641 076	0.637 464	125	130
130	130	0.680 653	0.676 583	0.672 553	0.668 566	0.664 621	0.660 716	0.656 853	0.653 030	0.649 247	0.645 502	0.641 795	130	135
135	135	0.685 190	0.681 018	0.676 892	0.672 810	0.668 772	0.664 779	0.660 829	0.656 920	0.653 054	0.649 229	0.645 444	135	140
140	140	0.688 991	0.684 737	0.680 530	0.676 369	0.672 255	0.668 188	0.664 165	0.660 186	0.656 251	0.652 358	0.648 507	140	145
145	145	0.692 160	0.687 837	0.683 563	0.679 338	0.675 161	0.671 031	0.666 947	0.662 909	0.658 916	0.654 968	0.651 065	145	150
150	150	0.694 775	0.690 395	0.686 066	0.681 787	0.677 558	0.673 378	0.669 244	0.665 159	0.661 119	0.657 124	0.653 173	150	155
155	155	0.696 896	0.692 471	0.688 097	0.683 776	0.679 504	0.675 282	0.671 108	0.666 983	0.662 903	0.658 873	0.654 887	155	160
160	160	0.698 574	0.694 115	0.689 703	0.685 347	0.681 042	0.676 788	0.672 583	0.668 427	0.664 318	0.660 257	0.656 242	160	165
165	165	0.699 843	0.695 352	0.690 920	0.686 538	0.682 208	0.677 929	0.673 700	0.669 520	0.665 388	0.661 305	0.657 268	165	170
170	170	0.700 731	0.696 223	0.691 769	0.687 369	0.683 021	0.678 725	0.674 479	0.670 284	0.666 137	0.662 038	0.657 986	170	175
175	175	0.701 257	0.696 738	0.692 273	0.687 862	0.683 503	0.679 197	0.674 941	0.670 736	0.666 580	0.662 472	0.658 412	175	180
180	180	0.701 430	0.696 907	0.692 439	0.688 025	0.683 664	0.679 354	0.675 095	0.670 886	0.666 726	0.662 615	0.658 552	180	

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ	z	1.41	1.42	1.43	1.44	1.45	1.46	1.47	1.48	1.49	1.50	1.51	z	ψ
0°	0°	9.581 11	8.801 03	8.096 16	7.457 84	6.878 58	6.351 84	5.871 95	5.433 95	5.033 46	4.666 67	4.330 18	0°	5
5	5	8.668 91	7.989 94	7.373 06	6.811 55	6.299 53	5.831 84	5.403 94	5.011 82	4.651 93	4.321 15	4.016 68	5	10
10	10	6.611 23	6.141 59	5.709 33	5.311 01	4.943 56	4.604 24	4.290 54	4.000 25	3.731 33	3.481 97	3.250 53	10	15
15	15	4.513 64	4.228 20	3.961 68	3.712 70	3.480 00	3.262 40	3.058 82	2.868 26	2.689 78	2.522 54	2.365 75	15	20
20	20	2.897 27	2.731 00	2.573 88	2.425 39	2.285 05	2.152 38	2.026 95	1.908 35	1.796 18	1.690 07	1.589 67	20	25
25	25	1.785 86	1.688 33	1.595 36	1.506 74	1.422 28	1.341 78	1.265 06	1.191 95	1.122 28	1.055 88	0.992 612	25	30
30	30	1.049 11	0.990 346	0.933 992	0.879 963	0.828 170	0.778 529	0.730 957	0.685 372	0.641 698	0.599 858	0.559 777	30	35
35	35	0.560 912	0.524 431	0.489 317	0.455 525	0.423 017	0.391 744	0.361 668	0.332 746	0.304 937	0.278 207	0.252 507	35	40
40	40	0.232 344	0.209 144	0.186 768	0.165 191	0.144 390	0.124 341	0.105 019	0.086 472	0.068 472	0.051 202	0.034 571	40	45
45	45	0.006 368	0.008 591	0.023 029	0.036 959	0.050 397	0.063 357	0.075 853	0.087 899	0.099 509	0.110 695	0.121 471	45	50
50	50	0.152 700	0.162 339	0.171 636	0.180 599	0.189 238	0.197 562	0.205 581	0.213 304	0.220 738	0.227 893	0.234 777	50	55
55	55	0.267 234	0.273 313	0.279 160	0.284 782	0.290 186	0.295 378	0.300 364	0.305 151	0.309 743	0.314 148	0.318 370	55	60
60	60	0.351 470	0.355 086	0.358 543	0.361 847	0.365 002	0.368 012	0.370 882	0.373 617	0.376 219	0.378 694	0.381 044	60	65
65	65	0.414 638	0.416 500	0.418 253	0.419 903	0.421 449	0.422 899	0.424 253	0.425 514	0.426 686	0.427 772	0.428 775	65	70
70	70	0.462 847	0.463 428	0.463 936	0.464 374	0.464 744	0.465 049	0.465 290	0.465 470	0.465 590	0.465 653	0.465 662	70	75
75	75	0.500 228	0.499 848	0.499 430	0.498 962	0.498 451	0.497 898	0.497 305	0.496 672	0.496 004	0.495 299	0.494 560	75	80
80	80	0.529 629	0.528 524	0.527 392	0.526 234	0.525 050	0.523 843	0.522 612	0.521 359	0.520 086	0.518 793	0.517 482	80	85
85	85	0.553 048	0.551 379	0.549 696	0.548 002	0.546 297	0.544 581	0.542 855	0.541 121	0.539 378	0.537 628	0.535 870	85	90
90	90	0.571 914	0.569 801	0.567 686	0.565 570	0.563 454	0.561 337	0.559 222	0.557 107	0.554 994	0.552 882	0.550 773	90	95
95	95	0.587 266	0.584 798	0.582 338	0.579 886	0.577 442	0.575 006	0.572 579	0.570 160	0.567 751	0.565 351	0.562 961	95	100
100	100	0.599 868	0.597 115	0.594 376	0.591 652	0.588 944	0.586 250	0.583 571	0.580 908	0.578 259	0.575 626	0.573 009	100	105
105	105	0.610 291	0.607 305	0.604 343	0.601 395	0.598 471	0.595 567	0.592 683	0.589 820	0.586 976	0.584 153	0.581 350	105	110
110	110	0.618 967	0.615 791	0.612 639	0.609 512	0.606 411	0.603 334	0.600 282	0.597 254	0.594 251	0.591 271	0.588 316	110	115
115	115	0.626 228	0.622 893	0.619 587	0.616 310	0.613 062	0.609 842	0.606 650	0.603 487	0.600 351	0.597 242	0.594 161	115	120
120	120	0.632 328	0.628 861	0.625 427	0.622 025	0.618 655	0.615 317	0.612 009	0.608 732	0.605 485	0.602 269	0.599 082	120	125
125	125	0.637 464	0.633 888	0.630 348	0.626 842	0.623 370	0.619 932	0.616 527	0.613 156	0.609 817	0.606 511	0.603 238	125	130
130	130	0.641 795	0.638 127	0.634 497	0.630 904	0.627 347	0.623 826	0.620 341	0.616 890	0.613 474	0.610 092	0.606 744	130	135
135	135	0.645 444	0.641 700	0.637 995	0.634 328	0.630 699	0.627 109	0.623 556	0.620 039	0.616 558	0.613 113	0.609 703	135	140
140	140	0.648 507	0.644 698	0.640 930	0.637 203	0.633 515	0.629 867	0.626 256	0.622 684	0.619 150	0.615 653	0.612 192	140	145
145	145	0.651 065	0.647 201	0.643 381	0.639 603	0.635 866	0.632 169	0.628 512	0.624 894	0.621 314	0.617 773	0.614 268	145	150
150	150	0.653 173	0.649 266	0.645 405	0.641 586	0.637 808	0.634 071	0.630 375	0.626 720	0.623 103	0.619 526	0.615 987	150	155
155	155	0.654 887	0.650 945	0.647 048	0.643 195	0.639 384	0.635 616	0.631 889	0.628 203	0.624 557	0.620 951	0.617 384	155	160
160	160	0.656 242	0.652 273	0.648 349	0.644 469	0.640 633	0.636 839	0.633 187	0.629 377	0.625 707	0.622 078	0.618 488	160	165
165	165	0.657 268	0.653 278	0.649 334	0.645 434	0.641 578	0.637 765	0.633 994	0.630 266	0.626 578	0.622 931	0.619 323	165	170
170	170	0.657 986	0.653 981	0.650 022	0.646 108	0.642 238	0.638 412	0.634 628	0.630 887	0.627 187	0.623 528	0.619 909	170	175
175	175	0.658 412	0.654 398	0.650 430	0.646 508	0.642 630	0.638 795	0.635 004	0.631 255	0.627 548	0.623 882	0.620 256	175	180
180	180	0.658 552	0.654 536	0.650 565	0.646 640	0.642 760	0.638 923	0.635 129	0.631 378	0.627 669	0.624 000	0.620 373	180	
ψ	z	1.41	1.42	1.43	1.44	1.45	1.46	1.47	1.48	1.49	1.50	1.51	z	ψ

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ	z	1.51	1.52	1.53	1.54	1.55	1.56	1.57	1.58	1.59	1.60	1.61	z	ψ
0°	0°	4.330 18	4.021 03	3.736 57	3.474 45	3.232 59	3.009 13	2.802 40	2.610 92	2.433 36	2.268 52	2.115 31	0°	0°
5	5	4.016 68	3.736 06	3.477 08	3.237 76	3.016 34	2.811 24	2.621 03	2.444 45	2.280 35	2.127 68	1.985 51	5	5
10	10	3.250 53	3.035 48	2.835 52	2.649 41	2.476 03	2.314 38	2.163 53	2.022 65	1.890 98	1.767 82	1.652 53	10	10
15	15	2.365 75	2.218 67	2.080 63	1.951 02	1.829 24	1.714 78	1.607 13	1.505 84	1.410 48	1.320 67	1.236 04	15	15
20	20	1.589 67	1.494 66	1.404 72	1.319 57	1.238 93	1.162 54	1.090 17	1.021 57	0.956 547	0.894 888	0.836 407	20	20
25	25	0.992 612	0.932 312	0.874 840	0.820 062	0.767 851	0.718 078	0.670 629	0.625 391	0.582 261	0.541 129	0.501 904	25	25
30	30	0.559 777	0.521 384	0.484 613	0.449 394	0.415 664	0.383 361	0.352 428	0.322 804	0.294 436	0.267 271	0.241 256	30	30
35	35	0.252 507	0.227 820	0.204 094	0.181 295	0.159 393	0.138 353	0.118 144	0.098 734	0.080 094	0.062 196	0.045 010	35	35
40	40	0.034 571	0.018 559	0.003 146	0.011 688	0.025 963	0.039 698	0.052 911	0.065 620	0.077 841	0.089 593	0.100 892	40	40
45	45	0.121 471	0.131 850	0.141 843	0.151 464	0.160 723	0.169 634	0.178 206	0.186 450	0.194 378	0.202 000	0.209 325	45	45
50	50	0.234 777	0.241 399	0.247 765	0.253 886	0.259 767	0.265 416	0.270 841	0.276 048	0.281 045	0.285 838	0.290 434	50	50
55	55	0.318 370	0.322 416	0.326 289	0.329 996	0.333 542	0.336 933	0.340 172	0.343 264	0.346 215	0.349 028	0.351 709	55	55
60	60	0.381 044	0.383 275	0.385 390	0.387 392	0.389 284	0.391 072	0.392 757	0.394 343	0.395 833	0.397 230	0.398 538	60	60
65	65	0.428 775	0.429 696	0.430 539	0.431 307	0.432 001	0.432 624	0.433 179	0.433 668	0.434 093	0.434 457	0.434 761	65	65
70	70	0.465 662	0.465 617	0.465 521	0.465 376	0.465 186	0.464 943	0.464 660	0.464 333	0.463 966	0.463 559	0.463 114	70	70
75	75	0.494 560	0.493 787	0.492 984	0.492 149	0.491 286	0.490 395	0.489 478	0.488 534	0.487 567	0.486 576	0.485 563	75	75
80	80	0.517 482	0.516 152	0.514 805	0.513 443	0.512 065	0.510 673	0.509 268	0.507 850	0.506 420	0.504 978	0.503 527	80	80
85	85	0.535 870	0.534 106	0.532 337	0.530 563	0.528 784	0.527 001	0.525 214	0.523 425	0.521 634	0.519 840	0.518 045	85	85
90	90	0.550 773	0.548 666	0.546 563	0.544 463	0.542 367	0.540 274	0.538 187	0.536 103	0.534 025	0.531 952	0.529 885	90	90
95	95	0.562 961	0.560 580	0.558 209	0.555 849	0.553 498	0.551 159	0.548 829	0.546 511	0.544 203	0.541 906	0.539 620	95	95
100	100	0.573 009	0.570 406	0.567 820	0.565 248	0.562 692	0.560 152	0.557 627	0.555 118	0.552 624	0.550 145	0.547 682	100	100
105	105	0.581 350	0.578 570	0.575 804	0.573 060	0.570 337	0.567 633	0.564 948	0.562 283	0.559 637	0.557 010	0.554 402	105	105
110	110	0.588 316	0.585 384	0.582 475	0.579 590	0.576 728	0.573 889	0.571 073	0.568 279	0.565 508	0.562 759	0.560 032	110	110
115	115	0.594 161	0.591 106	0.588 077	0.585 074	0.582 097	0.579 147	0.576 221	0.573 321	0.570 446	0.567 596	0.564 770	115	115
120	120	0.599 082	0.595 925	0.592 795	0.589 695	0.586 623	0.583 580	0.580 564	0.577 575	0.574 613	0.571 678	0.568 769	120	120
125	125	0.603 238	0.599 993	0.596 781	0.593 600	0.590 448	0.587 326	0.584 235	0.581 173	0.578 138	0.575 130	0.572 151	125	125
130	130	0.606 744	0.603 429	0.600 146	0.596 896	0.593 678	0.590 491	0.587 336	0.584 211	0.581 115	0.578 050	0.575 014	130	130
135	135	0.609 703	0.606 328	0.602 987	0.599 680	0.596 406	0.593 164	0.589 956	0.586 778	0.583 633	0.580 518	0.577 434	135	135
140	140	0.612 192	0.608 766	0.605 376	0.602 021	0.598 700	0.595 413	0.592 160	0.588 939	0.585 750	0.582 594	0.579 469	140	140
145	145	0.614 268	0.610 803	0.607 372	0.603 977	0.600 618	0.597 293	0.594 002	0.590 745	0.587 523	0.584 331	0.581 172	145	145
150	150	0.615 987	0.612 487	0.609 023	0.605 595	0.602 203	0.598 847	0.595 526	0.592 239	0.588 986	0.585 768	0.582 582	150	150
155	155	0.617 384	0.613 855	0.610 363	0.606 910	0.603 492	0.600 111	0.596 764	0.593 454	0.590 178	0.586 936	0.583 727	155	155
160	160	0.618 488	0.614 937	0.611 425	0.607 950	0.604 512	0.601 111	0.597 746	0.594 416	0.591 121	0.587 861	0.584 635	160	160
165	165	0.619 323	0.615 756	0.612 227	0.608 737	0.605 285	0.601 869	0.598 490	0.595 146	0.591 837	0.588 563	0.585 323	165	165
170	170	0.619 909	0.616 331	0.612 791	0.609 289	0.605 825	0.602 399	0.599 009	0.595 655	0.592 337	0.589 053	0.585 804	170	170
175	175	0.620 256	0.616 672	0.613 125	0.609 617	0.606 147	0.602 714	0.599 318	0.595 958	0.592 634	0.589 344	0.586 089	175	175
180	180	0.620 373	0.616 784	0.613 235	0.609 725	0.606 253	0.602 818	0.599 420	0.596 058	0.592 731	0.589 440	0.586 183	180	180

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ	$\frac{1}{m}$	1.61	1.62	1.63	1.64	1.65	1.66	1.67	1.68	1.69	1.70	1.71	z
0°	—	2.115 31	1.972 78	1.840 03	1.716 28	1.600 81	1.492 96	1.392 14	1.297 82	1.209 50	1.126 74	1.049 12	0°
5	—	1.985 51	1.852 98	1.729 33	1.613 87	1.505 95	1.405 00	1.310 48	1.221 93	1.138 90	1.060 99	0.987 826	5
10	—	1.652 53	1.544 53	1.443 29	1.348 31	1.259 16	1.175 41	1.096 69	1.022 65	0.952 978	0.887 368	0.825 550	10
15	—	1.236 04	1.156 25	1.081 00	1.009 98	0.942 943	0.879 624	0.819 795	0.763 238	0.709 752	0.659 151	0.611 258	15
20	—	0.836 407	0.780 925	0.728 276	0.678 301	0.630 853	0.585 792	0.542 987	0.502 316	0.463 663	0.426 918	0.391 979	20
25	—	0.501 904	0.464 492	0.428 807	0.394 764	0.362 286	0.331 294	0.301 719	0.273 495	0.246 555	0.220 838	0.196 285	25
30	—	0.241 256	0.216 348	0.192 496	0.169 655	0.147 783	0.126 838	0.106 782	0.087 576	0.069 185	0.051 572	0.034 707	30
35	—	0.045 010	0.028 511	0.012 670	0.002 535	0.017 130	0.031 138	0.044 581	0.057 482	0.069 861	0.081 738	0.093 133	35
40	+	0.100 892	0.111 752	0.122 190	0.132 220	0.141 858	0.151 116	0.160 008	0.168 548	0.176 747	0.184 619	0.192 174	40
45	+	0.209 325	0.216 365	0.223 127	0.229 621	0.235 857	0.241 843	0.247 587	0.253 098	0.258 384	0.263 451	0.268 309	45
50	+	0.290 434	0.294 838	0.299 058	0.303 099	0.306 967	0.310 668	0.314 207	0.317 589	0.320 820	0.323 905	0.326 848	50
55	+	0.351 709	0.354 260	0.356 688	0.358 994	0.361 184	0.363 261	0.365 229	0.367 091	0.368 852	0.370 513	0.372 079	55
60	+	0.398 538	0.399 759	0.400 897	0.401 953	0.402 931	0.403 832	0.404 661	0.405 419	0.406 108	0.406 732	0.407 291	60
65	+	0.434 761	0.435 007	0.435 198	0.435 336	0.435 420	0.435 456	0.435 444	0.435 385	0.435 281	0.435 134	0.434 946	65
70	+	0.463 114	0.462 633	0.462 116	0.461 565	0.460 982	0.460 368	0.459 723	0.459 050	0.458 348	0.457 620	0.456 867	70
75	+	0.485 563	0.484 528	0.483 473	0.482 399	0.481 306	0.480 195	0.479 068	0.477 925	0.476 767	0.475 594	0.474 408	75
80	+	0.503 527	0.502 065	0.500 594	0.499 115	0.497 628	0.496 133	0.494 632	0.493 124	0.491 611	0.490 092	0.488 569	80
85	+	0.518 045	0.516 249	0.514 452	0.512 655	0.510 858	0.509 061	0.507 267	0.505 472	0.503 680	0.501 889	0.500 100	85
90	+	0.529 885	0.527 824	0.525 768	0.523 719	0.521 676	0.519 640	0.517 610	0.515 588	0.513 573	0.511 565	0.509 565	90
95	+	0.539 620	0.537 345	0.535 081	0.532 830	0.530 589	0.528 360	0.526 143	0.523 937	0.521 742	0.519 560	0.517 389	95
100	+	0.547 682	0.545 235	0.542 803	0.540 386	0.537 985	0.535 600	0.533 229	0.530 874	0.528 534	0.526 209	0.523 899	100
105	+	0.554 402	0.551 813	0.549 243	0.546 692	0.544 159	0.541 645	0.539 149	0.536 672	0.534 212	0.531 771	0.529 347	105
110	+	0.560 032	0.557 327	0.554 642	0.551 980	0.549 339	0.546 718	0.544 119	0.541 541	0.538 983	0.536 445	0.533 928	110
115	+	0.564 770	0.561 966	0.559 187	0.556 433	0.553 701	0.550 994	0.548 310	0.545 648	0.543 008	0.540 390	0.537 794	115
120	+	0.568 769	0.565 886	0.563 029	0.560 198	0.557 392	0.554 610	0.551 853	0.549 121	0.546 412	0.543 728	0.541 067	120
125	+	0.572 151	0.569 201	0.566 278	0.563 383	0.560 514	0.557 671	0.554 854	0.552 064	0.549 300	0.546 558	0.543 841	125
130	+	0.575 014	0.572 008	0.569 030	0.566 080	0.563 158	0.560 264	0.557 397	0.554 556	0.551 742	0.548 954	0.546 192	130
135	+	0.577 434	0.574 380	0.571 355	0.568 360	0.565 394	0.562 457	0.559 547	0.556 665	0.553 811	0.550 984	0.548 183	135
140	+	0.579 469	0.576 376	0.573 313	0.570 280	0.567 276	0.564 303	0.561 358	0.558 442	0.555 554	0.552 694	0.549 861	140
145	+	0.581 172	0.578 046	0.574 951	0.571 886	0.568 852	0.565 849	0.562 875	0.559 930	0.557 014	0.554 127	0.551 267	145
150	+	0.582 582	0.579 428	0.576 306	0.573 216	0.570 157	0.567 129	0.564 130	0.561 162	0.558 223	0.555 313	0.552 431	150
155	+	0.583 727	0.580 551	0.577 407	0.574 297	0.571 218	0.568 169	0.565 151	0.562 164	0.559 207	0.556 278	0.553 379	155
160	+	0.584 635	0.581 442	0.578 282	0.575 154	0.572 058	0.568 994	0.565 961	0.562 958	0.559 985	0.557 043	0.554 130	160
165	+	0.585 323	0.582 116	0.578 943	0.575 803	0.572 695	0.569 618	0.566 574	0.563 560	0.560 577	0.557 623	0.554 699	165
170	+	0.585 804	0.582 589	0.579 408	0.576 258	0.573 141	0.570 057	0.566 904	0.563 782	0.560 691	0.557 630	0.554 597	170
175	+	0.586 089	0.582 868	0.579 681	0.576 527	0.573 405	0.570 316	0.567 258	0.564 232	0.561 236	0.558 270	0.555 334	175
180	+	0.586 183	0.582 961	0.579 772	0.576 617	0.573 494	0.570 403	0.567 344	0.564 315	0.561 317	0.558 349	0.555 411	180
ψ	z	1.61	1.62	1.63	1.64	1.65	1.66	1.67	1.68	1.69	1.70	1.71	$\frac{1}{m}$

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\phi}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

$\psi \backslash z$	1.71	1.72	1.73	1.74	1.75	1.76	1.77	1.78	1.79	1.80	1.81	$z \backslash \psi$
0°	—	0.976 270	—	0.843 547	—	0.783 069	—	0.672 555	—	0.574 434	—	0.487 114
5	—	0.919 078	—	0.793 612	—	0.736 347	—	0.631 547	—	0.538 321	—	0.455 211
10	—	0.767 272	—	0.660 426	—	0.611 444	—	0.521 436	—	0.440 959	—	0.368 875
15	—	0.565 912	—	0.482 263	—	0.443 686	—	0.372 407	—	0.308 232	—	0.250 374
20	—	0.358 749	—	0.297 059	—	0.268 433	—	0.215 236	—	0.166 991	—	0.123 202
25	—	0.172 842	—	0.129 080	—	0.108 663	—	0.070 531	—	0.035 725	—	0.003 947
30	—	0.018 556	—	0.003 090	—	0.025 904	—	0.052 492	—	0.076 874	—	0.099 231
35	+	0.104 065	+	0.124 608	+	0.134 254	+	0.152 373	+	0.169 028	+	0.184 330
40	+	0.199 425	+	0.213 055	+	0.219 456	+	0.231 477	+	0.242 521	+	0.252 656
45	+	0.268 309	+	0.272 963	+	0.285 775	+	0.293 422	+	0.300 412	+	0.306 787
50	+	0.326 848	+	0.329 654	+	0.337 298	+	0.341 791	+	0.345 838	+	0.349 469
55	+	0.372 079	+	0.373 553	+	0.377 452	+	0.379 644	+	0.381 535	+	0.383 146
60	+	0.407 291	+	0.407 789	+	0.408 934	+	0.409 206	+	0.409 718	+	0.409 825
65	+	0.434 946	+	0.434 717	+	0.433 806	+	0.433 024	+	0.432 114	+	0.431 614
70	+	0.456 867	+	0.456 093	+	0.453 620	+	0.451 869	+	0.450 043	+	0.448 148
75	+	0.474 408	+	0.473 208	+	0.469 540	+	0.467 042	+	0.464 508	+	0.461 942
80	+	0.488 569	+	0.487 042	+	0.482 439	+	0.479 357	+	0.476 268	+	0.473 176
85	+	0.500 100	+	0.498 314	+	0.492 973	+	0.489 430	+	0.485 902	+	0.482 392
90	+	0.509 565	+	0.507 572	+	0.501 642	+	0.497 729	+	0.493 850	+	0.490 005
95	+	0.517 389	+	0.515 230	+	0.508 824	+	0.504 613	+	0.500 449	+	0.496 333
100	+	0.523 899	+	0.521 605	+	0.514 811	+	0.510 357	+	0.505 961	+	0.501 623
105	+	0.529 347	+	0.526 941	+	0.519 829	+	0.515 175	+	0.510 588	+	0.506 068
110	+	0.533 928	+	0.531 430	+	0.524 054	+	0.519 234	+	0.514 489	+	0.509 818
115	+	0.537 794	+	0.535 220	+	0.527 626	+	0.522 667	+	0.517 790	+	0.512 993
120	+	0.541 067	+	0.538 429	+	0.530 651	+	0.525 578	+	0.520 591	+	0.515 689
125	+	0.543 841	+	0.541 149	+	0.535 838	+	0.528 049	+	0.522 969	+	0.517 979
130	+	0.546 192	+	0.543 456	+	0.535 397	+	0.530 145	+	0.524 988	+	0.519 925
135	+	0.548 183	+	0.545 409	+	0.537 242	+	0.531 922	+	0.526 700	+	0.521 573
140	+	0.549 861	+	0.547 056	+	0.538 798	+	0.533 423	+	0.528 146	+	0.522 966
145	+	0.551 267	+	0.548 436	+	0.540 103	+	0.534 679	+	0.529 357	+	0.524 134
150	+	0.552 431	+	0.549 578	+	0.541 183	+	0.535 720	+	0.530 361	+	0.525 102
155	+	0.553 379	+	0.550 508	+	0.542 063	+	0.536 568	+	0.531 178	+	0.525 891
160	+	0.554 130	+	0.551 245	+	0.542 761	+	0.537 241	+	0.531 827	+	0.526 516
165	+	0.554 699	+	0.551 804	+	0.543 289	+	0.537 750	+	0.532 318	+	0.526 991
170	+	0.555 097	+	0.552 195	+	0.543 660	+	0.538 108	+	0.532 664	+	0.527 324
175	+	0.555 334	+	0.552 427	+	0.543 879	+	0.538 320	+	0.532 868	+	0.527 521
180	+	0.555 411	+	0.552 504	+	0.543 952	+	0.538 390	+	0.532 934	+	0.527 585
$\psi \backslash z$	1.71	1.72	1.73	1.74	1.75	1.76	1.77	1.78	1.79	1.80	1.81	$z \backslash \psi$

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$

ψ	1.81	1.82	1.83	1.84	1.85	1.86	1.87	1.88	1.89	1.90	1.91	z
0°	0.487 114	0.447 072	0.409 236	0.373 469	0.339 639	0.307 628	0.277 324	0.248 623	0.221 430	0.195 654	0.171 212	0°
5	0.455 211	0.417 053	0.380 970	0.346 833	0.314 524	0.283 929	0.254 947	0.227 480	0.201 440	0.176 743	0.153 311	5
10	0.368 875	0.335 667	0.304 199	0.274 368	0.246 078	0.219 240	0.193 770	0.169 591	0.146 630	0.124 817	0.104 091	10
15	0.250 374	0.223 599	0.198 152	0.173 963	0.150 961	0.129 084	0.108 270	0.088 465	0.069 613	0.051 665	0.034 575	15
20	0.123 202	0.102 839	0.083 428	0.064 922	0.047 275	0.030 446	0.014 393	0.000 921	0.015 532	0.029 474	0.042 781	20
25	0.003 947	0.010 893	0.025 076	0.038 633	0.051 592	0.063 979	0.075 820	0.087 140	0.097 963	0.108 310	0.118 203	25
30	0.099 231	0.109 703	0.119 729	0.129 330	0.138 521	0.147 321	0.155 745	0.163 809	0.171 529	0.178 917	0.185 988	30
35	0.184 330	0.191 507	0.198 382	0.204 970	0.211 280	0.217 324	0.223 110	0.228 650	0.233 953	0.239 029	0.243 884	35
40	0.252 656	0.257 404	0.261 949	0.266 297	0.270 458	0.274 436	0.278 240	0.281 874	0.285 347	0.288 662	0.291 827	40
45	0.306 787	0.309 758	0.312 591	0.315 291	0.317 862	0.320 310	0.322 638	0.324 851	0.326 952	0.328 945	0.330 837	45
50	0.349 469	0.351 137	0.352 712	0.354 196	0.355 593	0.356 906	0.358 138	0.359 291	0.360 368	0.361 372	0.362 306	50
55	0.383 146	0.383 852	0.384 496	0.385 078	0.385 602	0.386 068	0.386 481	0.386 840	0.387 148	0.387 408	0.387 620	55
60	0.409 825	0.409 813	0.409 759	0.409 666	0.409 533	0.409 364	0.409 158	0.408 918	0.408 645	0.408 340	0.408 004	60
65	0.431 086	0.430 531	0.429 951	0.429 343	0.428 713	0.428 059	0.427 385	0.426 686	0.425 968	0.425 229	0.424 475	65
70	0.448 148	0.447 177	0.446 191	0.445 191	0.444 177	0.443 151	0.442 112	0.441 062	0.440 001	0.438 930	0.437 849	70
75	0.461 942	0.460 648	0.459 348	0.458 041	0.456 730	0.455 414	0.454 092	0.452 768	0.451 438	0.450 105	0.448 769	75
80	0.473 176	0.471 629	0.470 082	0.468 535	0.466 989	0.465 444	0.463 900	0.462 357	0.460 816	0.459 277	0.457 739	80
85	0.482 392	0.480 644	0.478 901	0.477 163	0.475 430	0.473 703	0.471 980	0.470 265	0.468 555	0.466 850	0.465 152	85
90	0.490 005	0.488 096	0.486 195	0.484 303	0.482 420	0.480 547	0.478 682	0.476 826	0.474 980	0.473 142	0.471 314	90
95	0.496 333	0.494 293	0.492 264	0.490 248	0.488 244	0.486 251	0.484 270	0.482 301	0.480 343	0.478 398	0.476 464	95
100	0.501 623	0.499 476	0.497 343	0.495 225	0.493 121	0.491 031	0.488 955	0.486 893	0.484 844	0.482 810	0.480 789	100
105	0.506 068	0.503 833	0.501 614	0.499 412	0.497 225	0.495 055	0.492 901	0.490 762	0.488 638	0.486 531	0.484 438	105
110	0.509 818	0.507 510	0.505 220	0.502 948	0.500 693	0.498 456	0.496 237	0.494 034	0.491 849	0.489 680	0.487 528	110
115	0.512 993	0.510 624	0.508 274	0.505 945	0.503 634	0.501 341	0.499 067	0.496 811	0.494 573	0.492 354	0.490 152	115
120	0.515 689	0.513 269	0.510 870	0.508 491	0.506 132	0.503 793	0.501 473	0.499 173	0.496 892	0.494 631	0.492 387	120
125	0.517 979	0.515 517	0.513 076	0.510 656	0.508 257	0.505 878	0.503 520	0.501 182	0.498 863	0.496 565	0.494 287	125
130	0.519 925	0.517 426	0.514 950	0.512 495	0.510 062	0.507 651	0.505 261	0.502 892	0.500 543	0.498 215	0.495 907	130
135	0.521 573	0.519 045	0.516 539	0.514 056	0.511 595	0.509 156	0.506 738	0.504 343	0.501 969	0.499 615	0.497 282	135
140	0.522 966	0.520 412	0.517 881	0.515 374	0.512 890	0.510 429	0.507 988	0.505 570	0.503 174	0.500 799	0.498 445	140
145	0.524 134	0.521 559	0.519 008	0.516 480	0.513 976	0.511 494	0.509 036	0.506 599	0.504 184	0.501 792	0.499 422	145
150	0.525 102	0.522 510	0.519 942	0.517 398	0.514 877	0.512 380	0.509 906	0.507 454	0.505 024	0.502 616	0.500 230	150
155	0.525 891	0.523 284	0.520 702	0.518 145	0.515 611	0.513 101	0.510 614	0.508 150	0.505 708	0.503 289	0.500 891	155
160	0.526 516	0.523 898	0.521 306	0.518 738	0.516 194	0.513 673	0.511 176	0.508 702	0.506 252	0.503 823	0.501 417	160
165	0.526 991	0.524 364	0.521 763	0.519 187	0.516 635	0.514 107	0.511 602	0.509 121	0.506 663	0.504 227	0.501 814	165
170	0.527 324	0.524 692	0.522 085	0.519 503	0.516 945	0.514 412	0.511 902	0.509 416	0.506 952	0.504 511	0.502 093	170
175	0.527 521	0.524 886	0.522 276	0.519 690	0.517 129	0.514 592	0.512 079	0.509 590	0.507 124	0.504 680	0.502 259	175
180	0.527 585	0.524 950	0.522 339	0.519 752	0.517 190	0.514 652	0.512 138	0.509 648	0.507 181	0.504 736	0.502 314	180
ψ	1.81	1.82	1.83	1.84	1.85	1.86	1.87	1.88	1.89	1.90	1.91	z

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$

ψ	z	1.91	1.92	1.93	1.94	1.95	1.96	1.97	1.98	1.99	2.00	2.01	z	ψ
0°	0°	—	—	—	—	—	—	—	—	—	—	—	0°	0°
5	5	0.171 212	0.148 027	0.126 026	0.105 140	0.085 308	0.066 470	0.048 570	0.031 557	0.015 382	0.000 000	0.014 632	5	5
10	10	0.153 311	0.131 071	0.109 957	0.089 904	0.070 852	0.052 747	0.035 536	0.019 171	0.003 605	0.011 203	0.025 294	10	10
15	15	—	—	—	—	—	—	—	—	—	—	—	15	15
20	20	0.104 091	0.084 389	0.065 658	0.047 843	0.030 896	0.014 771	0.000 575	0.015 185	0.029 096	0.042 344	0.054 963	20	20
25	25	0.034 575	0.018 296	0.002 790	0.011 985	0.026 066	0.039 487	0.052 282	0.064 480	0.076 113	0.087 207	0.097 789	25	25
30	30	0.042 781	0.055 480	0.067 603	0.079 176	0.090 226	0.100 776	0.110 849	0.120 469	0.129 657	0.138 431	0.146 810	30	30
35	35	0.118 203	0.127 661	0.136 705	0.145 351	0.153 619	0.161 523	0.169 080	0.176 304	0.183 210	0.189 812	0.196 122	35	35
40	40	0.182 988	0.192 755	0.199 230	0.205 425	0.211 352	0.217 020	0.222 442	0.227 626	0.232 582	0.237 320	0.241 849	40	40
45	45	—	—	—	—	—	—	—	—	—	—	—	45	45
50	50	0.243 884	0.248 528	0.252 970	0.257 217	0.261 276	0.265 155	0.268 860	0.272 398	0.275 776	0.278 999	0.282 073	50	50
55	55	0.291 827	0.294 846	0.297 725	0.300 469	0.303 083	0.305 571	0.307 939	0.310 190	0.312 329	0.314 360	0.316 287	55	55
60	60	0.330 837	0.332 628	0.334 322	0.335 923	0.337 435	0.338 860	0.340 202	0.341 463	0.342 646	0.343 754	0.344 790	60	60
65	65	0.362 306	0.363 171	0.363 970	0.364 706	0.365 381	0.365 997	0.366 556	0.367 060	0.367 511	0.367 910	0.368 261	65	65
70	70	0.387 620	0.387 786	0.387 908	0.387 988	0.388 026	0.388 026	0.387 987	0.387 911	0.387 800	0.387 655	0.387 475	70	70
75	75	0.408 004	0.407 639	0.407 245	0.406 823	0.406 375	0.405 902	0.405 404	0.404 883	0.404 340	0.403 775	0.403 189	75	75
80	80	0.424 475	0.423 701	0.422 911	0.422 103	0.421 280	0.420 442	0.419 590	0.418 724	0.417 845	0.416 953	0.416 049	80	80
85	85	0.437 849	0.436 759	0.435 659	0.434 552	0.433 437	0.432 315	0.431 186	0.430 050	0.428 909	0.427 762	0.426 610	85	85
90	90	0.448 769	0.447 431	0.446 090	0.444 747	0.443 402	0.442 056	0.440 709	0.439 361	0.438 013	0.436 664	0.435 315	90	90
95	95	0.457 739	0.456 205	0.454 672	0.453 142	0.451 616	0.450 092	0.448 571	0.447 054	0.445 541	0.444 031	0.442 525	95	95
100	100	0.465 152	0.463 460	0.461 775	0.460 095	0.458 422	0.456 756	0.455 096	0.453 443	0.451 797	0.450 158	0.448 525	100	100
105	105	0.471 314	0.469 495	0.467 686	0.465 885	0.464 094	0.462 312	0.460 540	0.458 777	0.457 023	0.455 279	0.453 544	105	105
110	110	0.476 464	0.474 542	0.472 631	0.470 732	0.468 845	0.466 969	0.465 104	0.463 251	0.461 409	0.459 579	0.457 752	110	110
115	115	0.480 789	0.478 782	0.476 789	0.474 809	0.472 842	0.470 889	0.468 948	0.467 021	0.465 107	0.463 206	0.461 318	115	115
120	120	0.484 438	0.482 361	0.480 299	0.478 252	0.476 220	0.474 202	0.472 199	0.470 211	0.468 237	0.466 277	0.464 331	120	120
125	125	0.487 528	0.485 393	0.483 273	0.481 171	0.479 084	0.477 013	0.474 958	0.472 918	0.470 895	0.468 885	0.466 892	125	125
130	130	0.490 152	0.487 968	0.485 801	0.483 653	0.481 519	0.479 404	0.477 305	0.475 223	0.473 157	0.471 107	0.469 075	130	130
135	135	0.492 387	0.490 161	0.487 954	0.485 765	0.483 594	0.481 442	0.479 307	0.477 189	0.475 087	0.473 002	0.470 935	135	135
140	140	0.494 287	0.492 028	0.489 788	0.487 567	0.485 364	0.483 179	0.481 012	0.478 863	0.476 732	0.474 620	0.472 525	140	140
145	145	0.495 907	0.493 618	0.491 349	0.489 100	0.486 870	0.484 659	0.482 467	0.480 293	0.478 137	0.475 999	0.473 879	145	145
150	150	0.497 282	0.494 969	0.492 676	0.490 404	0.488 150	0.485 917	0.483 702	0.481 507	0.479 330	0.477 172	0.475 032	150	150
155	155	0.498 445	0.496 111	0.493 799	0.491 507	0.489 235	0.486 982	0.484 749	0.482 536	0.480 342	0.478 166	0.476 009	155	155
160	160	0.499 422	0.497 071	0.494 742	0.492 433	0.490 145	0.487 877	0.485 629	0.483 400	0.481 191	0.479 001	0.476 830	160	160
165	165	0.500 230	0.497 867	0.495 524	0.493 202	0.490 901	0.488 620	0.486 359	0.484 119	0.481 897	0.479 695	0.477 513	165	165
170	170	0.500 891	0.498 516	0.496 161	0.493 829	0.491 517	0.489 226	0.486 954	0.484 703	0.482 472	0.480 261	0.478 068	170	170
175	175	0.501 417	0.499 032	0.496 669	0.494 327	0.492 007	0.489 707	0.487 428	0.485 169	0.482 930	0.480 711	0.478 511	175	175
180	180	0.501 814	0.499 423	0.497 053	0.494 705	0.492 378	0.490 072	0.487 787	0.485 522	0.483 277	0.481 053	0.478 847	180	180
		0.502 093	0.499 697	0.497 323	0.494 970	0.492 639	0.490 329	0.488 040	0.485 771	0.483 521	0.481 292	0.479 084		
		0.502 259	0.499 860	0.497 482	0.495 127	0.492 793	0.490 481	0.488 189	0.485 917	0.483 666	0.481 435	0.479 223		
		0.502 314	0.499 914	0.497 536	0.495 179	0.492 845	0.490 531	0.488 238	0.485 965	0.483 714	0.481 482	0.479 269		

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ	z	2.11	2.12	2.13	2.14	2.15	2.16	2.17	2.18	2.19	2.20	2.21	z	ψ
0°	0°	0.127 397	0.135 953	0.144 108	0.151 883	0.159 295	0.166 362	0.173 101	0.179 527	0.185 655	0.191 498	0.197 071	0°	0°
5	5	0.134 064	0.142 328	0.150 207	0.157 720	0.164 884	0.171 716	0.178 231	0.184 444	0.190 369	0.196 020	0.201 410	5	5
10	10	0.152 786	0.160 248	0.167 365	0.174 155	0.180 631	0.186 810	0.192 704	0.198 327	0.203 690	0.208 806	0.213 686	10	10
15	15	0.180 276	0.186 599	0.192 633	0.198 392	0.203 887	0.209 132	0.214 137	0.218 912	0.223 468	0.227 814	0.231 959	15	15
20	20	0.212 460	0.217 511	0.222 332	0.226 934	0.231 327	0.235 518	0.239 517	0.243 332	0.246 970	0.250 439	0.253 746	20	20
25	25	0.245 667	0.249 478	0.253 114	0.256 582	0.259 889	0.263 041	0.266 046	0.268 909	0.271 635	0.274 231	0.276 701	25	25
30	30	0.277 270	0.279 973	0.282 546	0.284 994	0.287 323	0.289 537	0.291 641	0.293 638	0.295 533	0.297 329	0.299 031	30	30
35	35	0.305 778	0.307 540	0.309 216	0.310 798	0.312 294	0.313 705	0.315 036	0.316 289	0.317 468	0.318 574	0.319 611	35	35
40	40	0.330 594	0.331 597	0.332 533	0.333 405	0.334 215	0.334 965	0.335 657	0.336 294	0.336 878	0.337 410	0.337 892	40	40
45	45	0.351 718	0.352 100	0.352 447	0.352 749	0.353 001	0.353 224	0.353 401	0.353 540	0.353 640	0.353 705	0.353 739	45	45
50	50	0.369 413	0.369 325	0.369 206	0.369 056	0.368 877	0.368 670	0.368 435	0.368 175	0.367 889	0.367 579	0.367 246	50	50
55	55	0.384 147	0.383 681	0.383 194	0.382 688	0.382 163	0.381 620	0.381 059	0.380 481	0.379 888	0.379 278	0.378 655	55	55
60	60	0.396 365	0.395 601	0.394 825	0.394 037	0.393 238	0.392 428	0.391 608	0.390 779	0.389 940	0.389 093	0.388 237	60	60
65	65	0.406 486	0.405 488	0.404 482	0.403 471	0.402 456	0.401 435	0.400 410	0.399 380	0.398 345	0.397 308	0.396 267	65	65
70	70	0.414 886	0.413 701	0.412 513	0.411 325	0.410 136	0.408 947	0.407 757	0.406 567	0.405 376	0.404 188	0.402 997	70	70
75	75	0.421 873	0.420 538	0.419 207	0.417 875	0.416 547	0.415 222	0.413 900	0.412 581	0.411 265	0.409 953	0.408 643	75	75
80	80	0.427 706	0.426 251	0.424 800	0.423 355	0.421 916	0.420 481	0.419 052	0.417 629	0.416 211	0.414 799	0.413 392	80	80
85	85	0.432 593	0.431 040	0.429 495	0.427 956	0.426 426	0.424 902	0.423 386	0.421 878	0.420 377	0.418 883	0.417 398	85	85
90	90	0.436 705	0.435 073	0.433 449	0.431 834	0.430 229	0.428 633	0.427 045	0.425 467	0.423 898	0.422 338	0.420 787	90	90
95	95	0.440 171	0.438 474	0.436 785	0.435 108	0.433 442	0.431 787	0.430 140	0.428 503	0.426 878	0.425 264	0.423 659	95	95
100	100	0.443 122	0.441 369	0.439 628	0.437 898	0.436 180	0.434 474	0.432 779	0.431 096	0.429 423	0.427 762	0.426 112	100	100
105	105	0.445 624	0.443 825	0.442 040	0.440 267	0.438 506	0.436 758	0.435 023	0.433 300	0.431 588	0.429 888	0.428 201	105	105
110	110	0.447 759	0.445 923	0.444 100	0.442 291	0.440 495	0.438 712	0.436 943	0.435 186	0.433 442	0.431 710	0.429 992	110	110
115	115	0.449 583	0.447 715	0.445 861	0.444 021	0.442 195	0.440 383	0.438 584	0.436 800	0.435 028	0.433 270	0.431 525	115	115
120	120	0.451 143	0.449 249	0.447 368	0.445 503	0.443 652	0.441 815	0.439 992	0.438 183	0.436 388	0.434 607	0.432 839	120	120
125	125	0.452 478	0.450 560	0.448 657	0.446 770	0.444 899	0.443 041	0.441 198	0.439 369	0.437 554	0.435 753	0.433 967	125	125
130	130	0.453 619	0.451 682	0.449 761	0.447 856	0.445 967	0.444 090	0.442 229	0.440 383	0.438 552	0.436 735	0.434 932	130	130
135	135	0.454 590	0.452 638	0.450 701	0.448 780	0.446 875	0.444 985	0.443 109	0.441 250	0.439 404	0.437 573	0.435 757	135	135
140	140	0.455 416	0.453 450	0.451 500	0.449 565	0.447 647	0.445 744	0.443 856	0.441 984	0.440 127	0.438 284	0.436 457	140	140
145	145	0.456 111	0.454 133	0.452 171	0.450 226	0.448 297	0.446 384	0.444 486	0.442 604	0.440 738	0.438 884	0.437 047	145	145
150	150	0.456 688	0.454 701	0.452 731	0.450 777	0.448 839	0.446 916	0.445 010	0.443 119	0.441 244	0.439 384	0.437 539	150	150
155	155	0.457 160	0.455 165	0.453 187	0.451 225	0.449 280	0.447 351	0.445 437	0.443 540	0.441 658	0.439 792	0.437 940	155	155
160	160	0.457 535	0.455 534	0.453 550	0.451 583	0.449 632	0.447 698	0.445 779	0.443 876	0.441 989	0.440 117	0.438 260	160	160
165	165	0.457 818	0.455 813	0.453 825	0.451 853	0.449 898	0.447 960	0.446 038	0.444 131	0.442 240	0.440 364	0.438 503	165	165
170	170	0.458 020	0.456 012	0.454 020	0.452 045	0.450 087	0.448 145	0.446 219	0.444 310	0.442 416	0.440 537	0.438 674	170	170
175	175	0.458 139	0.456 129	0.454 135	0.452 158	0.450 198	0.448 254	0.446 326	0.444 415	0.442 520	0.440 640	0.438 775	175	175
180	180	0.458 177	0.456 167	0.454 173	0.452 196	0.450 235	0.448 291	0.446 362	0.444 450	0.442 554	0.440 674	0.438 808	180	180

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{r}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ	$\frac{1}{m}$	2.21	2.22	2.23	2.24	2.25	2.26	2.27	2.28	2.29	2.30	2.31	z
0°	0°	0.197 071	0.202 384	0.207 451	0.212 282	0.216 889	0.221 280	0.225 467	0.229 457	0.233 260	0.236 884	0.240 337	0°
5	5	0.201 410	0.206 549	0.211 450	0.216 123	0.220 579	0.224 827	0.228 877	0.232 736	0.236 415	0.239 920	0.243 258	5
10	10	0.213 686	0.218 340	0.222 779	0.227 011	0.231 046	0.234 893	0.238 560	0.242 054	0.245 383	0.248 554	0.251 574	10
15	15	0.231 959	0.235 912	0.239 681	0.243 275	0.246 700	0.249 963	0.253 073	0.256 034	0.258 853	0.261 537	0.264 091	15
20	20	0.253 746	0.256 898	0.259 900	0.262 760	0.265 483	0.268 074	0.270 539	0.272 884	0.275 112	0.277 229	0.279 239	20
25	25	0.276 701	0.279 050	0.281 283	0.283 405	0.285 419	0.287 331	0.289 144	0.290 862	0.292 488	0.294 027	0.295 482	25
30	30	0.299 031	0.300 642	0.302 166	0.303 605	0.304 964	0.306 244	0.307 450	0.308 583	0.309 648	0.310 645	0.311 578	30
35	35	0.319 611	0.320 581	0.321 487	0.322 332	0.323 116	0.323 844	0.324 516	0.325 134	0.325 702	0.326 220	0.326 691	35
40	40	0.337 892	0.338 327	0.338 715	0.339 060	0.339 361	0.339 621	0.339 841	0.340 024	0.340 169	0.340 279	0.340 355	40
45	45	0.353 735	0.353 732	0.353 697	0.353 631	0.353 534	0.353 410	0.353 257	0.353 079	0.352 875	0.352 646	0.352 393	45
50	50	0.367 246	0.366 890	0.366 513	0.366 115	0.365 697	0.365 260	0.364 805	0.364 332	0.363 842	0.363 335	0.362 814	50
55	55	0.378 655	0.378 017	0.377 365	0.376 701	0.376 024	0.375 335	0.374 636	0.373 925	0.373 204	0.372 473	0.371 733	55
60	60	0.388 237	0.387 374	0.386 503	0.385 625	0.384 741	0.383 850	0.382 953	0.382 050	0.381 143	0.380 230	0.379 313	60
65	65	0.396 267	0.395 223	0.394 177	0.393 128	0.392 076	0.391 022	0.389 967	0.388 910	0.387 852	0.386 793	0.385 731	65
70	70	0.402 997	0.401 808	0.400 620	0.399 433	0.398 247	0.397 062	0.395 879	0.394 697	0.393 517	0.392 339	0.391 163	70
75	75	0.408 643	0.407 337	0.406 035	0.404 736	0.403 441	0.402 149	0.400 863	0.399 580	0.398 301	0.397 026	0.395 756	75
80	80	0.413 392	0.411 991	0.410 596	0.409 207	0.407 823	0.406 446	0.405 074	0.403 708	0.402 348	0.400 994	0.399 646	80
85	85	0.417 398	0.415 919	0.414 448	0.412 984	0.411 528	0.410 080	0.408 639	0.407 205	0.405 779	0.404 360	0.402 948	85
90	90	0.420 787	0.419 244	0.417 711	0.416 186	0.414 670	0.413 163	0.411 665	0.410 175	0.408 694	0.407 222	0.405 758	90
95	95	0.423 659	0.422 063	0.420 479	0.418 902	0.417 338	0.415 783	0.414 238	0.412 700	0.411 174	0.409 658	0.408 152	95
100	100	0.426 112	0.424 473	0.422 845	0.421 228	0.419 621	0.418 025	0.416 440	0.414 865	0.413 300	0.411 746	0.410 202	100
105	105	0.428 201	0.426 526	0.424 861	0.423 210	0.421 568	0.419 938	0.418 319	0.416 713	0.415 116	0.413 531	0.411 955	105
110	110	0.429 992	0.428 285	0.426 591	0.424 908	0.423 238	0.421 580	0.419 934	0.418 299	0.416 676	0.415 064	0.413 464	110
115	115	0.431 525	0.429 792	0.428 072	0.426 365	0.424 670	0.422 988	0.421 318	0.419 660	0.418 015	0.416 382	0.414 760	115
120	120	0.432 839	0.431 084	0.429 343	0.427 615	0.425 900	0.424 198	0.422 508	0.420 830	0.419 166	0.417 513	0.415 872	120
125	125	0.433 967	0.432 193	0.430 433	0.428 688	0.426 957	0.425 238	0.423 529	0.421 835	0.420 154	0.418 485	0.416 829	125
130	130	0.434 932	0.433 143	0.431 368	0.429 607	0.427 859	0.426 125	0.424 404	0.422 697	0.421 002	0.419 320	0.417 651	130
135	135	0.435 757	0.433 954	0.432 167	0.430 392	0.428 631	0.426 885	0.425 153	0.423 433	0.421 727	0.420 033	0.418 353	135
140	140	0.436 457	0.434 643	0.432 844	0.431 060	0.429 290	0.427 533	0.425 789	0.424 059	0.422 343	0.420 640	0.418 951	140
145	145	0.437 047	0.435 224	0.433 417	0.431 622	0.429 842	0.428 077	0.426 325	0.424 588	0.422 864	0.421 153	0.419 455	145
150	150	0.437 539	0.435 708	0.433 892	0.432 091	0.430 304	0.428 531	0.426 773	0.425 028	0.423 297	0.421 579	0.419 875	150
155	155	0.437 940	0.436 103	0.434 281	0.432 474	0.430 681	0.428 903	0.427 138	0.425 388	0.423 651	0.421 929	0.420 219	155
160	160	0.438 260	0.436 418	0.434 592	0.432 780	0.430 982	0.429 199	0.427 430	0.425 675	0.423 934	0.422 207	0.420 493	160
165	165	0.438 503	0.436 657	0.434 826	0.433 010	0.431 209	0.429 423	0.427 651	0.425 893	0.424 149	0.422 418	0.420 701	165
170	170	0.438 674	0.436 826	0.434 993	0.433 174	0.431 370	0.429 581	0.427 806	0.426 046	0.424 300	0.422 567	0.420 848	170
175	175	0.438 775	0.436 925	0.435 091	0.433 271	0.431 467	0.429 675	0.427 899	0.426 137	0.424 390	0.422 656	0.420 935	175
180	180	0.438 808	0.436 958	0.435 123	0.433 303	0.431 497	0.429 706	0.427 929	0.426 167	0.424 419	0.422 684	0.420 963	180
ψ	z	2.21	2.22	2.23	2.24	2.25	2.26	2.27	2.28	2.29	2.30	2.31	ψ

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ	z	2.31	2.32	2.33	2.34	2.35	2.36	2.37	2.38	2.39	2.40	2.41	z	ψ
0°	0°	0.240 337	0.243 626	0.246 758	0.249 739	0.252 578	0.255 279	0.257 848	0.260 291	0.262 614	0.264 820	0.266 916	0°	0°
5	5	0.243 258	0.246 439	0.249 467	0.252 349	0.255 093	0.257 703	0.260 185	0.262 545	0.264 788	0.266 918	0.268 941	5	5
10	10	0.251 574	0.254 449	0.257 186	0.259 789	0.262 266	0.264 620	0.266 858	0.268 984	0.271 002	0.272 917	0.274 733	10	10
15	15	0.264 091	0.266 518	0.268 827	0.271 021	0.273 104	0.275 082	0.276 959	0.278 738	0.280 424	0.282 021	0.283 531	15	15
20	20	0.279 239	0.281 146	0.282 955	0.284 669	0.286 293	0.287 829	0.289 281	0.290 653	0.291 947	0.293 167	0.294 315	20	20
25	25	0.295 482	0.296 855	0.298 151	0.299 372	0.300 521	0.301 601	0.302 614	0.303 563	0.304 451	0.305 279	0.306 050	25	25
30	30	0.311 578	0.312 450	0.313 262	0.314 016	0.314 716	0.315 362	0.315 957	0.316 504	0.317 003	0.317 456	0.317 866	30	30
35	35	0.326 691	0.327 115	0.327 496	0.327 834	0.328 132	0.328 390	0.328 160	0.328 794	0.328 942	0.329 057	0.329 135	35	35
40	40	0.340 355	0.340 398	0.340 409	0.340 390	0.340 341	0.340 264	0.340 160	0.340 030	0.339 875	0.339 695	0.339 492	40	40
45	45	0.352 393	0.352 117	0.351 820	0.351 501	0.351 163	0.350 804	0.350 427	0.350 033	0.349 621	0.349 192	0.348 747	45	45
50	50	0.362 814	0.362 277	0.361 726	0.361 161	0.360 583	0.359 993	0.359 390	0.358 776	0.358 150	0.357 515	0.356 869	50	50
55	55	0.371 733	0.370 984	0.370 226	0.369 460	0.368 687	0.367 906	0.367 118	0.366 324	0.365 523	0.364 717	0.363 905	55	55
60	60	0.379 313	0.378 392	0.377 466	0.376 537	0.375 605	0.374 669	0.373 730	0.372 789	0.371 845	0.370 899	0.369 951	60	60
65	65	0.385 731	0.384 672	0.383 611	0.382 549	0.381 487	0.380 425	0.379 364	0.378 302	0.377 243	0.376 181	0.375 122	65	65
70	70	0.391 163	0.389 988	0.388 817	0.387 647	0.386 480	0.385 315	0.384 153	0.382 994	0.381 840	0.380 683	0.379 533	70	70
75	75	0.395 756	0.394 489	0.393 227	0.391 969	0.390 716	0.389 467	0.388 223	0.386 984	0.385 750	0.384 518	0.383 293	75	75
80	80	0.399 646	0.398 304	0.396 968	0.395 638	0.394 315	0.392 997	0.391 685	0.390 380	0.389 080	0.387 787	0.386 500	80	80
85	85	0.402 948	0.401 544	0.400 148	0.398 758	0.397 376	0.396 002	0.394 634	0.393 274	0.391 921	0.390 576	0.389 238	85	85
90	90	0.405 758	0.404 302	0.402 855	0.401 417	0.399 987	0.398 565	0.397 151	0.395 746	0.394 349	0.392 960	0.391 579	90	90
95	95	0.408 152	0.406 632	0.405 164	0.403 685	0.402 214	0.400 753	0.399 301	0.397 850	0.396 427	0.394 999	0.393 582	95	95
100	100	0.410 202	0.408 668	0.407 145	0.405 631	0.404 127	0.402 633	0.401 149	0.399 675	0.398 210	0.396 754	0.395 308	100	100
105	105	0.411 955	0.410 392	0.408 839	0.407 296	0.405 765	0.404 243	0.402 732	0.401 230	0.399 739	0.398 260	0.396 789	105	105
110	110	0.413 464	0.411 875	0.410 298	0.408 731	0.407 175	0.405 630	0.404 096	0.402 573	0.401 060	0.399 557	0.398 065	110	110
115	115	0.414 760	0.413 149	0.411 549	0.409 962	0.408 386	0.406 821	0.405 269	0.403 725	0.402 194	0.400 674	0.399 164	115	115
120	120	0.415 872	0.414 244	0.412 628	0.411 024	0.409 431	0.407 849	0.406 280	0.404 721	0.403 173	0.401 637	0.400 112	120	120
125	125	0.416 829	0.415 186	0.413 554	0.411 935	0.410 328	0.408 732	0.407 148	0.405 576	0.404 015	0.402 466	0.400 927	125	125
130	130	0.417 651	0.415 994	0.414 350	0.412 719	0.411 099	0.409 491	0.407 896	0.406 312	0.404 740	0.403 179	0.401 630	130	130
135	135	0.418 353	0.416 686	0.415 031	0.413 388	0.411 758	0.410 140	0.408 534	0.406 941	0.405 359	0.403 789	0.402 230	135	135
140	140	0.418 951	0.417 274	0.415 610	0.413 959	0.412 320	0.410 693	0.409 079	0.407 477	0.405 887	0.404 309	0.402 743	140	140
145	145	0.419 455	0.417 770	0.416 099	0.414 440	0.412 794	0.411 160	0.409 539	0.407 930	0.406 333	0.404 748	0.403 175	145	145
150	150	0.419 875	0.418 184	0.416 506	0.414 841	0.413 189	0.411 550	0.409 923	0.408 308	0.406 706	0.405 115	0.403 537	150	150
155	155	0.420 219	0.418 523	0.416 839	0.415 169	0.413 513	0.411 868	0.410 237	0.408 617	0.407 010	0.405 415	0.403 832	155	155
160	160	0.420 493	0.418 792	0.417 105	0.415 431	0.413 770	0.412 122	0.410 486	0.408 863	0.407 252	0.405 654	0.404 068	160	160
165	165	0.420 701	0.418 997	0.417 306	0.415 630	0.413 967	0.412 316	0.410 677	0.409 051	0.407 437	0.405 836	0.404 246	165	165
170	170	0.420 848	0.419 142	0.417 449	0.415 770	0.414 104	0.412 451	0.410 811	0.409 183	0.407 567	0.405 964	0.404 373	170	170
175	175	0.420 935	0.419 228	0.417 534	0.415 854	0.414 187	0.412 532	0.410 890	0.409 261	0.407 645	0.406 041	0.404 449	175	175
180	180	0.420 963	0.419 256	0.417 562	0.415 881	0.414 213	0.412 558	0.410 916	0.409 287	0.407 670	0.406 066	0.404 473	180	180
ψ	z	2.31	2.32	2.33	2.34	2.35	2.36	2.37	2.38	2.39	2.40	2.41	z	ψ

1. TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ	$\frac{1}{m}$	2.41	2.42	2.43	2.44	2.45	2.46	2.47	2.48	2.49	2.50	2.51	$\frac{1}{m}$
ψ	$\frac{1}{m}$	2.41	2.42	2.43	2.44	2.45	2.46	2.47	2.48	2.49	2.50	2.51	$\frac{1}{m}$
0°	5	0.266 916	0.268 905	0.270 793	0.272 583	0.274 279	0.275 885	0.277 405	0.278 842	0.280 200	0.281 481	0.282 690	0°
5		0.268 941	0.270 860	0.272 681	0.274 406	0.276 040	0.277 587	0.279 050	0.280 433	0.281 738	0.282 969	0.284 129	5
10		0.274 733	0.276 454	0.278 085	0.279 628	0.281 087	0.282 466	0.283 768	0.284 995	0.286 152	0.287 240	0.288 262	10
15		0.283 531	0.284 959	0.286 307	0.287 579	0.288 778	0.289 907	0.290 968	0.291 965	0.292 898	0.293 772	0.294 589	15
20		0.294 315	0.295 395	0.296 408	0.297 359	0.298 248	0.299 078	0.299 853	0.300 572	0.301 240	0.301 858	0.302 427	20
25		0.306 050	0.306 766	0.307 430	0.308 043	0.308 607	0.309 124	0.309 596	0.310 024	0.310 411	0.310 757	0.311 064	25
30		0.317 866	0.318 233	0.318 560	0.318 847	0.319 097	0.319 311	0.319 490	0.319 635	0.319 748	0.319 830	0.319 881	30
35		0.329 135	0.329 189	0.329 210	0.329 201	0.329 164	0.329 101	0.329 011	0.328 897	0.328 758	0.328 597	0.328 413	35
40		0.339 492	0.339 266	0.339 019	0.338 751	0.338 463	0.338 156	0.337 830	0.337 487	0.337 127	0.336 750	0.336 357	40
45		0.348 747	0.348 287	0.347 812	0.347 325	0.346 822	0.346 308	0.345 780	0.345 240	0.344 689	0.344 126	0.343 554	45
50		0.356 869	0.356 213	0.355 548	0.354 874	0.354 192	0.353 502	0.352 804	0.352 099	0.351 387	0.350 669	0.349 944	50
55		0.363 905	0.363 088	0.362 266	0.361 439	0.360 608	0.359 772	0.358 933	0.358 091	0.357 245	0.356 396	0.355 544	55
60		0.369 951	0.369 001	0.368 050	0.367 097	0.366 143	0.365 188	0.364 232	0.363 276	0.362 319	0.361 362	0.360 404	60
65		0.375 122	0.374 063	0.373 006	0.371 950	0.370 895	0.369 841	0.368 789	0.367 738	0.366 690	0.365 643	0.364 599	65
70		0.379 533	0.378 385	0.377 241	0.376 100	0.374 962	0.373 827	0.372 696	0.371 568	0.370 444	0.369 323	0.368 206	70
75		0.383 293	0.382 072	0.380 856	0.379 645	0.378 439	0.377 237	0.376 041	0.374 849	0.373 662	0.372 480	0.371 304	75
80		0.386 500	0.385 219	0.383 944	0.382 675	0.381 412	0.380 155	0.378 904	0.377 660	0.376 421	0.375 189	0.373 962	80
85		0.389 238	0.387 906	0.386 583	0.385 266	0.383 956	0.382 653	0.381 358	0.380 069	0.378 788	0.377 513	0.376 245	85
90		0.391 579	0.390 207	0.388 842	0.387 485	0.386 136	0.384 795	0.383 462	0.382 137	0.380 819	0.379 509	0.378 207	90
95		0.393 582	0.392 175	0.390 778	0.389 388	0.388 005	0.386 632	0.385 269	0.383 912	0.382 564	0.381 224	0.379 894	95
100		0.395 308	0.393 872	0.392 444	0.391 026	0.389 617	0.388 217	0.386 827	0.385 445	0.384 071	0.382 707	0.381 351	100
105		0.396 789	0.395 328	0.393 875	0.392 433	0.391 002	0.389 580	0.388 166	0.386 763	0.385 368	0.383 984	0.382 606	105
110		0.398 065	0.396 583	0.395 111	0.393 650	0.392 198	0.390 756	0.389 324	0.387 902	0.386 488	0.385 086	0.383 692	110
115		0.399 164	0.397 664	0.396 175	0.394 697	0.393 229	0.391 771	0.390 323	0.388 885	0.387 457	0.386 039	0.384 630	115
120		0.400 112	0.398 598	0.397 094	0.395 601	0.394 118	0.392 646	0.391 185	0.389 734	0.388 292	0.386 861	0.385 440	120
125		0.400 927	0.399 401	0.397 885	0.396 380	0.394 886	0.393 402	0.391 929	0.390 466	0.389 013	0.387 571	0.386 139	125
130		0.401 630	0.400 092	0.398 565	0.397 050	0.395 546	0.394 052	0.392 569	0.391 096	0.389 634	0.388 182	0.386 741	130
135		0.402 230	0.400 683	0.399 148	0.397 624	0.396 110	0.394 608	0.393 117	0.391 636	0.390 165	0.388 707	0.387 259	135
140		0.402 743	0.401 188	0.399 645	0.398 113	0.396 592	0.395 083	0.393 585	0.392 097	0.390 621	0.389 155	0.387 699	140
145		0.403 175	0.401 614	0.400 065	0.398 527	0.397 000	0.395 485	0.393 980	0.392 487	0.391 005	0.389 533	0.388 072	145
150		0.403 537	0.401 971	0.400 416	0.398 872	0.397 341	0.395 821	0.394 312	0.392 813	0.391 326	0.389 850	0.388 385	150
155		0.403 832	0.402 261	0.400 702	0.399 155	0.397 619	0.396 095	0.394 582	0.393 080	0.391 589	0.390 109	0.388 640	155
160		0.404 068	0.402 494	0.400 932	0.399 381	0.397 842	0.396 314	0.394 798	0.393 293	0.391 799	0.390 316	0.388 844	160
165		0.404 246	0.402 669	0.401 104	0.399 551	0.398 010	0.396 480	0.394 962	0.393 454	0.391 958	0.390 472	0.388 998	165
170		0.404 373	0.402 795	0.401 228	0.399 673	0.398 129	0.396 598	0.395 077	0.393 568	0.392 070	0.390 583	0.389 107	170
175		0.404 449	0.402 869	0.401 301	0.399 746	0.398 201	0.396 668	0.395 146	0.393 636	0.392 137	0.390 649	0.389 172	175
180		0.404 473	0.402 893	0.401 325	0.399 768	0.398 223	0.396 690	0.395 168	0.393 658	0.392 158	0.390 670	0.389 193	180

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$

ψ	z	2.51	2.52	2.53	2.54	2.55	2.56	2.57	2.58	2.59	2.60	2.61	z	ψ
0°	0°	0.282 690	0.283 828	0.284 899	0.285 904	0.286 848	0.287 732	0.288 558	0.289 330	0.290 048	0.290 715	0.291 333	0°	0°
5	5	0.284 129	0.285 220	0.286 246	0.287 209	0.288 111	0.288 955	0.289 743	0.290 478	0.291 160	0.291 794	0.292 379	5	5
10	10	0.288 262	0.289 221	0.290 120	0.290 960	0.291 744	0.292 475	0.293 154	0.293 783	0.294 364	0.294 900	0.295 391	10	10
15	15	0.294 589	0.295 351	0.296 058	0.296 713	0.297 321	0.297 881	0.298 396	0.298 867	0.299 295	0.299 681	0.300 033	15	15
20	20	0.302 427	0.302 950	0.303 428	0.303 863	0.304 257	0.304 611	0.304 927	0.305 206	0.305 449	0.305 659	0.305 836	20	20
25	25	0.311 064	0.311 334	0.311 568	0.311 768	0.311 934	0.312 068	0.312 172	0.312 246	0.312 291	0.312 309	0.312 301	25	25
30	30	0.319 881	0.319 904	0.319 900	0.319 868	0.319 811	0.319 729	0.319 623	0.319 495	0.319 344	0.319 172	0.318 980	30	30
35	35	0.328 413	0.328 208	0.327 982	0.327 736	0.327 471	0.327 188	0.326 888	0.326 570	0.326 236	0.325 887	0.325 522	35	35
40	40	0.336 357	0.335 949	0.335 527	0.335 090	0.334 641	0.334 178	0.333 703	0.333 217	0.332 718	0.332 209	0.331 690	40	40
45	45	0.343 554	0.342 971	0.342 379	0.341 778	0.341 167	0.340 549	0.339 922	0.339 288	0.338 647	0.337 998	0.337 343	45	45
50	50	0.349 944	0.349 213	0.348 477	0.347 735	0.346 989	0.346 238	0.345 482	0.344 722	0.343 958	0.343 191	0.342 419	50	50
55	55	0.355 544	0.354 689	0.353 832	0.352 973	0.352 112	0.351 249	0.350 384	0.349 518	0.348 651	0.347 782	0.346 913	55	55
60	60	0.360 404	0.359 447	0.358 489	0.357 532	0.356 575	0.355 619	0.354 664	0.353 709	0.352 755	0.351 802	0.350 850	60	60
65	65	0.364 599	0.363 556	0.362 516	0.361 477	0.360 441	0.359 407	0.358 376	0.357 347	0.356 321	0.355 297	0.354 277	65	65
70	70	0.368 206	0.367 093	0.365 984	0.364 878	0.363 776	0.362 678	0.361 583	0.360 493	0.359 407	0.358 324	0.357 246	70	70
75	75	0.371 304	0.370 132	0.368 966	0.367 804	0.366 647	0.365 495	0.364 348	0.363 207	0.362 071	0.360 939	0.359 813	75	75
80	80	0.373 962	0.372 742	0.371 528	0.370 319	0.369 117	0.367 921	0.366 730	0.365 546	0.364 368	0.363 195	0.362 029	80	80
85	85	0.376 245	0.374 984	0.373 730	0.372 483	0.371 243	0.370 009	0.368 783	0.367 563	0.366 349	0.365 142	0.363 942	85	85
90	90	0.378 207	0.376 913	0.375 625	0.374 346	0.373 074	0.371 809	0.370 552	0.369 301	0.368 059	0.366 823	0.365 595	90	90
95	95	0.379 894	0.378 571	0.377 256	0.375 950	0.374 651	0.373 360	0.372 076	0.370 800	0.369 534	0.368 274	0.367 020	95	95
100	100	0.381 351	0.380 004	0.378 666	0.377 336	0.376 015	0.374 701	0.373 397	0.372 100	0.370 811	0.369 531	0.368 259	100	100
105	105	0.382 606	0.381 239	0.379 881	0.378 532	0.377 191	0.375 859	0.374 537	0.373 222	0.371 914	0.370 618	0.369 330	105	105
110	110	0.383 692	0.382 308	0.380 933	0.379 567	0.378 210	0.376 862	0.375 524	0.374 193	0.372 872	0.371 560	0.370 256	110	110
115	115	0.384 630	0.383 231	0.381 842	0.380 462	0.379 091	0.377 730	0.376 377	0.375 034	0.373 700	0.372 375	0.371 059	115	115
120	120	0.385 440	0.384 029	0.382 627	0.381 235	0.379 853	0.378 480	0.377 117	0.375 763	0.374 418	0.373 081	0.371 754	120	120
125	125	0.386 139	0.384 717	0.383 305	0.381 903	0.380 510	0.379 128	0.377 755	0.376 391	0.375 036	0.373 692	0.372 355	125	125
130	130	0.386 741	0.385 311	0.383 890	0.382 479	0.381 078	0.379 687	0.378 306	0.376 934	0.375 572	0.374 219	0.372 875	130	130
135	135	0.387 259	0.385 820	0.384 391	0.382 973	0.381 565	0.380 167	0.378 778	0.377 400	0.376 031	0.374 672	0.373 322	135	135
140	140	0.387 699	0.386 254	0.384 820	0.383 396	0.381 981	0.380 577	0.379 183	0.377 799	0.376 424	0.375 058	0.373 702	140	140
145	145	0.388 072	0.386 622	0.385 182	0.383 752	0.382 333	0.380 924	0.379 524	0.378 135	0.376 755	0.375 386	0.374 026	145	145
150	150	0.388 385	0.386 930	0.385 485	0.384 050	0.382 627	0.381 214	0.379 811	0.378 417	0.377 034	0.375 660	0.374 296	150	150
155	155	0.388 640	0.387 181	0.385 733	0.384 295	0.382 868	0.381 451	0.380 044	0.378 647	0.377 260	0.375 884	0.374 516	155	155
160	160	0.388 844	0.387 382	0.385 931	0.384 490	0.383 060	0.381 641	0.380 231	0.378 832	0.377 442	0.376 063	0.374 694	160	160
165	165	0.388 998	0.387 534	0.386 081	0.384 638	0.383 207	0.381 785	0.380 373	0.378 972	0.377 581	0.376 200	0.374 828	165	165
170	170	0.389 107	0.387 642	0.386 187	0.384 743	0.383 310	0.381 887	0.380 474	0.379 071	0.377 678	0.376 296	0.374 923	170	170
175	175	0.389 172	0.387 706	0.386 251	0.384 805	0.383 371	0.381 947	0.380 533	0.379 130	0.377 737	0.376 353	0.374 980	175	175
180	180	0.389 193	0.387 727	0.386 271	0.384 826	0.383 391	0.381 967	0.380 553	0.379 149	0.377 755	0.376 372	0.374 998	180	180
ψ	z	2.51	2.52	2.53	2.54	2.55	2.56	2.57	2.58	2.59	2.60	2.61	z	ψ

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ	z	2.61	2.62	2.63	2.64	2.65	2.66	2.67	2.68	2.69	2.70	2.71	z	ψ
0°	5	0.291 333	0.291 903	0.292 431	0.292 913	0.293 354	0.293 755	0.294 115	0.294 441	0.294 730	0.294 985	0.295 206	0°	5
10	10	0.292 379	0.292 917	0.293 414	0.293 867	0.294 279	0.294 652	0.294 987	0.295 286	0.295 551	0.295 782	0.295 980	10	10
15	15	0.295 391	0.295 840	0.296 248	0.296 617	0.296 948	0.297 242	0.297 502	0.297 728	0.297 922	0.298 084	0.298 217	15	15
20	20	0.300 033	0.300 345	0.300 620	0.300 862	0.301 069	0.301 245	0.301 390	0.301 505	0.301 591	0.301 651	0.301 683	20	20
25	25	0.305 836	0.305 981	0.306 096	0.306 182	0.306 239	0.306 270	0.306 274	0.306 254	0.306 209	0.306 141	0.306 051	25	25
30	30	0.312 301	0.312 267	0.312 209	0.312 127	0.312 022	0.311 896	0.311 748	0.311 580	0.311 393	0.311 187	0.310 963	30	30
35	35	0.318 980	0.318 768	0.318 537	0.318 288	0.318 021	0.317 738	0.317 439	0.317 124	0.316 794	0.316 450	0.316 092	35	35
40	40	0.325 522	0.325 144	0.324 752	0.324 345	0.323 926	0.323 495	0.323 052	0.322 599	0.322 133	0.321 657	0.321 172	40	40
45	45	0.331 690	0.331 161	0.330 622	0.330 074	0.329 517	0.328 952	0.328 378	0.327 798	0.327 210	0.326 614	0.326 013	45	45
50	50	0.337 343	0.336 682	0.336 015	0.335 343	0.334 664	0.333 981	0.333 292	0.332 600	0.331 903	0.331 202	0.330 498	50	50
55	55	0.342 419	0.341 645	0.340 868	0.340 088	0.339 305	0.338 520	0.337 733	0.336 943	0.336 152	0.335 359	0.334 565	55	55
60	60	0.346 913	0.346 043	0.345 172	0.344 301	0.343 429	0.342 559	0.341 686	0.340 813	0.339 942	0.339 070	0.338 199	60	60
65	65	0.350 850	0.349 899	0.348 950	0.348 002	0.347 055	0.346 110	0.345 167	0.344 225	0.343 285	0.342 347	0.341 411	65	65
70	70	0.354 277	0.353 258	0.352 243	0.351 231	0.350 221	0.349 215	0.348 212	0.347 211	0.346 214	0.345 219	0.344 228	70	70
75	75	0.357 246	0.356 171	0.355 101	0.354 035	0.352 973	0.351 915	0.350 861	0.349 811	0.348 766	0.347 724	0.346 687	75	75
80	80	0.359 813	0.358 691	0.357 575	0.356 464	0.355 358	0.354 256	0.353 160	0.352 069	0.350 983	0.349 902	0.348 826	80	80
85	85	0.362 029	0.360 868	0.359 713	0.358 564	0.357 421	0.356 284	0.355 152	0.354 027	0.352 907	0.351 792	0.350 684	85	85
90	90	0.363 942	0.362 749	0.361 561	0.360 381	0.359 207	0.358 039	0.356 878	0.355 723	0.354 574	0.353 432	0.352 296	90	90
95	95	0.365 595	0.364 373	0.363 159	0.361 952	0.360 752	0.359 558	0.358 372	0.357 192	0.356 020	0.354 853	0.353 694	95	95
100	100	0.367 020	0.365 777	0.364 540	0.363 311	0.362 088	0.360 873	0.359 666	0.358 466	0.357 272	0.356 086	0.354 906	100	100
105	105	0.368 259	0.366 994	0.365 738	0.364 489	0.363 248	0.362 015	0.360 789	0.359 571	0.358 361	0.357 158	0.355 962	105	105
110	110	0.369 330	0.368 047	0.366 774	0.365 509	0.364 253	0.363 004	0.361 762	0.360 530	0.359 305	0.358 086	0.356 878	110	110
115	115	0.370 256	0.368 930	0.367 673	0.366 394	0.365 124	0.363 862	0.362 608	0.361 362	0.360 124	0.358 895	0.357 673	115	115
120	120	0.371 059	0.369 751	0.368 452	0.367 162	0.365 880	0.364 606	0.363 341	0.362 085	0.360 837	0.359 597	0.358 365	120	120
125	125	0.371 754	0.370 436	0.369 127	0.367 827	0.366 535	0.365 252	0.363 978	0.362 712	0.361 454	0.360 205	0.358 963	125	125
130	130	0.372 355	0.371 029	0.369 711	0.368 403	0.367 104	0.365 812	0.364 529	0.363 255	0.361 988	0.360 732	0.359 483	130	130
135	135	0.372 875	0.371 541	0.370 215	0.368 899	0.367 592	0.366 294	0.365 005	0.363 724	0.362 452	0.361 188	0.359 933	135	135
140	140	0.373 322	0.371 981	0.370 649	0.369 327	0.368 014	0.366 709	0.365 414	0.364 127	0.362 849	0.361 580	0.360 319	140	140
145	145	0.373 702	0.372 356	0.371 019	0.369 692	0.368 373	0.367 064	0.365 763	0.364 472	0.363 190	0.361 916	0.360 650	145	145
150	150	0.374 026	0.372 676	0.371 335	0.370 003	0.368 680	0.367 365	0.366 060	0.364 764	0.363 477	0.362 199	0.360 930	150	150
155	155	0.374 296	0.372 941	0.371 596	0.370 260	0.368 934	0.367 617	0.366 309	0.365 009	0.363 719	0.362 437	0.361 165	155	155
160	160	0.374 516	0.373 159	0.371 810	0.370 472	0.369 142	0.367 823	0.366 512	0.365 210	0.363 917	0.362 633	0.361 358	160	160
165	165	0.374 694	0.373 334	0.371 983	0.370 642	0.369 310	0.367 988	0.366 675	0.365 371	0.364 076	0.362 789	0.361 512	165	165
170	170	0.374 828	0.373 466	0.372 114	0.370 771	0.369 438	0.368 114	0.366 799	0.365 493	0.364 196	0.362 908	0.361 629	170	170
175	175	0.374 923	0.373 560	0.372 206	0.370 862	0.369 527	0.368 202	0.366 886	0.365 579	0.364 281	0.362 992	0.361 712	175	175
180	180	0.374 980	0.373 615	0.372 261	0.370 915	0.369 580	0.368 254	0.366 938	0.365 630	0.364 331	0.363 041	0.361 761	180	180
		0.374 998	0.373 634	0.372 279	0.370 934	0.369 597	0.368 271	0.366 954	0.365 647	0.364 347	0.363 058	0.361 777		
ψ	z	2.61	2.62	2.63	2.64	2.65	2.66	2.67	2.68	2.69	2.70	2.71	ψ	z

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$

ψ	z	2.91	2.92	2.93	2.94	2.95	2.96	2.97	2.98	2.99	3.00	3.01	z	ψ
0°	0°	0.294 324	0.294 080	0.293 822	0.293 551	0.293 267	0.292 969	0.292 661	0.292 340	0.292 009	0.291 667	0.291 314	0°	0°
5	5	0.294 770	0.294 515	0.294 246	0.293 963	0.293 668	0.293 361	0.293 043	0.292 713	0.292 372	0.292 021	0.291 661	5	5
10	10	0.296 065	0.295 776	0.295 475	0.295 162	0.294 837	0.294 500	0.294 153	0.293 796	0.293 428	0.293 051	0.292 665	10	10
15	15	0.298 089	0.297 749	0.297 399	0.297 038	0.296 666	0.296 285	0.295 894	0.295 494	0.295 085	0.294 668	0.294 243	15	15
20	20	0.300 673	0.300 270	0.299 858	0.299 436	0.299 007	0.298 569	0.298 123	0.297 670	0.297 210	0.296 742	0.296 269	20	20
25	25	0.303 624	0.303 151	0.302 670	0.302 183	0.301 688	0.301 187	0.300 680	0.300 168	0.299 650	0.299 126	0.298 597	25	25
30	30	0.306 760	0.306 214	0.305 663	0.305 106	0.304 545	0.303 979	0.303 409	0.302 835	0.302 256	0.301 674	0.301 089	30	30
35	35	0.309 924	0.309 307	0.308 687	0.308 063	0.307 437	0.306 807	0.306 175	0.305 540	0.304 902	0.304 264	0.303 622	35	35
40	40	0.312 996	0.312 313	0.311 628	0.310 942	0.310 254	0.309 564	0.308 874	0.308 183	0.307 490	0.306 797	0.306 103	40	40
45	45	0.315 894	0.315 151	0.314 408	0.313 664	0.312 920	0.312 177	0.311 433	0.310 690	0.309 947	0.309 204	0.308 462	45	45
50	50	0.318 570	0.317 773	0.316 978	0.316 183	0.315 390	0.314 597	0.313 806	0.313 017	0.312 229	0.311 442	0.310 656	50	50
55	55	0.321 000	0.320 157	0.319 316	0.318 477	0.317 640	0.316 805	0.315 972	0.315 142	0.314 314	0.313 488	0.312 664	55	55
60	60	0.323 181	0.322 297	0.321 417	0.320 539	0.319 664	0.318 792	0.317 924	0.317 058	0.316 195	0.315 335	0.314 478	60	60
65	65	0.325 122	0.324 203	0.323 288	0.322 377	0.321 470	0.320 566	0.319 666	0.318 769	0.317 877	0.316 988	0.316 103	65	65
70	70	0.326 837	0.325 889	0.324 945	0.324 005	0.323 070	0.322 139	0.321 212	0.320 289	0.319 371	0.318 457	0.317 547	70	70
75	75	0.328 346	0.327 373	0.326 405	0.325 440	0.324 482	0.323 527	0.322 577	0.321 632	0.320 692	0.319 756	0.318 825	75	75
80	80	0.329 671	0.328 676	0.327 687	0.326 702	0.325 723	0.324 748	0.323 779	0.322 815	0.321 855	0.320 901	0.319 951	80	80
85	85	0.330 832	0.329 818	0.328 811	0.327 809	0.326 812	0.325 821	0.324 835	0.323 854	0.322 878	0.321 908	0.320 943	85	85
90	90	0.331 847	0.330 818	0.329 796	0.328 778	0.327 767	0.326 761	0.325 760	0.324 766	0.323 776	0.322 792	0.321 814	90	90
95	95	0.332 736	0.331 694	0.330 656	0.329 626	0.328 602	0.327 585	0.326 571	0.325 563	0.324 564	0.323 569	0.322 579	95	95
100	100	0.333 512	0.332 458	0.331 411	0.330 370	0.329 335	0.328 305	0.327 282	0.326 265	0.325 254	0.324 249	0.323 249	100	100
105	105	0.334 190	0.333 127	0.332 071	0.331 018	0.329 974	0.328 935	0.327 904	0.326 879	0.325 858	0.324 844	0.323 836	105	105
110	110	0.334 783	0.333 711	0.332 646	0.331 587	0.330 535	0.329 489	0.328 449	0.327 415	0.326 388	0.325 366	0.324 351	110	110
115	115	0.335 301	0.334 222	0.333 149	0.332 083	0.331 024	0.329 971	0.328 925	0.327 884	0.326 850	0.325 823	0.324 801	115	115
120	120	0.335 751	0.334 666	0.333 588	0.332 516	0.331 451	0.330 392	0.329 340	0.328 294	0.327 255	0.326 222	0.325 195	120	120
125	125	0.336 144	0.335 054	0.333 971	0.332 894	0.331 823	0.330 759	0.329 702	0.328 651	0.327 607	0.326 569	0.325 538	125	125
130	130	0.336 484	0.335 389	0.334 301	0.333 219	0.332 143	0.331 077	0.330 016	0.328 961	0.327 913	0.326 871	0.325 835	130	130
135	135	0.336 777	0.335 679	0.334 587	0.333 502	0.332 424	0.331 352	0.330 277	0.329 229	0.328 177	0.327 132	0.326 093	135	135
140	140	0.337 031	0.335 928	0.334 833	0.333 744	0.332 663	0.331 588	0.330 520	0.329 459	0.328 404	0.327 356	0.326 314	140	140
145	145	0.337 245	0.336 140	0.335 042	0.333 950	0.332 866	0.331 789	0.330 719	0.329 654	0.328 598	0.327 546	0.326 502	145	145
150	150	0.337 425	0.336 317	0.335 217	0.334 123	0.333 037	0.331 957	0.330 885	0.329 819	0.328 759	0.327 706	0.326 660	150	150
155	155	0.337 571	0.336 462	0.335 359	0.334 265	0.333 177	0.332 096	0.331 022	0.329 954	0.328 893	0.327 838	0.326 790	155	155
160	160	0.337 688	0.336 579	0.335 476	0.334 379	0.333 289	0.332 206	0.331 131	0.330 062	0.329 000	0.327 943	0.326 894	160	160
165	165	0.337 780	0.336 668	0.335 562	0.334 464	0.333 374	0.332 291	0.331 214	0.330 144	0.329 081	0.328 024	0.326 974	165	165
170	170	0.337 844	0.336 731	0.335 626	0.334 527	0.333 434	0.332 350	0.331 271	0.330 201	0.329 137	0.328 080	0.327 029	170	170
175	175	0.337 882	0.336 768	0.335 662	0.334 562	0.333 470	0.332 385	0.331 307	0.330 236	0.329 172	0.328 114	0.327 062	175	175
180	180	0.337 894	0.336 780	0.335 674	0.334 575	0.333 483	0.332 398	0.331 320	0.330 247	0.329 183	0.328 125	0.327 074	180	180

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ	z	3.01	3.02	3.03	3.04	3.05	3.06	3.07	3.08	3.09	3.10	3.11	z	ψ
0°	0°	0.291 314	0.290 953	0.290 581	0.290 201	0.289 812	0.289 414	0.289 009	0.288 596	0.288 176	0.287 748	0.287 315	0°	0°
5	5	0.291 661	0.291 289	0.290 910	0.290 521	0.290 125	0.289 720	0.289 307	0.288 887	0.288 460	0.288 026	0.287 586	5	5
10	10	0.292 665	0.292 270	0.291 867	0.291 456	0.291 037	0.290 610	0.290 176	0.289 736	0.289 289	0.288 835	0.288 376	10	10
15	15	0.294 243	0.293 811	0.293 371	0.292 924	0.292 470	0.292 010	0.291 543	0.291 071	0.290 592	0.290 109	0.289 621	15	15
20	20	0.296 269	0.295 788	0.295 302	0.294 810	0.294 313	0.293 810	0.293 302	0.292 790	0.292 273	0.291 751	0.291 226	20	20
25	25	0.298 597	0.298 064	0.297 526	0.296 984	0.296 437	0.295 886	0.295 332	0.294 774	0.294 213	0.293 649	0.293 082	25	25
30	30	0.301 089	0.300 500	0.299 909	0.299 314	0.298 716	0.298 116	0.297 513	0.296 909	0.296 302	0.295 692	0.295 082	30	30
35	35	0.303 622	0.302 980	0.302 336	0.301 688	0.301 041	0.300 392	0.299 740	0.299 090	0.298 436	0.297 785	0.297 129	35	35
40	40	0.306 103	0.305 409	0.304 714	0.304 018	0.303 323	0.302 627	0.301 931	0.301 235	0.300 539	0.299 844	0.299 148	40	40
45	45	0.308 462	0.307 721	0.306 980	0.306 239	0.305 500	0.304 761	0.304 023	0.303 286	0.302 550	0.301 815	0.301 082	45	45
50	50	0.310 656	0.309 872	0.309 090	0.308 309	0.307 530	0.306 753	0.305 978	0.305 204	0.304 432	0.303 662	0.302 894	50	50
55	55	0.312 664	0.311 843	0.311 024	0.310 208	0.309 394	0.308 583	0.307 774	0.306 967	0.306 163	0.305 362	0.304 563	55	55
60	60	0.314 478	0.313 625	0.312 774	0.311 926	0.311 082	0.310 241	0.309 403	0.308 567	0.307 736	0.306 907	0.306 081	60	60
65	65	0.316 103	0.315 221	0.314 342	0.313 468	0.312 597	0.311 730	0.310 866	0.310 006	0.309 150	0.308 297	0.307 448	65	65
70	70	0.317 547	0.316 641	0.315 738	0.314 841	0.313 947	0.313 057	0.312 172	0.311 290	0.310 413	0.309 539	0.308 670	70	70
75	75	0.318 825	0.317 898	0.316 975	0.316 058	0.315 144	0.314 235	0.313 331	0.312 431	0.311 535	0.310 644	0.309 757	75	75
80	80	0.319 951	0.319 007	0.318 067	0.317 132	0.316 202	0.315 276	0.314 356	0.313 440	0.312 528	0.311 622	0.310 720	80	80
85	85	0.320 943	0.319 984	0.319 029	0.318 079	0.317 134	0.316 194	0.315 260	0.314 330	0.313 405	0.312 486	0.311 570	85	85
90	90	0.321 814	0.320 841	0.319 873	0.318 911	0.317 954	0.317 002	0.316 056	0.315 114	0.314 178	0.313 247	0.312 321	90	90
95	95	0.322 579	0.321 594	0.320 614	0.319 643	0.318 674	0.317 712	0.316 756	0.315 804	0.314 858	0.313 917	0.312 982	95	95
100	100	0.323 249	0.322 255	0.321 267	0.320 285	0.319 308	0.318 336	0.317 371	0.316 410	0.315 456	0.314 506	0.313 562	100	100
105	105	0.323 836	0.322 834	0.321 838	0.320 847	0.319 864	0.318 884	0.317 911	0.316 943	0.315 980	0.315 024	0.314 074	105	105
110	110	0.324 351	0.323 342	0.322 339	0.321 341	0.320 350	0.319 364	0.318 384	0.317 410	0.316 442	0.315 479	0.314 522	110	110
115	115	0.324 801	0.323 786	0.322 777	0.321 774	0.320 777	0.319 786	0.318 800	0.317 820	0.316 846	0.315 878	0.314 915	115	115
120	120	0.325 195	0.324 175	0.323 160	0.322 151	0.321 149	0.320 153	0.319 162	0.318 178	0.317 200	0.316 227	0.315 260	120	120
125	125	0.325 538	0.324 513	0.323 494	0.322 481	0.321 474	0.320 473	0.319 478	0.318 490	0.317 506	0.316 529	0.315 560	125	125
130	130	0.325 835	0.324 806	0.323 783	0.322 767	0.321 757	0.320 753	0.319 754	0.318 762	0.317 775	0.316 795	0.315 820	130	130
135	135	0.326 093	0.325 061	0.324 035	0.323 015	0.322 001	0.320 994	0.319 992	0.318 997	0.318 007	0.317 024	0.316 047	135	135
140	140	0.326 314	0.325 279	0.324 250	0.323 228	0.322 211	0.321 201	0.320 197	0.319 200	0.318 208	0.317 222	0.316 242	140	140
145	145	0.326 502	0.325 464	0.324 433	0.323 408	0.322 390	0.321 377	0.320 371	0.319 371	0.318 377	0.317 389	0.316 407	145	145
150	150	0.326 660	0.325 620	0.324 587	0.323 560	0.322 540	0.321 526	0.320 518	0.319 516	0.318 520	0.317 531	0.316 547	150	150
155	155	0.326 790	0.325 749	0.324 714	0.323 686	0.322 664	0.321 648	0.320 638	0.319 635	0.318 637	0.317 646	0.316 661	155	155
160	160	0.326 894	0.325 851	0.324 815	0.323 786	0.322 762	0.321 745	0.320 734	0.319 730	0.318 732	0.317 740	0.316 753	160	160
165	165	0.326 974	0.325 930	0.324 893	0.323 863	0.322 839	0.321 820	0.320 808	0.319 802	0.318 803	0.317 810	0.316 823	165	165
170	170	0.327 029	0.325 985	0.324 947	0.323 916	0.322 891	0.321 873	0.320 860	0.319 854	0.318 854	0.317 861	0.316 873	170	170
175	175	0.327 062	0.326 017	0.324 979	0.323 947	0.322 922	0.321 903	0.320 891	0.319 885	0.318 884	0.317 890	0.316 902	175	175
180	180	0.327 074	0.326 029	0.324 991	0.323 959	0.322 933	0.321 914	0.320 901	0.319 895	0.318 894	0.317 900	0.316 912	180	180

$\frac{1}{m}$ TABLE FOR INTERPOLATING $\varepsilon = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ	$\frac{1}{m}$	3.11	3.12	3.13	3.14	3.15	3.16	3.17	3.18	3.19	3.20	3.21	$\frac{1}{m}$
0°	+	0.287 315	0.286 874	0.286 428	0.285 975	0.285 517	0.285 054	0.284 586	0.284 112	0.283 634	0.283 152	0.282 665	0°
5	+	0.287 586	0.287 139	0.286 686	0.286 228	0.285 764	0.285 295	0.284 821	0.284 343	0.283 858	0.283 370	0.282 880	5
10	+	0.288 376	0.287 911	0.287 441	0.286 965	0.286 484	0.285 999	0.285 509	0.285 015	0.284 517	0.284 014	0.283 508	10
15	+	0.289 621	0.289 127	0.288 629	0.288 127	0.287 620	0.287 109	0.286 594	0.286 076	0.285 554	0.285 029	0.284 501	15
20	+	0.291 226	0.290 696	0.289 627	0.289 627	0.289 087	0.288 543	0.287 997	0.287 448	0.286 896	0.286 342	0.285 786	20
25	+	0.293 082	0.292 512	0.291 939	0.291 364	0.290 786	0.290 206	0.289 624	0.289 040	0.288 455	0.287 868	0.287 279	25
30	+	0.295 082	0.294 469	0.293 855	0.293 239	0.292 622	0.292 004	0.291 385	0.290 764	0.290 143	0.289 521	0.288 898	30
35	+	0.297 129	0.296 473	0.295 817	0.295 162	0.294 507	0.293 850	0.293 194	0.292 536	0.291 879	0.291 223	0.290 567	35
40	+	0.299 148	0.298 453	0.297 758	0.297 064	0.296 370	0.295 677	0.294 985	0.294 293	0.293 602	0.292 912	0.292 223	40
45	+	0.301 082	0.300 350	0.299 619	0.298 889	0.298 160	0.297 433	0.296 707	0.295 983	0.295 260	0.294 539	0.293 820	45
50	+	0.302 894	0.302 128	0.301 364	0.300 602	0.299 842	0.299 084	0.298 328	0.297 574	0.296 822	0.296 073	0.295 325	50
55	+	0.304 563	0.303 767	0.302 973	0.302 183	0.301 395	0.300 609	0.299 826	0.299 046	0.298 268	0.297 494	0.296 721	55
60	+	0.306 081	0.305 259	0.304 439	0.303 623	0.302 810	0.302 000	0.301 193	0.300 390	0.299 589	0.298 792	0.297 998	60
65	+	0.307 448	0.306 603	0.305 761	0.304 923	0.304 088	0.303 257	0.302 429	0.301 605	0.300 785	0.299 968	0.299 155	65
70	+	0.308 670	0.307 805	0.306 944	0.306 086	0.305 232	0.304 383	0.303 537	0.302 695	0.301 858	0.301 024	0.300 194	70
75	+	0.309 757	0.308 875	0.307 997	0.307 122	0.306 252	0.305 387	0.304 526	0.303 668	0.302 816	0.301 967	0.301 122	75
80	+	0.310 720	0.309 822	0.308 930	0.308 041	0.307 157	0.306 278	0.305 403	0.304 533	0.303 667	0.302 806	0.301 948	80
85	+	0.311 570	0.310 660	0.309 755	0.308 854	0.307 959	0.307 067	0.306 181	0.305 299	0.304 422	0.303 549	0.302 681	85
90	+	0.312 321	0.311 399	0.310 483	0.309 572	0.308 666	0.307 765	0.306 868	0.305 976	0.305 089	0.304 207	0.303 330	90
95	+	0.312 982	0.312 051	0.311 126	0.310 205	0.309 290	0.308 380	0.307 475	0.306 574	0.305 679	0.304 788	0.303 903	95
100	+	0.313 562	0.312 624	0.311 690	0.310 762	0.309 839	0.308 921	0.308 009	0.307 101	0.306 198	0.305 301	0.304 408	100
105	+	0.314 074	0.313 128	0.312 186	0.311 252	0.310 322	0.309 398	0.308 479	0.307 565	0.306 656	0.305 752	0.304 852	105
110	+	0.314 522	0.313 570	0.312 624	0.311 683	0.310 747	0.309 817	0.308 892	0.307 973	0.307 059	0.306 150	0.305 246	110
115	+	0.314 915	0.313 958	0.313 006	0.312 060	0.311 119	0.310 185	0.309 255	0.308 331	0.307 412	0.306 499	0.305 590	115
120	+	0.315 260	0.314 298	0.313 344	0.312 391	0.311 446	0.310 508	0.309 573	0.308 645	0.307 722	0.306 805	0.305 892	120
125	+	0.315 560	0.314 594	0.313 634	0.312 680	0.311 731	0.310 789	0.309 851	0.308 920	0.307 993	0.307 072	0.306 156	125
130	+	0.315 820	0.314 852	0.313 889	0.312 932	0.311 980	0.311 034	0.310 093	0.309 159	0.308 229	0.307 306	0.306 386	130
135	+	0.316 047	0.315 076	0.314 110	0.313 150	0.312 195	0.311 247	0.310 303	0.309 366	0.308 434	0.307 508	0.306 586	135
140	+	0.316 242	0.315 268	0.314 300	0.313 338	0.312 381	0.311 430	0.310 484	0.309 545	0.308 610	0.307 682	0.306 758	140
145	+	0.316 407	0.315 432	0.314 462	0.313 498	0.312 539	0.311 586	0.310 638	0.309 697	0.308 760	0.307 830	0.306 905	145
150	+	0.316 547	0.315 570	0.314 597	0.313 631	0.312 670	0.311 716	0.310 767	0.309 825	0.308 886	0.307 954	0.307 027	150
155	+	0.316 661	0.315 682	0.314 709	0.313 742	0.312 781	0.311 826	0.310 875	0.309 931	0.308 991	0.308 058	0.307 129	155
160	+	0.316 753	0.315 773	0.314 798	0.313 830	0.312 867	0.311 910	0.310 958	0.310 013	0.309 073	0.308 140	0.307 210	160
165	+	0.316 823	0.315 842	0.314 867	0.313 898	0.312 934	0.311 977	0.311 025	0.310 079	0.309 138	0.308 203	0.307 273	165
170	+	0.316 873	0.315 891	0.314 915	0.313 945	0.312 981	0.312 023	0.311 070	0.310 124	0.309 182	0.308 247	0.307 316	170
175	+	0.316 902	0.315 920	0.314 944	0.313 974	0.313 009	0.312 051	0.311 098	0.310 151	0.309 209	0.308 273	0.307 342	175
180	+	0.316 912	0.315 930	0.314 953	0.313 983	0.313 018	0.312 060	0.311 107	0.310 160	0.309 218	0.308 282	0.307 351	180
ψ	$\frac{1}{m}$	3.11	3.12	3.13	3.14	3.15	3.16	3.17	3.18	3.19	3.20	3.21	$\frac{1}{m}$

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{r}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ	$\frac{1}{m}$	3.21	3.22	3.23	3.24	3.25	3.26	3.27	3.28	3.29	3.30	3.31	z	ψ
0°	5	0.282 665	0.282 174	0.281 680	0.281 181	0.280 680	0.280 174	0.279 666	0.279 155	0.278 641	0.278 124	0.277 605	0°	5
10	10	0.282 880	0.282 385	0.281 885	0.281 383	0.280 876	0.280 367	0.279 854	0.279 339	0.278 821	0.278 301	0.277 778	10	10
15	15	0.283 508	0.282 999	0.282 486	0.281 970	0.281 451	0.280 929	0.280 405	0.279 877	0.279 348	0.278 816	0.278 282	15	15
20	20	0.284 501	0.283 970	0.283 436	0.282 899	0.282 360	0.281 819	0.281 275	0.280 729	0.280 182	0.279 632	0.279 081	20	20
25	25	0.285 786	0.285 227	0.284 666	0.284 103	0.283 539	0.282 972	0.282 404	0.281 835	0.281 264	0.280 692	0.280 119	25	25
30	30	0.287 279	0.286 689	0.286 097	0.285 505	0.284 911	0.284 316	0.283 720	0.283 124	0.282 527	0.281 930	0.281 332	30	30
35	35	0.288 898	0.288 275	0.287 651	0.287 027	0.286 403	0.285 778	0.285 153	0.284 528	0.283 903	0.283 278	0.282 654	35	35
40	40	0.290 567	0.289 910	0.289 254	0.288 599	0.287 943	0.287 287	0.286 633	0.285 980	0.285 328	0.284 674	0.284 023	40	40
45	45	0.292 223	0.291 534	0.290 847	0.290 161	0.289 476	0.288 792	0.288 109	0.287 428	0.286 748	0.286 069	0.285 391	45	45
50	50	0.293 820	0.293 102	0.292 385	0.291 671	0.290 958	0.290 247	0.289 538	0.288 830	0.288 124	0.287 420	0.286 718	50	50
55	55	0.295 325	0.294 580	0.293 838	0.293 097	0.292 359	0.291 623	0.290 889	0.290 158	0.289 429	0.288 702	0.287 978	55	55
60	60	0.296 721	0.295 952	0.295 185	0.294 422	0.293 660	0.292 902	0.292 146	0.291 394	0.290 643	0.289 896	0.289 152	60	60
65	65	0.297 998	0.297 207	0.296 420	0.295 635	0.294 854	0.294 075	0.293 300	0.292 528	0.291 760	0.290 994	0.290 231	65	65
70	70	0.299 155	0.298 345	0.297 539	0.296 736	0.295 937	0.295 141	0.294 349	0.293 560	0.292 774	0.291 993	0.291 214	70	70
75	75	0.300 194	0.299 367	0.298 545	0.297 727	0.296 912	0.296 101	0.295 294	0.294 490	0.293 690	0.292 894	0.292 102	75	75
80	80	0.301 122	0.300 282	0.299 445	0.298 613	0.297 785	0.296 961	0.296 140	0.295 324	0.294 512	0.293 703	0.292 899	80	80
85	85	0.301 948	0.301 096	0.300 247	0.299 403	0.298 563	0.297 727	0.296 895	0.296 068	0.295 245	0.294 425	0.293 610	85	85
90	90	0.302 681	0.301 818	0.300 959	0.300 104	0.299 254	0.298 408	0.297 567	0.296 730	0.295 897	0.295 068	0.294 244	90	90
95	95	0.303 330	0.302 457	0.301 589	0.300 725	0.299 866	0.299 012	0.298 162	0.297 317	0.296 476	0.295 639	0.294 807	95	95
100	100	0.303 903	0.303 022	0.302 146	0.301 275	0.300 408	0.299 546	0.298 689	0.297 836	0.296 989	0.296 145	0.295 306	100	100
105	105	0.304 408	0.303 521	0.302 638	0.301 760	0.300 887	0.300 018	0.299 155	0.298 296	0.297 442	0.296 592	0.295 747	105	105
110	110	0.304 852	0.303 960	0.303 072	0.302 188	0.301 311	0.300 434	0.299 565	0.298 701	0.297 842	0.296 987	0.296 136	110	110
115	115	0.305 246	0.304 347	0.303 453	0.302 564	0.301 681	0.300 802	0.299 928	0.299 059	0.298 195	0.297 336	0.296 481	115	115
120	120	0.305 590	0.304 687	0.303 790	0.302 896	0.302 008	0.301 125	0.300 247	0.299 374	0.298 506	0.297 642	0.296 784	120	120
125	125	0.305 892	0.304 986	0.304 083	0.303 187	0.302 294	0.301 408	0.300 526	0.299 650	0.298 777	0.297 911	0.297 050	125	125
130	130	0.306 156	0.305 246	0.304 340	0.303 441	0.302 545	0.301 656	0.300 771	0.299 892	0.299 016	0.298 147	0.297 283	130	130
135	135	0.306 386	0.305 473	0.304 565	0.303 663	0.302 765	0.301 873	0.300 985	0.300 103	0.299 225	0.298 353	0.297 485	135	135
140	140	0.306 586	0.305 670	0.304 759	0.303 855	0.302 954	0.302 060	0.301 171	0.300 286	0.299 407	0.298 532	0.297 663	140	140
145	145	0.306 758	0.305 840	0.304 927	0.304 021	0.303 118	0.302 222	0.301 330	0.300 444	0.299 562	0.298 686	0.297 814	145	145
150	150	0.306 905	0.305 986	0.305 071	0.304 163	0.303 259	0.302 361	0.301 467	0.300 579	0.299 695	0.298 818	0.297 944	150	150
155	155	0.307 027	0.306 107	0.305 191	0.304 281	0.303 375	0.302 476	0.301 581	0.300 692	0.299 807	0.298 928	0.298 053	155	155
160	160	0.307 129	0.306 208	0.305 290	0.304 379	0.303 472	0.302 572	0.301 675	0.300 785	0.299 900	0.299 020	0.298 144	160	160
165	165	0.307 210	0.306 287	0.305 369	0.304 457	0.303 549	0.302 648	0.301 751	0.300 860	0.299 973	0.299 093	0.298 216	165	165
170	170	0.307 273	0.306 349	0.305 430	0.304 517	0.303 608	0.302 706	0.301 808	0.300 917	0.300 029	0.299 148	0.298 271	170	170
175	175	0.307 316	0.306 392	0.305 472	0.304 559	0.303 650	0.302 747	0.301 850	0.300 957	0.300 070	0.299 188	0.298 310	175	175
180	180	0.307 342	0.306 417	0.305 497	0.304 584	0.303 675	0.302 772	0.301 873	0.300 981	0.300 093	0.299 211	0.298 333	180	180
180	180	0.307 351	0.306 426	0.305 506	0.304 593	0.303 684	0.302 782	0.301 882	0.300 989	0.300 101	0.299 219	0.298 341	180	180
ψ	z	3.21	3.22	3.23	3.24	3.25	3.26	3.27	3.28	3.29	3.30	3.31	z	ψ

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ	z	3.31	3.32	3.33	3.34	3.35	3.36	3.37	3.38	3.39	3.40	3.41	z
0°	0°	0.277 605	0.277 084	0.276 560	0.276 034	0.275 506	0.274 976	0.274 445	0.273 912	0.273 378	0.272 842	0.272 305	0°
5	5	0.277 778	0.277 253	0.276 725	0.276 196	0.275 665	0.275 132	0.274 597	0.274 061	0.273 523	0.272 984	0.272 444	5
10	10	0.278 282	0.277 746	0.277 208	0.276 669	0.276 128	0.275 585	0.275 041	0.274 496	0.273 949	0.273 402	0.272 853	10
15	15	0.279 081	0.278 528	0.277 974	0.277 419	0.276 862	0.276 304	0.275 746	0.275 186	0.274 625	0.274 062	0.273 502	15
20	20	0.280 119	0.279 545	0.278 970	0.278 394	0.277 818	0.277 241	0.276 663	0.276 085	0.275 506	0.274 927	0.274 348	20
25	25	0.281 332	0.280 733	0.280 134	0.279 535	0.278 936	0.278 337	0.277 737	0.277 138	0.276 539	0.275 940	0.275 341	25
30	30	0.282 654	0.282 029	0.281 405	0.280 781	0.280 157	0.279 534	0.278 912	0.278 290	0.277 668	0.277 048	0.276 428	30
35	35	0.284 023	0.283 372	0.282 722	0.282 072	0.281 425	0.280 779	0.280 133	0.279 489	0.278 845	0.278 200	0.277 560	35
40	40	0.285 391	0.284 715	0.284 040	0.283 367	0.282 696	0.282 025	0.281 357	0.280 690	0.280 024	0.279 360	0.278 698	40
45	45	0.286 718	0.286 018	0.285 320	0.284 624	0.283 930	0.283 238	0.282 548	0.281 860	0.281 174	0.280 490	0.279 808	45
50	50	0.287 978	0.287 256	0.286 536	0.285 819	0.285 104	0.284 392	0.283 682	0.282 974	0.282 269	0.281 567	0.280 867	50
55	55	0.289 152	0.288 410	0.287 671	0.286 935	0.286 201	0.285 471	0.284 743	0.284 018	0.283 295	0.282 576	0.281 859	55
60	60	0.290 231	0.289 472	0.288 716	0.287 963	0.287 212	0.286 466	0.285 722	0.284 981	0.284 243	0.283 509	0.282 777	60
65	65	0.291 214	0.290 439	0.289 668	0.288 900	0.288 135	0.287 373	0.286 615	0.285 859	0.285 109	0.284 361	0.283 617	65
70	70	0.292 102	0.291 313	0.290 528	0.289 747	0.288 969	0.288 195	0.287 424	0.286 657	0.285 894	0.285 134	0.284 378	70
75	75	0.292 899	0.292 098	0.291 301	0.290 509	0.289 719	0.288 934	0.288 152	0.287 375	0.286 601	0.285 831	0.285 064	75
80	80	0.293 610	0.292 799	0.291 993	0.291 190	0.290 391	0.289 596	0.288 805	0.288 018	0.287 235	0.286 455	0.285 680	80
85	85	0.294 244	0.293 424	0.292 608	0.291 797	0.290 989	0.290 186	0.289 387	0.288 592	0.287 801	0.287 013	0.286 230	85
90	90	0.294 807	0.293 979	0.293 156	0.292 337	0.291 522	0.290 711	0.289 905	0.289 102	0.288 304	0.287 510	0.286 720	90
95	95	0.295 306	0.294 471	0.293 641	0.292 816	0.291 994	0.291 177	0.290 364	0.289 556	0.288 752	0.287 952	0.287 156	95
100	100	0.295 747	0.294 907	0.294 071	0.293 239	0.292 412	0.291 590	0.290 772	0.289 958	0.289 148	0.288 343	0.287 542	100
105	105	0.296 136	0.295 291	0.294 450	0.293 617	0.292 782	0.291 956	0.291 132	0.290 315	0.289 500	0.288 690	0.287 884	105
110	110	0.296 481	0.295 631	0.294 786	0.293 945	0.293 109	0.292 277	0.291 450	0.290 628	0.289 810	0.288 996	0.288 187	110
115	115	0.296 784	0.295 930	0.295 081	0.294 237	0.293 397	0.292 562	0.291 731	0.290 905	0.290 084	0.289 267	0.288 454	115
120	120	0.297 050	0.296 193	0.295 341	0.294 493	0.293 650	0.292 812	0.291 978	0.291 149	0.290 325	0.289 505	0.288 690	120
125	125	0.297 283	0.296 423	0.295 568	0.294 718	0.293 872	0.293 031	0.292 195	0.291 363	0.290 536	0.289 714	0.288 896	125
130	130	0.297 485	0.296 624	0.295 765	0.294 913	0.294 065	0.293 222	0.292 384	0.291 550	0.290 721	0.289 896	0.289 076	130
135	135	0.297 663	0.296 798	0.295 939	0.295 084	0.294 234	0.293 389	0.292 548	0.291 712	0.290 881	0.290 054	0.289 232	135
140	140	0.297 814	0.296 948	0.296 086	0.295 230	0.294 379	0.293 532	0.292 690	0.291 852	0.291 019	0.290 191	0.289 367	140
145	145	0.297 944	0.297 077	0.296 214	0.295 357	0.294 504	0.293 656	0.292 812	0.291 973	0.291 138	0.290 308	0.289 483	145
150	150	0.298 053	0.297 184	0.296 319	0.295 461	0.294 606	0.293 757	0.292 912	0.292 072	0.291 237	0.290 406	0.289 580	150
155	155	0.298 144	0.297 274	0.296 408	0.295 548	0.294 693	0.293 842	0.292 996	0.292 155	0.291 319	0.290 487	0.289 660	155
160	160	0.298 216	0.297 345	0.296 479	0.295 618	0.294 762	0.293 910	0.293 064	0.292 222	0.291 385	0.290 552	0.289 724	160
165	165	0.298 271	0.297 400	0.296 532	0.295 671	0.294 813	0.293 962	0.293 115	0.292 273	0.291 435	0.290 602	0.289 774	165
170	170	0.298 310	0.297 438	0.296 570	0.295 709	0.294 852	0.293 999	0.293 152	0.292 309	0.291 471	0.290 637	0.289 808	170
175	175	0.298 333	0.297 461	0.296 593	0.295 731	0.294 874	0.294 021	0.293 173	0.292 330	0.291 492	0.290 658	0.289 828	175
180	180	0.298 341	0.297 469	0.296 600	0.295 738	0.294 881	0.294 028	0.293 180	0.292 337	0.291 499	0.290 665	0.289 838	180
ψ	z	3.31	3.32	3.33	3.34	3.35	3.36	3.37	3.38	3.39	3.40	3.41	z

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ	$\frac{1}{m}$	3.41	3.42	3.43	3.44	3.45	3.46	3.47	3.48	3.49	3.50	3.51	z
0°	5	0.272 305	0.271 766	0.271 227	0.270 686	0.270 145	0.269 603	0.269 060	0.268 517	0.267 973	0.267 429	0.266 884	0°
5		0.272 444	0.271 903	0.271 361	0.270 818	0.270 274	0.269 729	0.269 184	0.268 638	0.268 092	0.267 545	0.266 998	5
10		0.272 853	0.272 303	0.271 753	0.271 202	0.270 651	0.270 098	0.269 546	0.268 993	0.268 439	0.267 886	0.267 332	10
15		0.273 502	0.272 939	0.272 376	0.271 813	0.271 249	0.270 685	0.270 121	0.269 556	0.268 992	0.268 427	0.267 863	15
20		0.274 348	0.273 769	0.273 189	0.272 610	0.272 031	0.271 451	0.270 872	0.270 293	0.269 715	0.269 136	0.268 558	20
25		0.275 341	0.274 742	0.274 144	0.273 546	0.272 949	0.272 352	0.271 756	0.271 160	0.270 565	0.269 971	0.269 378	25
30		0.276 428	0.275 809	0.275 190	0.274 573	0.273 956	0.273 340	0.272 726	0.272 112	0.271 499	0.270 888	0.270 277	30
35		0.277 560	0.276 919	0.276 282	0.275 644	0.275 007	0.274 374	0.273 738	0.273 107	0.272 478	0.271 848	0.271 220	35
40		0.278 698	0.278 038	0.277 379	0.276 722	0.276 066	0.275 413	0.274 761	0.274 111	0.273 463	0.272 817	0.272 172	40
45		0.279 808	0.279 128	0.278 450	0.277 775	0.277 101	0.276 430	0.275 761	0.275 094	0.274 429	0.273 766	0.273 106	45
50		0.280 867	0.280 169	0.279 474	0.278 781	0.278 091	0.277 403	0.276 718	0.276 035	0.275 354	0.274 676	0.274 001	50
55		0.281 859	0.281 145	0.280 434	0.279 726	0.279 020	0.278 317	0.277 617	0.276 920	0.276 225	0.275 534	0.274 844	55
60		0.282 777	0.282 049	0.281 323	0.280 601	0.279 881	0.279 165	0.278 452	0.277 741	0.277 034	0.276 329	0.275 628	60
65		0.283 617	0.282 875	0.282 137	0.281 402	0.280 670	0.279 942	0.279 217	0.278 495	0.277 776	0.277 060	0.276 348	65
70		0.284 378	0.283 625	0.282 876	0.282 130	0.281 387	0.280 648	0.279 913	0.279 181	0.278 452	0.277 726	0.277 004	70
75		0.285 064	0.284 302	0.283 543	0.282 787	0.282 035	0.281 286	0.280 542	0.279 801	0.279 063	0.278 328	0.277 598	75
80		0.285 680	0.284 908	0.284 141	0.283 377	0.282 616	0.281 860	0.281 107	0.280 358	0.279 612	0.278 871	0.278 132	80
85		0.286 230	0.285 451	0.284 676	0.283 904	0.283 137	0.282 373	0.281 613	0.280 857	0.280 105	0.279 356	0.278 612	85
90		0.286 720	0.285 934	0.285 153	0.284 375	0.283 601	0.282 831	0.282 065	0.281 303	0.280 545	0.279 790	0.279 040	90
95		0.287 156	0.286 364	0.285 577	0.284 793	0.284 014	0.283 239	0.282 467	0.281 700	0.280 937	0.280 177	0.279 422	95
100		0.287 542	0.286 746	0.285 953	0.285 165	0.284 381	0.283 601	0.282 825	0.282 053	0.281 285	0.280 521	0.279 761	100
105		0.287 884	0.287 084	0.286 287	0.285 493	0.284 706	0.283 923	0.283 142	0.282 366	0.281 594	0.280 826	0.280 063	105
110		0.288 187	0.287 382	0.286 582	0.285 786	0.284 994	0.284 206	0.283 423	0.282 643	0.281 868	0.281 097	0.280 330	110
115		0.288 454	0.287 646	0.286 844	0.286 043	0.285 248	0.284 457	0.283 670	0.282 888	0.282 110	0.281 336	0.280 566	115
120		0.288 690	0.287 878	0.287 072	0.286 270	0.285 472	0.284 678	0.283 889	0.283 104	0.282 323	0.281 547	0.280 774	120
125		0.288 896	0.288 082	0.287 273	0.286 468	0.285 668	0.284 872	0.284 081	0.283 294	0.282 511	0.281 732	0.280 957	125
130		0.289 076	0.288 260	0.287 449	0.286 642	0.285 840	0.285 042	0.284 248	0.283 459	0.282 674	0.281 894	0.281 117	130
135		0.289 232	0.288 414	0.287 601	0.286 793	0.285 989	0.285 190	0.284 394	0.283 603	0.282 815	0.282 034	0.281 256	135
140		0.289 367	0.288 548	0.287 734	0.286 924	0.286 118	0.285 317	0.284 520	0.283 728	0.282 941	0.282 158	0.281 377	140
145		0.289 483	0.288 662	0.287 846	0.287 035	0.286 227	0.285 424	0.284 627	0.283 834	0.283 045	0.282 260	0.281 479	145
150		0.289 580	0.288 759	0.287 942	0.287 129	0.286 321	0.285 518	0.284 719	0.283 924	0.283 133	0.282 347	0.281 565	150
155		0.289 660	0.288 838	0.288 020	0.287 207	0.286 398	0.285 594	0.284 794	0.283 998	0.283 207	0.282 420	0.281 637	155
160		0.289 724	0.288 901	0.288 083	0.287 269	0.286 459	0.285 654	0.284 854	0.284 058	0.283 266	0.282 478	0.281 695	160
165		0.289 774	0.288 950	0.288 131	0.287 316	0.286 506	0.285 700	0.284 899	0.284 102	0.283 310	0.282 522	0.281 739	165
170		0.289 808	0.288 984	0.288 165	0.287 350	0.286 539	0.285 733	0.284 932	0.284 135	0.283 342	0.282 554	0.281 770	170
175		0.289 828	0.289 004	0.288 184	0.287 369	0.286 558	0.285 752	0.284 951	0.284 154	0.283 361	0.282 573	0.281 788	175
180		0.289 838	0.289 012	0.288 192	0.287 376	0.286 565	0.285 759	0.284 957	0.284 160	0.283 367	0.282 578	0.281 794	180

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ	z	3.71	3.72	3.73	3.74	3.75	3.76	3.77	3.78	3.79	3.80	3.81	z	ψ
0°	5	0.255 999	0.255 459	0.254 920	0.254 382	0.253 844	0.253 308	0.252 772	0.252 237	0.251 703	0.251 170	0.250 638	0°	5
		0.256 076	0.255 535	0.254 995	0.254 455	0.253 916	0.253 379	0.252 842	0.252 305	0.251 770	0.251 236	0.250 703		
10		0.256 305	0.255 759	0.255 215	0.254 671	0.254 129	0.253 587	0.253 046	0.252 506	0.251 968	0.251 430	0.250 893	10	
15		0.256 669	0.256 117	0.255 566	0.255 017	0.254 468	0.253 920	0.253 373	0.252 828	0.252 284	0.251 740	0.251 198	15	
20		0.257 148	0.256 588	0.256 029	0.255 471	0.254 914	0.254 359	0.253 804	0.253 251	0.252 700	0.252 149	0.251 600	20	
25		0.257 716	0.257 146	0.256 577	0.256 010	0.255 444	0.254 880	0.254 316	0.253 755	0.253 195	0.252 636	0.252 079	25	
30		0.258 347	0.257 766	0.257 187	0.256 609	0.256 033	0.255 458	0.254 885	0.254 314	0.253 744	0.253 176	0.252 610	30	
35		0.259 010	0.258 420	0.257 830	0.257 241	0.256 655	0.256 070	0.255 486	0.254 906	0.254 327	0.253 749	0.253 174	35	
40		0.259 690	0.259 087	0.258 486	0.257 887	0.257 290	0.256 695	0.256 102	0.255 511	0.254 922	0.254 336	0.253 751	40	
45		0.260 361	0.259 748	0.259 136	0.258 527	0.257 920	0.257 315	0.256 712	0.256 112	0.255 514	0.254 918	0.254 324	45	
50		0.261 011	0.260 387	0.259 766	0.259 147	0.258 531	0.257 916	0.257 305	0.256 695	0.256 088	0.255 484	0.254 881	50	
55		0.261 629	0.260 996	0.260 365	0.259 738	0.259 112	0.258 489	0.257 869	0.257 251	0.256 636	0.256 023	0.255 413	55	
60		0.262 208	0.261 566	0.260 928	0.260 291	0.259 658	0.259 027	0.258 399	0.257 774	0.257 151	0.256 531	0.255 914	60	
65		0.262 744	0.262 095	0.261 449	0.260 806	0.260 165	0.259 527	0.258 892	0.258 260	0.257 630	0.257 004	0.256 379	65	
70		0.263 237	0.262 581	0.261 928	0.261 278	0.260 631	0.259 987	0.259 345	0.258 707	0.258 071	0.257 439	0.256 809	70	
75		0.263 686	0.263 024	0.262 365	0.261 709	0.261 056	0.260 407	0.259 759	0.259 116	0.258 474	0.257 837	0.257 202	75	
80		0.264 093	0.263 426	0.262 761	0.262 100	0.261 442	0.260 787	0.260 136	0.259 487	0.258 841	0.258 198	0.257 558	80	
85		0.264 460	0.263 788	0.263 119	0.262 454	0.261 791	0.261 132	0.260 476	0.259 823	0.259 172	0.258 525	0.257 882	85	
90		0.264 790	0.264 114	0.263 441	0.262 771	0.262 105	0.261 442	0.260 782	0.260 125	0.259 471	0.258 820	0.258 173	90	
95		0.265 086	0.264 406	0.263 730	0.263 057	0.262 387	0.261 720	0.261 056	0.260 396	0.259 739	0.259 085	0.258 434	95	
100		0.265 351	0.264 668	0.263 988	0.263 312	0.262 639	0.261 969	0.261 302	0.260 639	0.259 979	0.259 322	0.258 669	100	
105		0.265 588	0.264 900	0.264 218	0.263 540	0.262 862	0.262 190	0.261 522	0.260 857	0.260 194	0.259 534	0.258 877	105	
110		0.265 796	0.265 108	0.264 423	0.263 742	0.263 064	0.262 389	0.261 717	0.261 049	0.260 384	0.259 723	0.259 065	110	
115		0.265 982	0.265 292	0.264 605	0.263 922	0.263 241	0.262 565	0.261 891	0.261 221	0.260 554	0.259 891	0.259 230	115	
120		0.266 147	0.265 455	0.264 765	0.264 080	0.263 398	0.262 720	0.262 044	0.261 373	0.260 704	0.260 040	0.259 377	120	
125		0.266 292	0.265 598	0.264 907	0.264 220	0.263 536	0.262 857	0.262 180	0.261 507	0.260 837	0.260 171	0.259 507	125	
130		0.266 419	0.265 724	0.265 032	0.264 344	0.263 659	0.262 978	0.262 299	0.261 625	0.260 953	0.260 286	0.259 621	130	
135		0.266 530	0.265 834	0.265 140	0.264 451	0.263 765	0.263 083	0.262 403	0.261 728	0.261 055	0.260 387	0.259 721	135	
140		0.266 627	0.265 930	0.265 235	0.264 544	0.263 857	0.263 174	0.262 493	0.261 817	0.261 143	0.260 474	0.259 807	140	
145		0.266 708	0.266 010	0.265 315	0.264 624	0.263 936	0.263 252	0.262 570	0.261 893	0.261 218	0.260 548	0.259 880	145	
150		0.266 778	0.266 079	0.265 383	0.264 691	0.264 002	0.263 317	0.262 635	0.261 957	0.261 282	0.260 611	0.259 943	150	
155		0.266 836	0.266 136	0.265 440	0.264 747	0.264 057	0.263 372	0.262 689	0.262 011	0.261 335	0.260 664	0.259 995	155	
160		0.266 882	0.266 182	0.265 484	0.264 791	0.264 101	0.263 416	0.262 732	0.262 053	0.261 377	0.260 706	0.260 036	160	
165		0.266 916	0.266 216	0.265 518	0.264 825	0.264 136	0.263 448	0.262 767	0.262 086	0.261 410	0.260 738	0.260 068	165	
170		0.266 942	0.266 241	0.265 543	0.264 850	0.264 159	0.263 473	0.262 789	0.262 110	0.261 433	0.260 760	0.260 090	170	
175		0.266 955	0.266 256	0.265 558	0.264 864	0.264 173	0.263 487	0.262 803	0.262 123	0.261 446	0.260 774	0.260 104	175	
180		0.266 962	0.266 261	0.265 563	0.264 869	0.264 178	0.263 491	0.262 807	0.262 128	0.261 451	0.260 778	0.260 108	180	
ψ	z	3.71	3.72	3.73	3.74	3.75	3.76	3.77	3.78	3.79	3.80	3.81	ψ	z

1. TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ	z	3.81	3.82	3.83	3.84	3.85	3.86	3.87	3.88	3.89	3.90	3.91	z	ψ
0°	0°	0.250 638	0.250 107	0.249 577	0.249 048	0.248 520	0.247 993	0.247 467	0.246 943	0.246 419	0.245 897	0.245 376	0°	0°
5	5	0.250 703	0.250 171	0.249 639	0.249 109	0.248 580	0.248 052	0.247 526	0.247 000	0.246 476	0.245 952	0.245 430	5	5
10	10	0.250 893	0.250 358	0.249 823	0.249 290	0.248 758	0.248 227	0.247 697	0.247 169	0.246 641	0.246 115	0.245 590	10	10
15	15	0.251 198	0.250 659	0.250 118	0.249 580	0.249 043	0.248 507	0.247 972	0.247 439	0.246 907	0.246 376	0.245 847	15	15
20	20	0.251 600	0.251 053	0.250 506	0.249 961	0.249 418	0.248 876	0.248 335	0.247 795	0.247 258	0.246 721	0.246 186	20	20
25	25	0.252 079	0.251 523	0.250 968	0.250 415	0.249 864	0.249 315	0.248 767	0.248 220	0.247 675	0.247 132	0.246 590	25	25
30	30	0.252 610	0.252 046	0.251 483	0.250 922	0.250 362	0.249 804	0.249 248	0.248 694	0.248 142	0.247 591	0.247 042	30	30
35	35	0.253 174	0.252 600	0.252 028	0.251 458	0.250 890	0.250 324	0.249 761	0.249 198	0.248 638	0.248 078	0.247 522	35	35
40	40	0.253 751	0.253 168	0.252 587	0.252 009	0.251 432	0.250 857	0.250 285	0.249 714	0.249 146	0.248 579	0.248 015	40	40
45	45	0.254 324	0.253 732	0.253 143	0.252 556	0.251 971	0.251 388	0.250 808	0.250 229	0.249 653	0.249 079	0.248 507	45	45
50	50	0.254 881	0.254 281	0.253 684	0.253 089	0.252 496	0.251 905	0.251 317	0.250 731	0.250 147	0.249 566	0.248 987	50	50
55	55	0.255 413	0.254 806	0.254 200	0.253 598	0.252 998	0.252 400	0.251 804	0.251 212	0.250 621	0.250 033	0.249 447	55	55
60	60	0.255 914	0.255 299	0.254 687	0.254 077	0.253 470	0.252 866	0.252 264	0.251 665	0.251 068	0.250 474	0.249 882	60	60
65	65	0.256 379	0.255 759	0.255 140	0.254 524	0.253 911	0.253 301	0.252 692	0.252 087	0.251 485	0.250 885	0.250 288	65	65
70	70	0.256 809	0.256 182	0.255 558	0.254 936	0.254 317	0.253 702	0.253 088	0.252 478	0.251 870	0.251 265	0.250 663	70	70
75	75	0.257 202	0.256 569	0.255 940	0.255 314	0.254 690	0.254 069	0.253 451	0.252 836	0.252 224	0.251 614	0.251 007	75	75
80	80	0.257 558	0.256 922	0.256 288	0.255 657	0.255 029	0.254 404	0.253 782	0.253 162	0.252 546	0.251 932	0.251 321	80	80
85	85	0.257 882	0.257 241	0.256 603	0.255 968	0.255 336	0.254 707	0.254 081	0.253 458	0.252 838	0.252 221	0.251 606	85	85
90	90	0.258 173	0.257 528	0.256 887	0.256 249	0.255 613	0.254 981	0.254 352	0.253 725	0.253 102	0.252 481	0.251 864	90	90
95	95	0.258 434	0.257 787	0.257 142	0.256 501	0.255 863	0.255 227	0.254 595	0.253 966	0.253 340	0.252 716	0.252 096	95	95
100	100	0.258 669	0.258 019	0.257 371	0.256 727	0.256 086	0.255 448	0.254 813	0.254 182	0.253 553	0.252 927	0.252 304	100	100
105	105	0.258 877	0.258 225	0.257 577	0.256 930	0.256 286	0.255 645	0.255 009	0.254 374	0.253 744	0.253 116	0.252 491	105	105
110	110	0.259 065	0.258 410	0.257 758	0.257 110	0.256 464	0.255 822	0.255 183	0.254 547	0.253 914	0.253 284	0.252 657	110	110
115	115	0.259 230	0.258 574	0.257 920	0.257 270	0.256 622	0.255 979	0.255 337	0.254 700	0.254 065	0.253 434	0.252 805	115	115
120	120	0.259 377	0.258 719	0.258 063	0.257 412	0.256 763	0.256 118	0.255 475	0.254 836	0.254 200	0.253 567	0.252 936	120	120
125	125	0.259 507	0.258 848	0.258 191	0.257 538	0.256 887	0.256 241	0.255 596	0.254 956	0.254 318	0.253 684	0.253 052	125	125
130	130	0.259 621	0.258 960	0.258 302	0.257 648	0.256 996	0.256 348	0.255 703	0.255 062	0.254 423	0.253 788	0.253 155	130	130
135	135	0.259 721	0.259 059	0.258 399	0.257 744	0.257 091	0.256 443	0.255 796	0.255 154	0.254 514	0.253 879	0.253 245	135	135
140	140	0.259 807	0.259 144	0.258 484	0.257 828	0.257 174	0.256 525	0.255 877	0.255 234	0.254 593	0.253 957	0.253 322	140	140
145	145	0.259 880	0.259 217	0.258 556	0.257 899	0.257 245	0.256 595	0.255 947	0.255 303	0.254 661	0.254 024	0.253 388	145	145
150	150	0.259 943	0.259 279	0.258 617	0.257 959	0.257 304	0.256 654	0.256 005	0.255 361	0.254 719	0.254 081	0.253 445	150	150
155	155	0.259 995	0.259 330	0.258 668	0.258 010	0.257 354	0.256 703	0.256 054	0.255 409	0.254 766	0.254 128	0.253 491	155	155
160	160	0.260 036	0.259 371	0.258 708	0.258 050	0.257 394	0.256 742	0.256 092	0.255 447	0.254 804	0.254 165	0.253 528	160	160
165	165	0.260 068	0.259 403	0.258 740	0.258 081	0.257 424	0.256 772	0.256 122	0.255 476	0.254 833	0.254 194	0.253 557	165	165
170	170	0.260 090	0.259 424	0.258 761	0.258 102	0.257 445	0.256 793	0.256 143	0.255 498	0.254 854	0.254 215	0.253 577	170	170
175	175	0.260 104	0.259 438	0.258 775	0.258 116	0.257 459	0.256 807	0.256 157	0.255 511	0.254 867	0.254 227	0.253 589	175	175
180	180	0.260 108	0.259 442	0.258 779	0.258 120	0.257 463	0.256 810	0.256 160	0.255 514	0.254 870	0.254 231	0.253 593	180	180
ψ	z	3.81	3.82	3.83	3.84	3.85	3.86	3.87	3.88	3.89	3.90	3.91	z	ψ

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ	z	3.91	3.92	3.93	3.94	3.95	3.96	3.97	3.98	3.99	4.00	z
0°	5	0.245 376	0.244 856	0.244 337	0.243 819	0.243 303	0.242 788	0.242 274	0.241 762	0.241 251	0.240 741	0°
10	15	0.245 430	0.244 909	0.244 390	0.243 871	0.243 354	0.242 838	0.242 324	0.241 810	0.241 298	0.240 788	5
20	25	0.245 590	0.245 067	0.244 544	0.244 023	0.243 504	0.242 985	0.242 468	0.241 953	0.241 438	0.240 925	10
30	35	0.245 847	0.245 319	0.244 792	0.244 267	0.243 744	0.243 220	0.242 701	0.242 179	0.241 663	0.241 148	15
40	45	0.246 186	0.245 653	0.245 121	0.244 590	0.244 061	0.243 533	0.243 007	0.242 483	0.241 960	0.241 438	20
50	55	0.246 590	0.246 050	0.245 512	0.244 975	0.244 440	0.243 906	0.243 374	0.242 844	0.242 315	0.241 788	25
60	65	0.247 042	0.246 494	0.245 949	0.245 405	0.244 863	0.244 322	0.243 784	0.243 247	0.242 712	0.242 179	30
70	75	0.247 522	0.246 966	0.246 413	0.245 864	0.245 313	0.244 767	0.244 221	0.243 676	0.243 136	0.242 596	35
80	85	0.248 015	0.247 452	0.246 892	0.246 334	0.245 777	0.245 223	0.244 671	0.244 120	0.243 572	0.243 026	40
90	95	0.248 507	0.247 937	0.247 369	0.246 804	0.246 241	0.245 680	0.245 121	0.244 564	0.244 009	0.243 456	45
100	105	0.248 987	0.248 410	0.247 836	0.247 264	0.246 694	0.246 126	0.245 560	0.244 997	0.244 436	0.243 877	50
110	115	0.249 447	0.248 864	0.248 283	0.247 704	0.247 128	0.246 554	0.245 983	0.245 414	0.244 847	0.244 282	55
120	125	0.249 882	0.249 292	0.248 706	0.248 121	0.247 539	0.246 960	0.246 383	0.245 808	0.245 236	0.244 666	60
130	135	0.250 288	0.249 693	0.249 100	0.248 511	0.247 924	0.247 339	0.246 757	0.246 178	0.245 601	0.245 025	65
140	145	0.250 663	0.250 063	0.249 466	0.248 871	0.248 280	0.247 690	0.247 104	0.246 520	0.245 938	0.245 359	70
150	155	0.251 007	0.250 403	0.249 802	0.249 202	0.248 607	0.248 013	0.247 423	0.246 835	0.246 249	0.245 666	75
160	165	0.251 321	0.250 713	0.250 108	0.249 505	0.248 905	0.248 308	0.247 714	0.247 122	0.246 533	0.245 947	80
170	175	0.251 606	0.250 994	0.250 386	0.249 780	0.249 177	0.248 576	0.247 979	0.247 384	0.246 792	0.246 202	85
180	185	0.251 864	0.251 249	0.250 637	0.250 028	0.249 422	0.248 819	0.248 218	0.247 621	0.247 026	0.246 433	90
190	195	0.252 096	0.251 478	0.250 864	0.250 252	0.249 644	0.249 038	0.248 434	0.247 834	0.247 237	0.246 642	95
200	205	0.252 304	0.251 684	0.251 067	0.250 453	0.249 842	0.249 234	0.248 629	0.248 026	0.247 427	0.246 830	100
210	215	0.252 491	0.251 869	0.251 250	0.250 634	0.250 020	0.249 410	0.248 803	0.248 199	0.247 598	0.246 999	105
220	225	0.252 657	0.252 033	0.251 412	0.250 795	0.250 180	0.249 567	0.248 958	0.248 352	0.247 749	0.247 152	110
230	235	0.252 805	0.252 180	0.251 557	0.250 938	0.250 321	0.249 708	0.249 097	0.248 491	0.247 886	0.247 285	115
240	245	0.252 936	0.252 310	0.251 685	0.251 065	0.250 446	0.249 832	0.249 219	0.248 611	0.248 004	0.247 402	120
250	255	0.253 052	0.252 425	0.251 799	0.251 178	0.250 558	0.249 943	0.249 329	0.248 720	0.248 112	0.247 508	125
260	265	0.253 155	0.252 526	0.251 900	0.251 277	0.250 656	0.250 040	0.249 425	0.248 814	0.248 205	0.247 600	130
270	275	0.253 245	0.252 615	0.251 988	0.251 365	0.250 743	0.250 126	0.249 510	0.248 899	0.248 288	0.247 682	135
280	285	0.253 322	0.252 691	0.252 063	0.251 439	0.250 816	0.250 198	0.249 581	0.248 969	0.248 358	0.247 752	140
290	295	0.253 388	0.252 757	0.252 128	0.251 503	0.250 880	0.250 261	0.249 644	0.249 031	0.248 420	0.247 813	145
300	305	0.253 445	0.252 813	0.252 183	0.251 557	0.250 933	0.250 314	0.249 696	0.249 083	0.248 471	0.247 864	150
310	315	0.253 491	0.252 859	0.252 229	0.251 603	0.250 978	0.250 358	0.249 740	0.249 127	0.248 515	0.247 907	155
320	325	0.253 528	0.252 896	0.252 265	0.251 639	0.251 014	0.250 394	0.249 775	0.249 161	0.248 549	0.247 941	160
330	335	0.253 557	0.252 924	0.252 293	0.251 667	0.251 042	0.250 421	0.249 802	0.249 188	0.248 575	0.247 967	165
340	345	0.253 577	0.252 944	0.252 313	0.251 686	0.251 061	0.250 441	0.249 822	0.249 207	0.248 594	0.247 985	170
350	355	0.253 589	0.252 956	0.252 325	0.251 698	0.251 073	0.250 452	0.249 833	0.249 218	0.248 605	0.247 996	175
360	365	0.253 593	0.252 960	0.252 329	0.251 702	0.251 077	0.250 456	0.249 837	0.249 222	0.248 609	0.248 000	180
ψ	z	3.91	3.92	3.93	3.94	3.95	3.96	3.97	3.98	3.99	4.00	z

APPLICATIONS OF LEUSCHNER'S SHORT METHODS OF DETERMINING ORBITS.

~~1888. 1888. 1888~~

By RUSSELL TRACY CRAWFORD,
ASSOCIATE PROFESSOR OF PRACTICAL ASTRONOMY,
BERKELEY ASTRONOMICAL DEPARTMENT.

EXAMPLES A. DIRECT SOLUTIONS—*Continued.*

	Page.
8. Example showing change from Parabola to General Orbit during a Differential Correction. Computations by R. T. CRAWFORD and A. J. CHAMPREUX - - - - -	422
9. Differential Correction of a Direct Solution of the Orbit of a Minor Planet using the Series for f , g , ∂f , and ∂g . Computation by R. K. YOUNG - - - - -	425
10. Differential Correction of a Direct Solution of a Comet Orbit using the Series for f , g , ∂f , and ∂g . Computation by R. T. CRAWFORD - - - - -	428

EXAMPLES B. ORBITS BASED UPON PREVIOUS APPROXIMATIONS USING CLOSED EXPRESSIONS FOR f , g , ∂f , AND ∂g .

11. Case of an Initial Parabolic Orbit. Computation by W. F. MEYER and MISS S. H. LEVY - - - - -	431
12. Case of an Initial Elliptic Orbit of a Comet or a Minor Planet. Computation by S. EINARSSON and R. K. YOUNG - - - - -	440
13. Case of an Initial Nearly Parabolic Orbit. Computation by R. T. CRAWFORD and W. F. MEYER - - - - -	448

INTRODUCTION.

The purpose of this Part is to show in complete detail the computation necessary for determining orbits of all kinds. Among other things there will be shown (1) when parallax should be taken into consideration and how, in such cases, this is accomplished; (2) how to determine, in the course of the computation, whether a parabola is in the range of possible solutions; (3) the ease and rapidity with which transference may be made from a parabola to a general orbit; (4) the small amount of labor involved in making successive approximations in order to remove residuals completely; (5) how arbitrary methods of "cooking" to shorten the computations may sometimes be employed, due principally to the perfect perspicuity of LEUSCHNER'S methods, which enables the computer to see the meaning of every step in the computation, nothing being hidden in abstruse analytical developments.

In order to study the efficiency of LEUSCHNER'S methods as compared with others, all the available orbits of comets have been collected and arranged in the order of the length of arc from which they were computed. In most of the cases the quantities tabulated are ΔT , $\Delta \omega$, etc., these being the differences between the elements of the particular orbit and the elements from the longest arc (considered to be the best). In a few cases these are not given, but the elements themselves are tabulated for reasons stated under each case.

The orbits taken are all those computed here beginning with that of Comet *a*, 1905 (GIACOBINI), as this is the first case of an orbit computed here by the *Short Method* in its present form as set forth in Part 7.

The data for orbits computed in the Berkeley Astronomical Department are given in bolder type.

COMET *a* 1905 (GIACOBINI).

The orbit of reference in this case is that of WEDEMEYER who used a 33 day arc. His elements are given in *A. N.* 168 243.

No.	ΔT	$\Delta \omega$	$\Delta \Omega$	Δi	$\Delta \log q$	Arc.	Computer.	Reference.
I	+ 0.49	+ 1° 0'	+ 3° 15'	- 2° 44'	- 0.0137	2 days	MORGAN LAMSON	<i>A. N.</i> 168 14
II	+ 0.89	+ 1 4	+ 1 15	- 1 23	- 0.0050	4 days	STRÖMGREN	<i>A. N.</i> 168 14
III	- 0.07	- 0 5	+ 0 16	- 0 10	- 0.0010	4 days	GIACOBINI	<i>A. N.</i> 168 27
IV	+ 0.33	+ 0 24	+ 0 38	- 0 37	- 0.0025	4 days	Crawford Maddrell	<i>L. O. B.</i> 73
V	+ 0.30	+ 0 20	+ 0 32	- 0 28	- 0.0021	5 days	MAUBANT	<i>A. N.</i> 168 27
VI	+ 0.01	+ 0 1	+ 0 16	- 0 9	- 0.0011	6 days	LAMSON	<i>A. N.</i> 168 63
VII	+ 0.09	+ 0 6	+ 0 21	- 0 16	- 0.0014	14 days	EBELL <i>et al.</i>	<i>A. N.</i> 168 46

There are two other sets of elements from longer arcs (27 days and 31 days) which are omitted.

There are three orbits from 4 day arcs. Ours (IV) is considerably better than No. II, but not so good as No. III. The reason for the latter is to be found in the fact that GIACOBINI had equal intervals (each two days) while we had intervals of one day and three days, respectively. Our orbit, however, is practically the same as No. V in which a 5 day arc was used. Comparing Nos. IV and V we have more nearly similar conditions (unequal intervals), and it is seen that our orbit is of the same order of accuracy as that in which an arc of one more day is used.

COMET *b* 1905 (SCHAER).

No.	T (B. M. T.)	ω	Ω	i	$\log q$	φ	Arc.	Computer.	Reference.
I	1905 Oct. 18.93	120° 6'	219° 50'	145° 32'	9.9997		2 days	MORGAN	<i>A. N.</i> 169 413
II	" 27.49	135 39	223 45	138 55	0.0263		2 days	EBELL	<i>A. N.</i> 169 415
III	" 25.71	132 35	222 55	140 37	0.0219		5 days	EBELL	<i>A. N.</i> 170 13
IV	" 25.77	132 40	222 56	140 36	0.0220		5 days	LAMSON	<i>A. N.</i> 170 112
V	" 25.06	132 7	223 4	140 27	0.0209	74° 41'	7 days	Crawford Champreux	<i>L. O. B.</i> 86
VI	" 25.80	132 43	222 56	140 35	0.0221		25 days	WEDEMEYER	<i>A. N.</i> 170 97

In this case we had no parabola. For this reason the differences in the elements are not given. The elements themselves are exhibited, however, in order that it may be noted that the longer the arc used the nearer are the various parabolic elements approaching our elliptical elements.

Orbit No. VI is based upon observations of November 18, December 1, and December 13, and does not represent two observations within this period (November 25 and November 28). The residual for the middle place is 8".

Assuming that the orbit is ultimately a parabola it would seem as though the parabolas of EBELL and LAMSON from 5 day arcs were better than our orbit from a

7 day arc. Comparison with the last available observation, however, does not bear this out. Such an observation by E. SMITH at the Lick Observatory on December 29 is represented practically exactly by the elements of WEDEMEYER. Our orbit, however, is closer to this position than the parabola No. III of EBELL. The residuals being:

$$O - C \begin{cases} \Delta \alpha & \text{CR. and CH.} & \text{EBELL.} \\ & + 4'' & - 3'' \\ \Delta \delta & + 0'.3 & - 2'.5 \end{cases}$$

COMET ϵ 1905 (GIACOBINI).

The orbit of reference in this case is one by SCHOENBERG and BUSS. They used a 38 day arc. Their elements have been taken from the *Astronomischer Jahresbericht* for the year 1906.

No.	ΔT	$\Delta \omega$	$\Delta \Omega$	Δi	$\Delta \log q$	Arc.	Computer.	Reference.
I	+6.16	-14° 44'.4	- 1° 15'.3	- 0° 43'.1	+ 0.3670	2 days	MORGAN	<i>A. N.</i> 170 27
II	-9.26	+27 48.0	+ 2 23.1	+ 0 55.3	- 0.3930	2 days	GIACOBINI	<i>A. N.</i> 170 68
III	-3.50	+ 8 43.4	+ 0 56.1	+ 0 22.5	- 0.1530	3 days	THIELE	<i>A. N.</i> 170 82
IV	-2.36	+ 5 46.2	+ 0 29.3	+ 0 13.4	- 0.1040	3 days	Crawford Champreux	<i>L. O. B.</i> 87
V	+4.15	- 9 59.0	- 0 59.7	- 0 29.4	+ 0.2285	4 days	STRÖMGREN	<i>A. N.</i> 170 67
VI	+1.12	- 2 41.8	- 0 2.0	- 0 3.7	+ 0.0526	5 days	MAUBANT	<i>A. N.</i> 170 84
VII	-0.27	+ 0 50.0	+ 0 10.0	+ 0 2.5	- 0.0155	10 days	STRÖMGREN	<i>A. N.</i> 170 100
VIII	-0.06	+ 0 10.2	+ 0 4.1	+ 0 1.0	- 0.0034	11 days	Crawford	<i>L. O. B.</i> 88

Our orbit (IV) from a 3 day arc is considerably better than that of THIELE (III) from a like arc which in turn is better than that of STRÖMGREN (V) from a 4 day arc. Our orbit from an 11 day arc (VIII) is decidedly better than that of STRÖMGREN (VII) from a 10 day arc.

COMET α 1906 (BROOKS).

The orbit of reference in this case is that of EBELL from a 14 day arc. His elements are given in *A. N.* 170 323.

No.	ΔT	$\Delta \omega$	$\Delta \Omega$	Δi	$\Delta \log q$	Arc.	Computer.	Reference.
I	+2.71	+ 3° 21'	+ 0° 55'	- 0° 21'	+ 0.0043	2 days	Crawford Champreux	<i>L. O. B.</i> 90
II	-9.38	-11 14	- 3 15	+ 1 28	- 0.0117	2 days	MAUBANT	<i>A. N.</i> 170 308
III	-2.31	- 2 53	+ 0 54	- 0 23	- 0.0032	2 days	LAMSON	<i>Pop. Ast.</i> 14 172
IV	+2.20	+ 2 43	+ 0 43	- 0 18	+ 0.0040	4 days	EBELL	<i>A. N.</i> 170 276
V	+0.20	+ 0 14	+ 0 3	- 0 1	+ 0.0004	5 days	LAMSON	<i>A. J.</i> 25 60
VI	+0.20	+ 0 15	+ 0 4	+ 0 1	+ 0.0003	6 days	MORGAN	<i>Pop. Ast.</i> 14 170
VII	-0.11	- 0 8	- 0 2.3	+ 0 1.6	- 0.00029	10 days	Crawford Champreux	<i>L. O. B.</i> 91
VIII	-0.14	- 0 11	- 0 3.2	+ 0 1.8	- 0.00035	12 days	MORGAN	<i>A. J.</i> 25 75

Our orbit (I), given above as from a 2 day arc was computed from observations actually covering *only 1.76 days*. It is considerably better than that of MAUBANT (II), and of the same order but not quite so good as that of LAMSON (III), both of which are based upon a 2 day arc. It is further of practically the same order as that of EBELL (IV), who used an arc of four days. Our orbit (VII) from a 10 day arc is a little better than that of MORGAN (VIII) from a 12 day arc.

COMET *b* 1906 (KOPFF).

The orbit of reference is that of EBELL from a 51 day arc. His elements are given in *A. N.* 171 III.

No.	ΔT	$\Delta \omega$	$\Delta \Omega$	Δi	$\Delta \log q$	Arc.	Computer.	Reference.
I	+67.5	-15°.0	-12°.8	-3° 6'	-0.404	2 days	Crawford Champreux	<i>L. O. B.</i> 92
II	+77.5	-20.3	-13.8	-3 21	-0.487	3 days	EBELL	<i>A. N.</i> 170 376
III	+17.8	+1.9	-1.2	-0 33	-0.031	14 days	EBELL	<i>A. N.</i> 170 389
IV	+2.2	+0.4	-0.06	-0 2	-0.0015	22 days	Crawford Champreux	<i>L. O. B.</i> 97

This comet had a very slow motion, and, as noted in *L. O. B.* 97 the computations indicated a wide range of possible solutions. Our orbit (I) from a 2 day arc is closer than that of EBELL (II) from a 3 day arc.

COMET *e* 1906 (KOPFF).

No.	T (B. M. T.)	ω	Ω	i	$\log q$	φ	Arc.	Computer.	Reference.
I	1907 Apr. 12.3	221° 38'	230° 2'	12° 44'	0.0485		2 days	Crawford	<i>L. O. B.</i> 99
II	1906 Dec. 7.3	243 13	179 19	15 18	9.9143		2 days	MORGAN	<i>A. N.</i> 172 224
III	1906 May 14.69	353 32	279 28	9 46	0.1457		4 days	EBELL	<i>A. N.</i> 172 223
IV	1905 Oct. 16.05	292 25	297 33	17 28	0.0842		4 days	EBELL	<i>A. N.</i> 172 223
V	1906 May 3.09	19 33.7	263 48.7	8 42.1	0.22875	31° 18'.2	20 days	EBELL	<i>A. N.</i> 172 303
VI	1906 May 2.125	19 28.7	263 45.4	8 44.2	0.23011	31 21.6	22 days	Crawford Champreux	<i>L. O. B.</i> 100

This comet presents a peculiar case. As indicated in *L. O. B.* 99 a wide range of solutions was expected. The orbit was ultimately shown to be a short period ($6\frac{2}{3}$ years) ellipse, announcement of which was first made in *L. O. B.* 100. The geocentric motion at the time of discovery was very slow. With the exception of T and ω in which there is a great mutual uncertainty our elements (I) from a 2 day arc are closer than those of MORGAN (II) from a like arc.

Elements III and IV by EBELL are two solutions from the same observations.

COMET α 1907 (GIACOBINI).

The orbit of reference is that of TRINGALI who used a 31 day arc. His elements are given in *A. N.* 176 73.

No.	ΔT	$\Delta \omega$	$\Delta \Omega$	Δi	$\Delta \log q$	Arc.	Computer.	Reference.
I	- 4.35	- 2° 28'	- 0° 30'	+ 0° 19'	+ 0.0003	2 days	EBELL	<i>A. N.</i> 174 143
II	+ 10.81	+ 5 55	+ 1 16	- 0 48	+ 0.0008	2 days	LAMSON	<i>Pop. Ast.</i> 15 248
III	+ 6.52	+ 3 38	+ 0 45	- 0 29	+ 0.0001	3 days	DUNCAN WILLIAMS	<i>A. N.</i> 174 159
IV	+ 6.49	+ 3 48	+ 0 53	- 0 34	+ 0.0002	3 days	Einarsson Glancy	<i>L. O. B.</i> 111
V	+ 3.48	+ 1 57	+ 0 21	- 0 14	0.0000	4 days	GIACOBINI	<i>A. N.</i> 174 173
VI	- 2.82	- 1 36	- 0 19	+ 0 14	0.0000	5 days	BRÜCK	<i>C. R.</i> 144 613
VII	- 6.89	- 3 51	- 0 44	+ 0 32	0.0000	7 days	EBELL	<i>A. N.</i> 174 207
VIII	+ 1.47	+ 0 50	+ 0 10	+ 0 7	+ 0.0001	8 days	LAMSON Einarsson	<i>A. N.</i> 174 335
IX	- 0.10	- 0 4	- 0 1	0 0	0.0000	28 days	Glancy Joy	<i>L. O. B.</i> 113

The data here present a peculiar state of affairs. EBELL's orbit (I) from a 2 day arc is very good and considerably better than his later orbit (VII) from a 7 day arc. Our orbit (IV) from a 3 day arc is practically the same as that of DUNCAN and WILLIAMS (III) from a similar arc. It may be remarked that our orbit could have been improved by removing a residual of 0'.2. As a second orbit was contemplated it was not considered advisable at the time to make the improvement to the preliminary orbit.

COMET α 1907 (DANIEL).

The orbit of reference is that of KRITZINGER from a 74 day arc. His elements are given in *A. N.* 176 15.

No.	ΔT	$\Delta \omega$	$\Delta \Omega$	Δi	$\Delta \log q$	Arc.	Computer.	Reference.
I	+ 1.97	+ 52° 23'	- 0° 39'	+ 2° 43'	- 0.4047	3 days	STRÖMGREN	<i>A. N.</i> 175 155
II	+ 0.18	+ 0 25	0 0	- 0 11	+ 0.0052	6 days	Crawford Einarsson Glancy	<i>L. O. B.</i> 119
III	- 5.43	+ 14 59	- 0 22	+ 1 4	- 0.1574	8 days	STRÖMGREN	<i>A. N.</i> 175 191
IV	- 0.29	+ 0 32	0 0	+ 0 2	- 0.0062	19 days	KRITZINGER	<i>A. N.</i> 175 259
V	- 0.04	+ 0 7	+ 0 2	+ 0 1	- 0.0016	27 days	DYBECK	<i>A. N.</i> 175 307

Our orbit (II) from a 6 day arc is considerably better than that of STRÖMGREN (III) from an 8 day arc and slightly better than that of KRITZINGER (IV) from a 19 day arc. No improvement is found until DYBECK's orbit (V) from a 27 day arc. Another orbit, not given here, by MILLOSEVICH from a 67 day arc is practically the same as the orbit of reference.

COMET ϵ 1907 (MELLISH).

The orbit of reference is one by KOBOLD who used a 42 day arc. His elements are given in *A. N.* 177 240.

No.	ΔT	$\Delta \omega$	$\Delta \Omega$	Δi	$\Delta \log q$	Arc.	Computer.	Reference.
I	+ 1.89	+ 2° 40'	- 0° 57'	+ 0° 45'	+ 0.0044	2 days	LAMSON	<i>A. N.</i> 176 127
II	- 1.77	- 2 24	+ 0 59	- 0 34	- 0.0020	2 days	Crawford	<i>L. O. B.</i> 121
III	- 2.28	- 3 22	+ 0 38	- 0 17	- 0.0058	2 days	Glancy	<i>Pop. Ast.</i> 15 570
IV	+ 0.65	+ 0 55	- 0 19	+ 0 14	+ 0.0013	3 days	Morgan	
V	- 0.52	- 0 44	+ 0 13	- 0 9	- 0.0011	6 days	WILSON	<i>A. N.</i> 176 147
VI	- 0.107	- 0 10.0	+ 0 1.5	- 0 3.1	- 0.00051	15 days	EBELL	<i>Pop. Ast.</i> 15 632
VII	- 0.009	- 0 0.6	0 0.0	+ 0 0.3	- 0.00001	18 days	Crawford	<i>L. O. B.</i> 124
							Glancy	
							EBELL	<i>A. N.</i> 176 195

Of the three orbits from 2 day arcs ours (II) is better than those of LAMSON (I) and WILSON (III). EBELL'S (IV) from a 3 day arc is a decided improvement. Our orbit (VI) from a 15 day arc is not so close as that of EBELL (VII) from an 18 day arc.

COMET ϵ 1908 (MOREHOUSE).

The orbit of reference is one by KOBOLD who used an 83 day arc. His elements are given in *A. N.* 179 273.

No.	ΔT	$\Delta \omega$	$\Delta \Omega$	Δi	$\Delta \log q$	Arc.	Computer.	Reference.
I	+ 1.51	- 2° 53'	- 1° 53'	- 0° 26'	+ 0.0112	2 days	KOBOLD	<i>A. N.</i> 179 15
II	- 10.92	+ 19 34	+ 12 49	+ 4 15	- 0.0921	2 days	Einarsson	<i>L. O. B.</i> 138
III	- 0.023	+ 0 6.7	+ 0 5.1	+ 0 0.6	- 0.00036	15 days	Meyer	<i>L. O. B.</i> 139
IV	- 0.015	+ 0 14.8	+ 0 8.1	+ 0 0.8	- 0.00103	15 days	Einarsson	
V	+ 0.011	- 0 1.9	- 0 2.1	- 0 0.2	+ 0.00014	21 days	Meyer	<i>Mem. s. a. Mex.</i> 25 292
VI	- 0.078	- 0 15.2	- 0 6.6	+ 0 1.0	+ 0.00086	26 days	CHACÓN	<i>A. N.</i> 179 47
							KOBOLD	
							GABBA	<i>Lomb. Ist. Rend.</i> (2) 42

Our preliminary orbit (II) from a 2 day arc is decidedly in error. After it was issued an error was discovered in one of the observations upon which it was based. Our orbit (III) from a 15 day arc is better than that of CHACÓN (IV) from a like arc, and also better than that of GABBA (VI) from a 26 day arc. It is not so good as KOBOLD'S (V) from a 21 day arc.

COMET α 1909 (DANIEL).

No.	T (B. M. T.)	ω	Ω	i	$\log q$	Arc.	Computer.	Reference.
I	1909 June 5.35	5° 4'	305° 21'	51° 32'	9.9252	2 days	KOBOLD	<i>A. N.</i> 181 211
II	1909 June 5.21	4 59	306 19	52 26	9.9275	5 days	Crawford	<i>L. O. B.</i> 159
III	1909 June 5.16	4 56	306 40	52 39	9.9281	5 days	BOSS	<i>A. N.</i> 181 363
IV	1909 June 5.30	5 1	305 38	52 4	9.9258	8 days	KOBOLD	<i>A. N.</i> 181 283

In this case the elements themselves are given as there was no orbit from an arc long enough to be a standard for reference. Of these four the one from the longest arc (8 days) is that of KOBOLD (IV). His first orbit (I) is slightly nearer his second orbit than is our orbit (II) from a 5 day arc; but it should be noted that as KOBOLD'S arc is lengthened his elements *proceed in the direction from his first set toward our elements*. An orbit from a still longer arc might then well come still further toward the elements (II), so that, in this case, the data are rather insufficient from which to make a comparison or draw a conclusion. Our orbit could have been improved by the removal of some rather large residuals, but they were left with the idea that a second orbit would be computed. Owing to pressure of time this second orbit was not determined.

COMET *e* 1909 (DANIEL).

No.	T (B. M. T.)	ω	Ω	i	$\log q$	e	Arc.	Computer.	Reference.
I	1909 Dec. 5.60	8° 16'	73° 33'	26° 57'	0.1967	1.0000	2 days	EBELL	<i>A. N.</i> 183 127
II	1909 Dec. 1.78	5 31	73 31	26 50	0.1944	1.0000	3 days	RUSSELL	<i>A. J.</i> 26 46
III	1909 Nov. 27.67	2 31	73 27	26 36	0.1911	1.0000	5 days	EBELL	<i>A. N.</i> 183 175
IV	1909 Nov. 29.80	10 52	68 58	19 44	0.1614	1.0000	7 days	KRASSOWSKI	<i>A. N.</i> 183 191
V	1909 Nov. 28.23	3 2	71 10	19 55	0.1433	0.6252	7 days	{ Einarsson Young	<i>L. O. B.</i> 179
VI	1909 Nov. 28.91	3 37	70 58	19 23	0.1402	0.5995	37 days	EBELL	<i>A. N.</i> 183 263
VII	1909 Nov. 28.76	3 29	71 0	19 27	0.1404	0.6025	86 days	{ Einarsson Young	<i>L. O. B.</i> 179

The elements themselves are given in this case because the orbit is a short period ellipse. The periods of the orbits (V), (VI) and (VII) are 7.^s15, 6.^s40 and 6.^s48 respectively. Our ellipse (V) is sufficiently close to establish the character of the orbit from a 7 day arc.

COMET *b* 1910 (METCALF).

The orbit of reference for this comet is the one by KOBOLD, which is based on an arc extending from August 11 to August 25 and then corrected by means of an observation taken September 18th. These elements are given in *A. N.* 186 80.

No.	ΔT	$\Delta \omega$	$\Delta \Omega$	Δi	$\Delta \log q$	Arc.	Computer.	Reference.
I	- 5.73	- 2° 55'	+ 0° 15'	+ 0° 11'	+ 0.0019	2 days	Young	<i>Pop. Ast.</i> 18 9
II	+ 66.25	+ 72 9	- 17 4	+ 1 35	+ 0.2687	2 days	KOBOLD	<i>A. N.</i> 185 296
III	+ 17.31	+ 8 46	- 0 48	- 0 23	+ 0.0012	4 days	KOBOLD	<i>A. N.</i> 185 308
IV	+ 21.28	+ 10 55	- 1 10	- 0 22	+ 0.0069	14 days	KOBOLD	<i>A. N.</i> 185 408

Our orbit (I) from a 2 day arc is very much closer to the orbit of reference than any of KOBOLD'S orbits which were computed from arcs of 2, 4, and 14

days, respectively. It should be noted that KOBOLD attained an improvement over our orbit from a 2 day arc only by extending his arc to 38 days. As the arc is lengthened by KOBOLD his orbits (II), (III), and that of reference successively approach ours from a 2 day arc. Orbit (IV), however, may be affected by some error, probably of observation.

COMET c 1910 (CERULLI-FAYE).

No.	T (B. M. T.)	ω	Ω	i	$\log q$	e	Arc.	Computer.	Reference.
I	1910 Sept. 15.06	168° 33'	211° 39'	15° 36'	0.0512	1.0000	2 days	EBELL	<i>A. N.</i> 186 209
II	1910 Nov. 19.52	200 17	212 58	18 17	0.3416	1.0000	4 days	Meyer Levy	<i>L. O. B.</i> 186
III	1910 Nov. 12.45	206 20	205 29	10 14	0.2175	0.5459	4 days		
IV	1910 Nov. 2.31	199 51	206 7	10 30	0.2178	0.5608	12 days	FAYET	<i>A. N.</i> 186 351
V	1910 Oct. 30.08	197 55	205 39	9 48	0.2009	0.5169	14 days	PRAGER RISTANPART	<i>A. N.</i> 186 370
VI	1910 Nov. 1.50	199 17	206 14	10 36	0.2189	0.5656	33 days		
								Meyer Levy	<i>L. O. B.</i> 187

The elements themselves are given in this case because the orbit is a short period ellipse. The periods of the orbits (III), (IV), (V) and (VI) are 6.^s93, 7.^s29, 5.^s96 and 7.^s44 respectively. Our orbit (III) from a 4 day arc is better than (V) from a 14 day arc.

It was from orbit (II) that Professor LEUSCHNER announced the probable identity of the comet discovered by CERULLI with FAYE's comet. The elements (III) from the same observations made the identification certain.

In every case of orbits computed here the geocentric parallax for the middle observation has been eliminated.

In the first three cases (the comets for 1905) differential corrections have been used to improve the first solutions. In all other cases for short arcs the orbits represent a direct solution (no differential correction) for first orbits.

CONCLUSION.

It is to be noted from these data that, in general, orbits computed by LEUSCHNER'S Methods are better than other orbits computed from the same arc, and frequently better even than orbits from longer arcs, although in no case of unequal intervals was the epoch chosen at the mean date as might have been done, with little additional labor, for the purpose of still further increasing the accuracy of the direct solution.

In view of this, and also of the unquestionable brevity of the computations as compared with other methods, the value of LEUSCHNER'S Methods, as established theoretically by him in Part 7, is fully substantiated by actual experience.

EXAMPLES A. DIRECT SOLUTIONS.

NOTE.—In the computations given in the examples the formulæ referred to by numbers are those of the *Synopsis of Formulæ* in Part 7. In the discussions of the examples the formulæ and pages referred to are those of Part 7.

In the arguments log I means the logarithm of the first term in a formulæ composed of several terms. Log II means the logarithm of the second term of such an expression, etc. SUM means the logarithm of the sum of two terms of an expression. Similarly DIFF means the logarithm of the difference. ADD and SUB refer to the numbers taken from an addition and subtraction table. Log [] or { } means the logarithm of the expression contained within the square brackets or braces.

EXAMPLE NO. 1.

DIRECT SOLUTION OF A PARABOLIC ORBIT.

As an example of the details of the computation of an ordinary preliminary orbit, the complete calculation of the orbit of Comet α 1909 (Daniel) is given here. The computation was made by R. T. CRAWFORD, and has since been done in duplicate by P. W. MERRILL, Fellow in the Lick Observatory.

The computation is based on the following observations:

1909 Gr. M. T.	α (app.)	δ (app.)	Observer.
I June 16.5306	1 ^h 41 ^m 54 ^s .1	+ 29° 58' 18"	JAVELLE—Nice
II 18.9809	48 49.5	+ 33 26 15	CAMPBELL—Lick
III June 21.9659	1 57 51.0	+ 37 25 9	ALBRECHT—Lick

A Ia

Reduced to the beginning of the year 1909.0, including the aberration terms, these are

α_1	25° 28' 38"	δ_1	+ 29° 58' 25"
α_u	27 12 29	δ_u	+ 33 26 22
α_{III}	29 27 51	δ_{III}	+ 37 25 17

Using the data of the *American Ephemeris and Nautical Almanac*, the interpolation for the Sun's coördinates, referred to 1909.0, gives for the three dates of observation:

N_1	+0.085434	V_1	+0.928875	Z_1	+0.402945
N_u	+0.044042	V_u	+0.931466	Z_u	+0.404071
N_{III}	-0.006472	V_{III}	+0.932482	Z_{III}	+0.404513

Velocities of the Solar Coördinates at the Instant and Middle Date ($w = 1$).

1909	N	f'	f''	f'''
June 17.5	+0.0690722			
18.5	521741	- 168981	- 145	
19.5	352615	169126	- 92	+ 53
20.5	+0.0183397	- 169218		
	V			
June 17.5	+0.9300897			
18.5	9310827	+ 9930	- 2640	
19.5	9318117	7290	- 2639	+ 1
20.5	+0.9322768	+ 4651		

1909	Z	f^1	f^m
June 17.5	+0.4034724		
18.5	4039040	+ 4316	
19.5	4042209	3169	- 1147
20.5	+0.4044232	+ 2023	+ 1

Date: June 18.9809	$f(l)$	X'	Y'	Z'
	$f^1(a + \frac{1}{2}w)$	-0.0169126	+0.0007290	+0.0003169
$m = -0.0191$	$m f^1(a + \frac{1}{2}w)$	+ 2	+ 50	+ 22
	$M_{f^1(m)}^1 f^m(a + \frac{1}{2}w)$	- 2	0	0
$M_{f^1(m)}^3 = -0.041$	$k^{(m)} d l$	-0.0169126	+0.0007340	+0.0003191
	$\log k^{(m)} d l$	8.22821n	6.86570	6.50393
	$\log X'_0; Y'_0; Z'_0$	9.99263n	8.63012	8.26835

Correction of the Solar Coördinates of the Middle Date for the Partial Elimination
of the Geocentric Parallax.

The parallax factors for the middle observation were taken from the tables of Volume I, *Publications of the Lick Observatory*. The logarithms of these, expressed in circular measure, are:

	$\log p_{\alpha\rho}$	5.5810n		
	$\log p_{\delta\rho}$	5.1699		
Then				
	$\sin \alpha_u$	9.6601	$R \cos D \cos A$	8.64380
	$\cos \alpha_u$	9.9491	$R \cos D \sin A$	9.96918
	$\sin \delta_u$	9.7412	$\tan A$	1.32538
	$\cos \delta_u$	9.9214	A	87°.2934
			$A - \alpha_u$	60.0853
	$\log I$	5.1625n	$\sin A$	9.99951
	$\log II$	4.8602	$\cos A$	8.67414
	X_u	+0.044042	$\log S = R \cos D$	9.96966
	I	- 14	$R \sin D$	9.60644
	II	+ 7	$\tan D$	9.63678
	$(X)_u$	+0.044035	$\sin D$	9.59942
			$\cos D$	9.96264
			$\log R$	0.00702
	$\log I$	5.4515		
	$\log II$	4.5712		
	Y_u	+0.931466	$\log(t_{uu} - t_u)$	0.474944
	I	+ 28	$\log(t_{uu} - t_u)$	0.389219
	II	+ 4	$\log(t_{uu} - t_u)$	0.735224
	$(Y)_u$	+0.931498		
			$\log \theta_u$	8.7105
			$\log \theta_{uu}$	8.6248
	$\log AZ_u$	+5.0913n		
	Z_u	+0.404071		
	AZ_u	- 12		
	$(Z)_u$	+0.404059		

A II₃

ϕ	α	δ		
$\phi'' - \phi'$	+ 6231"	+ 12467"		
$\phi''' - \phi''$	+ 8122	+ 14335		
$\log (\phi'' - \phi')$	3.794558	4.096110		
$\log (\phi''' - \phi'')$	3.909663	4.156398		
$\log \phi'''$	3.405339	3.706891		
$\log \phi'$	3.434719	3.681454	$\sec^2 \delta''$	0.157180
sub	8.845043	8.780467	$\log (\tan c)'$	0.302780
$\log (\phi' - \phi''')$	2.250382	2.461921n	$\log \delta''$	0.291200
$\log \frac{2}{k^2} \sin 1''$	8.515442		$\tan \delta''$	9.819785
$\text{colog } (t''' - t')$	9.264776		$\log I$	0.412015
$\log \phi''$	0.030600	0.242139n	$\log II$	0.242139n
			add	9.680050
$\log (t''' - t'') \phi'''$	3.880283	4.181835	$\log (I + II)$	9.922189
$\log (t'' - t') \phi'$	3.823938	4.070673	$\log (\tan \delta)''$	0.079369
add	0.273771	0.248996		
sum	4.154054	4.430831		
$\log \frac{\sin 1''}{k}$	6.449993			
$\text{colog } (t''' - t')$	9.264776			
$\log \phi'$	9.868823	0.145600		

A IIIa

$\log \alpha_0''$	9.73765	$\log \alpha''$	0.03060	$\sin \alpha''$	9.66013
$\tan \delta''$	9.81978	$\log n I'$	1.11455n	$\cos \alpha''$	9.94907
$\log \alpha_0'' \tan \delta''$	9.55743	add	9.96265	$\log \lambda/\kappa \cos \alpha''$	9.52105n
$\log (\tan \delta)''$	0.07937	$\log \text{numer.}$	1.07720n	$\log \alpha_0' \sin \alpha''$	9.52895
add	0.11416	$\log (-2 \Phi)$	1.50522	sub	0.29710
$\log n$	0.19353	$\log \lambda/\kappa$	9.57198n	$\log a_z$	9.82605n
$\cos (A - \alpha'')$	9.69785	$\sin \delta''$	9.74120	$\log \lambda/\kappa \sin \alpha''$	9.23211n
$\tan \delta'' \cos (A - \alpha'')$	9.51763	$\sin D$	9.59942	$\log \alpha_0' \cos \alpha''$	9.81789
$\tan D$	9.63678	$\log I$	9.34062	add	9.86949
sub	9.49925	$\cos \delta''$	9.92141	$\log a_y$	9.68738
$\log C_1$	9.01688n	$\cos D$	9.96264	$\log \lambda/\kappa \tan \delta''$	9.39176n
$\log C_2$	9.93790	$\cos (A - \alpha'')$	9.69785	$\log (\tan \delta)''$	0.30278
$\log I'$	0.92102n	$\log II$	9.58190	add	9.94313
$\log \alpha_0'$	9.86882	add	0.19694	$\log a_z$	0.24591
$\log I' (\tan \delta)''$	1.22380n	$\log c$	9.77884		
add	9.98039	ψ	53°.062		
$\log \Phi$	1.20419n	$\log s$	9.90270		

A IVa

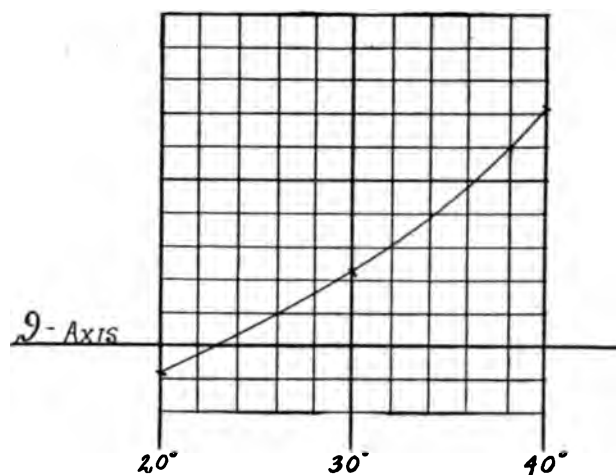
$\log a_z^2$	9.65210	$\log X_0'^2$	9.98526	$\log a_z X_0'$	9.81868
$\log a_y^2$	9.37476	$\log Y_0'^2$	7.26024	$\log a_y Y_0'$	8.31750
add	0.18413	add	0.00082	add	0.01348
sum	9.83623	sum	9.98608	sum	9.83216
$\log a_z^2$	0.49182	$\log Z_0'^2$	6.53670	$\log a_z Z_0'$	8.51426
add	0.08672	add	0.00015	add	0.02040
$\log []$	0.57854	$\log G_z^2$	9.98623	$\log []$	9.85256
$\cos^2 \delta''$	9.84282			$\cos \delta''$	9.92141
$\log a^2$	0.42136			$\log b$	9.77397

A IVa—Continued.

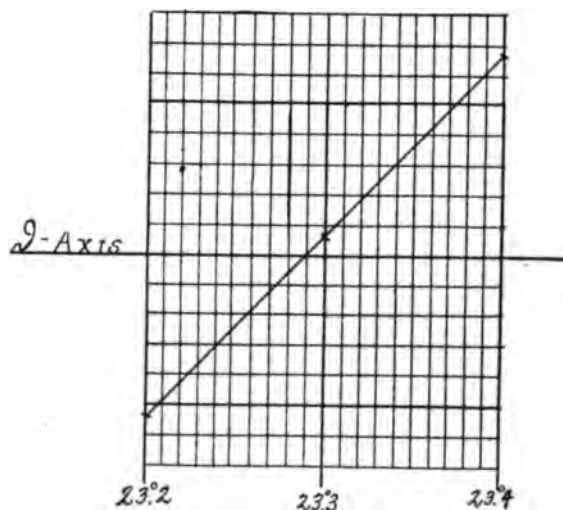
$\log a^2 G_1^2$	0.40759	$\log 2$	0.30103	$\log s$	9.9027
$\log b^2$	9.54794	$\text{colog } a^2$	9.57864	$\log a^2$	0.4214
sub	9.93543	$\text{colog } R^2$	9.97894	$\log q'^2$	9.4863
diff	0.34302	$\log h$	9.85861	$\text{colog } 2$	9.6990
$\log a^4 R^2$	0.85676			$\log (m' = \cos \vartheta_m)$	9.5094
$\log q'^2$	9.48626	$\log c$	9.77884	ϑ_m	71°
q'^2	+0.30638	$\log p'$	9.34559	$\log h/s$	9.95591
		sub	9.80019	h/s	+0.90346
$\log a^2 R$	0.42838	$\log c'$	9.57903		
$\log p'$	9.34559	c'	+0.37934	s	+0.79928

Graphical Determination of z_1 .

ϑ	z'^2	y	$f(\vartheta)$
70°	6.7	0.0	+ 6.7
40	1.10	0.39	+ 0.71
30	0.71	0.48	+ 0.23
20	0.45	0.54	- 0.09



ϑ	z'^2	y	$f(\vartheta)$
23°.4	+0.5261	+0.5228	+0.0033
23.3	+0.5237	+0.5234	+0.0003
23.2	+0.5213	+0.5240	-0.0027



ϑ	23°.290
$\log \tan \vartheta$	9.63393
$\log s \tan \vartheta$	9.53663
$\log c$	9.77884
add	0.19659
$\log z_1$	9.97543

NOTE.—In this case a second figure is not really necessary as a simple inspection of the numbers in the column (ϑ) shows that $f(\vartheta) = 0$ for $\vartheta = 23°.290$.

Final Solution for z .			
i	I	2	
$\log z_i$	9.97543	9.97548	
sub	9.88394	9.88396	
$\log (z_i - p')$	9.85937	9.85944	
$\log I$	9.71874	9.71888	
sub		9.75792	
$\log (z_i - c)$	9.53663	9.53676	
$\log (z_i - c)^2$	9.07326	9.07352	
add	0.07383 -	0.07387 -	
$\log \mu$	9.87923	9.87927	
$\log \mu^{\frac{1}{2}}$	9.93961	9.93963	
$\log (III = h \mu - \frac{1}{2})$	9.91900	9.91898	
I	+0.52329	+0.52346	
-III	-0.82985	-0.82982	
M_i	-0.00018	+0.00002 check	
			i
			I
			$\log h \mu - \frac{1}{2}$
			9.9190
			$\log (z_i - c)$
			9.5366
			$\log \mu^{-1}$
			0.1208
			$\log II$
			9.5764
			$\log I$
			0.1604
			add
			0.1006
			$\log \text{denom.}$
			0.2610
			$\log (-M_i)$
			6.2553
			$\log \partial z$
			5.9943
			add
			0.00005
			$\log z_i + 1$
			9.97548

It is to be noted that the graphical solution gave the value of $\log z$ correct to within one half a unit of the fourth decimal place and that a single differential correction gave the final value to 5 decimals.

A Va

$\log R$	0.00702	$\log \lambda/\kappa \theta_{III}$	8.1968n	$\log \lambda/\kappa \theta_i$	8.2825n
$\log z$	9.97548	$\log (1 - \lambda/\kappa \theta_{III})$	0.0068	$\log (1 + \lambda/\kappa \theta_{II})$	0.9916
$\log \rho_0$	9.98250	$\sec \delta_i$	0.0624	$\sec \delta_{III}$	0.1001
		$\log \rho_i$	9.9731	$\log \rho_{III}$	9.9956
$\log \alpha \rho_0$	7.7438n	$\log \alpha \rho_i$	7.7344n	$\log \alpha \rho_{III}$	7.7569n
$\alpha \rho_0$	-0.0055	$\alpha \rho_i$	-0.0054	$\alpha \rho_{III}$	-0.0057
t_{II} (obs.)	18.9809	t_i (obs.)	16.5306	t_{III} (obs.)	21.9659
t_{II}	18.9754	t_i	16.5252	t_{III}	21.9602
		$\log (t_{II} - t_i)$	0.38920	$\log (t_{III} - t_{II})$	0.47491
		$\log \theta_{III}$	8.62478	$\log \theta_i$	8.71049
$\cos \delta_{II}$	9.92141				
$\log \sigma_0$	9.90391				
$\cos \alpha_{II}$	9.94907	$\log a_z \sigma_0$	9.72996n	$\log x_0^2$	9.65056
$\log \sigma_0 \cos \alpha_{II}$	9.85298	$\log X_0'$	9.99263n	$\log y_0^2$	9.50414
$\log (X)_{II}$	8.64380	sub	9.91956	add	0.23396
sub	9.97230	$\log x_0'$	9.64952	sum	9.88452
$\log x_0$	9.82528			$\log z_0^2$	8.19540
$\sin \alpha_{II}$	9.66013	$\log a_y \sigma_0$	9.59129	add	0.00880
$\log \sigma_0 \sin \alpha_{II}$	9.56404	$\log Y_0'$	8.63012	$\log r_0^2$	8.89332
$\log (Y)_{II}$	9.96918	sub	9.94970	$\log r_0$	9.94666
sub	0.18803	$\log y_0'$	9.54099	Check	
$\log y_0$	9.75207n			$\log R^2$	0.01404
$\tan \delta_{II}$	9.81978	$\log a_z \sigma_0$	0.14982	$\log \mu$	9.87927
$\log \sigma_0 \tan \delta_{II}$	9.72369	$\log Z_0'$	8.26835	$\log r_0^2$	9.89331
$\log (Z)_{II}$	9.60644	sub	9.99425		
sub	9.49126	$\log z_0'$	0.14407		
$\log z_0$	9.09770			$\log x_0 x_0'$	9.47480
		$\log x_0'^2$	9.29904	$\log y_0 y_0'$	9.29306n
		$\log y_0'^2$	9.08198	add	9.71571
		add	0.20592	sum	9.00877
		sum	9.50496	$\log z_0 z_0'$	9.24177
		$\log z_0'^2$	0.28814	add	0.19997
		add	0.06623	$\log r_0 r_0'$	9.44174
		$\log G_0^2$	0.35437	$\log r_0'$	9.49508
		Check			
		$\log^2 r_0$	0.35437		

A VIa

Representation of the First and Third Places.

	I	III		I	III
$\log \theta$	8.62478	8.71049	$\log f x_0$	9.82471	9.82447
$\log \theta^2$	7.24956	7.42098	$\log f y_0$	9.75150n	9.75126n
$\log \theta^3$	5.8743	6.1315	$\log f z_0$	9.09713	9.09689
$\log \theta^4$	4.499	4.842	$\log g x_0'$	8.27411n	8.35974
			$\log g y_0'$	8.16558n	8.25121
$\log r_0^3$	9.83998		$\log g z_0'$	8.76866n	8.85429
$\log f_1$	9.85899n		X	+0.085434	-0.006472
			$f x_0$	+0.667900	+0.667519
$\log r_0'$	9.49508		$g x_0'$	-0.018798	+0.022895
$\log 2 r_0^4$	0.08767		$\frac{1}{2}$	+0.734536	+0.683942
$\log f_3$	9.4074		Y	+0.928875	+0.932482
$\log (r_0 r_0')^2$	8.8835		$f y_0$	-0.564288	+0.563975
$\text{colog } r_0$	0.0533		$g y_0'$	-0.014641	+0.017832
$\log \frac{1}{4} \delta$	0.5740		η	+0.349946	+0.386339
$\log II$	9.5108				
$\log (I - II)$	9.8298		Z	+0.402945	+0.404513
$\log 6 r_0^8$	0.4581		$f z_0$	+0.125063	+0.124994
$\log f_4$	9.372		$g z_0'$	-0.058703	+0.071497
			$\frac{1}{5}$	+0.469305	+0.601004
$\log g_1$	9.3819n		$\rho \cos \delta \cos \alpha$	9.86601	9.83502
			$\rho \cos \delta \sin \alpha$	9.54400	9.58697
$\log 4 r_0^4$	0.3887		$\tan \alpha$	9.67799	9.75195
$\log g_4$	9.106		α	25°.4740	29°.4611
			α (Obs.)	25.4750	29.4617
$\log \theta^2 f_2$	7.10855n	7.27997n	(O - C) $\begin{cases} \Delta \alpha \\ \Delta \alpha \cos \delta \end{cases}$	+0.0010 +0.0009	+0.0006 +0.0005
$\log \theta^3 f_3$	5.2817	5.5389	$\sin \alpha$	9.63357	9.69182
$\log \theta^4 f_4$	3.871	4.214	$\cos \alpha$	9.95558	9.93987
			$\rho \cos \delta$	9.91043	9.89515
$\log \theta^3 g_3$	5.2562n	5.5134n	$\rho \sin \delta$	9.67145	9.77887
$\log \theta^4 g_4$	3.605	3.948	$\tan \delta$	9.76102	9.88372
			δ	+29°.9761	+37°.4200
	1.000000	1.000000	δ (Obs.)	+29.9753	+37.4222
$\theta^2 f_2$	- 1284	- 1905	(O - C) $\Delta \delta$	- 0.0008	+ 0.0022
$\mp \theta^3 f_3$	- 19	+ 35			
$\theta^4 f_4$	+ 1	+ 2	$\sin \delta$	9.69866	9.78366
f_1, f_{III}	+0.998698	+0.998132	$\cos \delta$	9.93764	9.89993
$\log f_1; \log f_{III}$	9.99943	9.99919	$\log \rho$	9.97279	9.99522
$\mp \theta$	-0.042148	+0.051344	$\log p_a \rho$	0.864n	0.932n
$\mp \theta^3 g_3$	+ 18	- 33	$\log p_s \rho$	0.748	0.493
$\theta^4 g_4$	0	+ 1			
$g_1; g_{III}$	-0.042130	+0.051312	$\log p_a$	0.891n	0.937n
$\log g_1; \log g_{III}$	8.62459n	8.71022	$\log p_s$	0.775	0.498
			p_a	- 8"	- 9"
			p_s	+ 6	+ 3

A VIIIa

Constants for the Equator 1909.0, and Elements.

d	w	D'	a	x	A'	b	y	B'	c	z	C'
$\log r_0 w_0'$			9.59618			9.48765			0.09073		
$\log w_0 r_0'$			9.32036			9.24715n			8.59278		
sub			9.94802			0.19722			9.98598		
$\log [r_0 w' - w_0 r_0']$			9.26838			9.68487			0.07671		
$\sin d \cos (D' + v_0)$			9.15413			9.57062			9.96246		
$\sin d \sin (D' + v_0)$			9.87862			9.80541n			9.15104		
$\tan (D' + v_0)$			0.72449			0.23479n			9.18858		
$D' + v_0$			79°.3202			300°.2159			8°.7758		
$D' = A'; B'; C'$			55.3191			276.2148			344.7747		
$\sin (D' + v_0)$			9.99241			9.93658n			9.18346		
$\cos (D' + v_0)$			9.26792			9.70180			9.99488		
$\sin d = \sin a; \sin b; \sin c$			9.88621			9.86882			9.96758		
$\log r_0'$	9.49508	$\tan \varepsilon$		9.63728		A			50°.3324		
$\log G$	0.17718					B			271.2281		
$\sin \frac{1}{2} v_0$	9.31790	$\log I$		9.96246		C			339.7880		
$\frac{1}{2} v_0$	12°.00056	$\log II$		9.20790		$\sin B$			9.99990n		
v_0	24.0011	sub		9.91594		$\sin b$			9.86882		
$\log M$	1.24892	$\sec \varepsilon \sin i \cos u$		9.87840		$\sec \varepsilon$			0.03744		
$\log 2 r_0$	0.24769	$\log I$		9.15104		$\sin \Omega$			9.90616n		
$\log (r_0 r_0')^2$	8.88348	$\log II$		9.44269n		$\sin a$			9.88621		
sub	9.98081	sub		0.17924		$\sin A$			9.88635		
$\log \phi$	0.22850	$\sec \varepsilon \sin i \sin u$		9.62193		$\cos \Omega$			9.77256		
$\log q$	9.92747	$\tan u$		9.74353		Ω			306°.3220		
		u		28°.9878							
$\log q \frac{1}{2}$	9.89120	ω		4°.9867		$(-\sin a)$			9.88621n		
$\log M q \frac{1}{2}$	1.14012	$\sin u$		9.68540		$\cos A$			9.80505		
$M q \frac{1}{2}$	13.8077	$\cos u$		9.94187		$\operatorname{cosec} \Omega$			0.09384n		
t_0	June 18.9754	$\sec \varepsilon \sin i$		9.93653		$\cos i$			9.78510		
T	June 5.1677 Gr. M. T.	$\cos \varepsilon$		9.96256		i			52°.4340		
		$\sin i$		9.89909							
			Check			Check					
			$\sin b \sin c$	9.83640		$\sin c$			9.96758		
			$\sin (C - B)$	9.96886		$\sin C$			9.53844n		
			$\operatorname{cosec} a$	0.11379		$\operatorname{cosec} \varepsilon$			0.40016		
			$\sec A$	0.19495		$\sin \Omega$			9.90618n		
			$\tan i$	0.11400							

COLLECTION OF RESULTS.

Elements.		
T	1909 June 5.1677 Gr. M. T.	
ω	4°.9867	1909.0
Ω	306°.3220	
i	52°.4340	
$\log q$	9.92747	

Constants for the Equator 1909.0.	
$x = r [9.88621] \sin (55°.3191 + v)$	
$y = r [9.86882] \sin (276.2148 + v)$	
$z = r [9.96758] \sin (344.7747 + v)$	

SUPPLEMENTARY DISCUSSION.

This supplementary discussion is given in full as a sample of the analysis that should be made in connection with every preliminary orbit. The experienced computer can arrive at the conclusions very quickly without going into such elaborate detail as is given here for the sake of making the process clear.

First of all we should investigate the number of mathematical solutions through the use of the criteria given on page 288, so as to avoid the adoption of a fictitious solution in case of multiple solutions. Since p' and c are positive, three solutions are possible only if $\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2$ is negative. For this expression to be negative p must be negative, *i. e.* $9(2s^2 + q'^2) < 7c'^2$. But a glance at the numerical values of s , q'^2 , and c' shows that $9(2s^2 + q'^2) > 7c'^2$. Hence only a single solution is possible.

It is further essential to investigate whether the parabolic hypothesis is admissible within the uncertainty of the computation or whether it should be rejected. For this we have to compute only approximate values of N , κ and $\frac{1}{m}$ according to A IIIa, A IVa and determine the degree of accuracy to which $\frac{1}{m}$ is known.

In this let us place a dot over the last definitely known figure. If we assume that the error of observation is a matter of only a few seconds of arc, then the differences in α and δ are known to 3 and 4 places respectively. The corresponding differences and intervals are:

$$\begin{array}{lll} t_{\text{II}} - t_{\text{I}} = 2.4503 & \alpha_{\text{II}} - \alpha_{\text{I}} = 6231'' & \delta_{\text{II}} - \delta_{\text{I}} = 12467'' \\ t_{\text{III}} - t_{\text{II}} = 2.9850 & \alpha_{\text{III}} - \alpha_{\text{II}} = 8122 & \delta_{\text{III}} - \delta_{\text{II}} = 14335 \end{array}$$

The intervals (not exactly equal) are in the neighborhood of 3 days. In order to get a better idea of the run of the differences they can be reduced by proportion to a three-day basis. On this basis, assigning the differences to the instants of the middle of the intervals, we would have

	f_a^{I}	Diff.	f_a^{II}	f_b^{I}	Diff.	f_b^{II}
June 17.7557	7629''		589''	15276''		960''
		(534'')			(870'')	
June 20.4734	8163			14406		

The numbers in parentheses are the differences between the f_a^{I} and f_b^{I} respectively which correspond to a time interval of 2.7177 days. These reduced to correspond to a three-day interval are the quantities shown as f_a^{II} and f_b^{II} respectively. Since the principal or governing term in α'' and δ'' is the corresponding f'' it is seen that the accelerations are known only to two places. But now we should consider the error introduced by neglecting f''' . In α , f'' is somewhat smaller than $\frac{1}{10} f'$. If now we assume the differences to decrease at the same rate we would have as an approximation $f_a''' = 59''$, $f_b''' = 96''$.

According to (2) page 267, or also according to the discussion, pages 274 and 275, the error produced by neglecting the third differences is $\frac{\theta, \theta''' \alpha'''}{6}$ and $(\theta''' - \theta,)' \frac{\alpha'''}{3}$ in the velocities and accelerations respectively. Or, since $\alpha''' = \frac{f'''}{\theta^2}$ approximately, and further $\theta''' - \theta, = 0.5 k$ approximately, and since (on our three-day basis) we can place $\theta, = \theta''' = \theta = 3 k$, the error created in the velocities is approximately $\frac{f'''}{6 \theta}$, and in the accelerations $\frac{0.5 k}{3 \theta^2} f'''$ or $\frac{0.5}{9 \theta^2} f'''$. For the velocities, therefore, $\frac{f'''}{6}$ should be compared with f' , and for the accelerations $\frac{0.5}{9} f'''$ with f'' . For $\frac{f'''}{6}$ we have $\frac{59''}{6} = 10''$ in α , and $\frac{96''}{6} = 16''$ in δ . So that, through neglecting the third differences, the uncertainty gets slightly into the tens of seconds and consequently the uncertainty of the velocities in α and δ is in the third and fourth places respectively. For the accelerations, $\frac{0.5}{9} f'''$ becomes $\frac{59''}{18}$ in α and $\frac{96''}{18}$ in δ , or $3'' +$ and $5'' +$ respectively. It is seen, therefore, that the neglect of the third differences introduces no greater uncertainty into the accelerations than was assumed for them arising from errors of observation. These two errors, however, might be cumulative so that their combination might affect the last two places, and since in f'' only three places are available we can say that we know *absolutely* only one place.

Collecting these results then it is seen that we have *certain*

in α' 2 places δ' 3 places α'' 1 place δ'' 1 place.

It should be noted that the conditions are somewhat more favorable for δ'' than for α'' . From the computation for $(\tan \delta)'$ and $(\tan \delta)''$ it follows that these quantities are of the same order of certainty as δ' and δ'' respectively.

After the investigation into the accuracy of the velocities and accelerations the subject of parallax should be studied. In this case there is no necessity for the *complete* elimination of the parallax, since the correction for parallax can scarcely affect more than the already uncertain last two places. As should always be done, the partial elimination of the geocentric parallax was applied as a correction to the middle solar place merely to give the correct residuals for the first and third places, in that, through this elimination, the geocentric parallax in the α and δ of the middle date is taken into account.

There remains the investigation into the character of the orbit. Through an approximate computation we find for the three terms in the quantity N , according to AIIIa, the following:

$$N = 0.267 - 2.155 + 0.888 = -1.000$$

Since α'' is the least certain, so is the second term, the conclusive one for the sufficiency of N . Therefore, the uncertainty of N is one unit of the first decimal place.

Further, according to A IIIa, $\log \kappa = 0.191$ and, according to A IVa, $\frac{1}{m} = -0.57$. κ and $\frac{1}{m}$ are of the order of accuracy of N . If we place the uncertainty of N at about 0.2, *i. e.* $\frac{1}{5}$ of its value, then the uncertainty of $\frac{1}{m}$ will be a little more than 0.1. If we interpolate the general solution z_g from the Table at the end of Part 7 with the arguments $\psi = 53^\circ.062$ and a somewhat closer value of $\frac{1}{m}$, *viz.*, $\frac{1}{m} = -0.5739$, we find $z_g = 0.936$. From the appropriate part of the Table in Part 7 we find a variation of 0.01 in z corresponds to a variation of 0.03 in $\frac{1}{m}$. The difference of 0.009 between $z = 0.945$ and z_g corresponds, therefore, to an uncertainty of 0.03 in $\frac{1}{m}$. This is so much less than the determined uncertainty of more than 0.1 in $\frac{1}{m}$ that we are not to expect a general orbit from the observational material at hand.

One might draw the conclusion from the fair agreement between the parabolic and the general values of z , that, in practice, it would be better to take the first approximation for z from the Table than to use the graphical method which was employed in the example. But this is not the case, since, generally, in using the tabular value several applications of the differential formulæ AIVa would be required to get the final value of z .

In conclusion it is to be noted that the greatest residual ($O - C$) is about $8''$, and that on account of the difficulty of observing this comet accurately, it is not advisable to attempt an improvement of the orbit through a differential correction to remove the residuals.

EXAMPLE NO. 2.

DETERMINATION OF α' , δ' ; α'' , δ'' FROM FIVE OBSERVATIONS.

It sometimes happens that the computer has reason to believe, at the start, that three observations can not give velocities and accelerations in α and δ accurately enough, and that, therefore, five or more observations should be used according to the scheme outlined in the footnote, page 314. To illustrate the workings of these formulæ the case of Comet ϵ 1906 (KOPFF) is selected. The computations were made by R. T. CRAWFORD and A. J. CHAMPREUX.

A preliminary orbit of this comet was available, but because of the very slow motion of the object large changes in the orbit were expected. It was deemed advisable, therefore, not to attempt to derive a second orbit based upon the preliminary, but to make another *direct solution* starting with more accurate geocentric velocities and accelerations as determined from five observations.

The five observations used were made by FATH at the Lick Observatory. They are:

Gr. M. T. 1906	ϵ (1906.0)	δ (1906.0)
I Aug. 24.7063	341° 59' 0".6	+ 10° 18' 6".2
II Aug. 29.9817	341 " " 2	10 " 0 49.3
III Sept. 5.6771	339 43 57.0	9 28 40.3
IV Sept. 8.7203	339 11 54.0	9 11 13.0
V Sept. 15.7371	338 6 4.4	+ 8 25 45.4

Using the first, third, and fifth observations, approximate values of the velocities and accelerations were found, according to A IIa (unit of time being one Mean Solar Day). They are:

$\log \alpha'_0$	7.482973n	$\log \alpha''_0$	5.621814
$\log \delta'_0$	7.186673n	$\log \delta''_0$	5.749006n

The more accurate determination depending upon the five observations, according to the formulæ, page 314, proceeds as follows:

i	1	2	4	5
t_i	24.7063	29.9817	39.7203	46.7371
$t_i - t_0$	- 11.9708	- 6.6954	+ 3.0432	+ 10.0600
$\log (t_i - t_0)$	1.078123n	0.825777n	0.483330	1.002598
$\log (t_i - t_0)^2$	2.156246	1.651554	0.966660	2.005196
$\log (t_i - t_0)^3$	3.234369n	2.477331n	1.449990	3.007794
$\log a_i$	1.854158n	1.097120n	0.069779	1.627583
$\log b_i$	1.378095	0.873403	0.188509	1.227045
$\log c_i$	0.777093n	0.524747n	0.182300	0.701568

i	2	4	i	2	4
$\log 206265''$	5.314425	5.314425	$\log 2 c_i \delta'_0$	3.326875	2.984428n
$\log 2 c_i \alpha'_0$	3.623175	3.280728n	$\log 3 b_i \delta''_0$	2.413955n	1.729061n
$\log 3 b_i \alpha''_0$	2.286763	1.601869			
α_i	341° 0' 0".2	339° 11' 54".0	δ_i	+ 10° 0' 49".3	+ 9° 11' 13".0
$-\alpha_0$	- 339 43 57.0		$-\delta_0$	- 9 28 40.3	
$- 2 c_i \alpha'_0$	- 1 9 59.3	+ 0 31 48.7	$- 2 c_i \delta'_0$	- 0 35 22.6	+ 0 16 4.8
$- 3 b_i \alpha''_0$	- 0 3 13.5	- 0 0 40.0	$- 3 b_i \delta''_0$	+ 0 4 19.4	+ 0 0 53.6
sum	+ 0 2 50.4	- 0 0 54.3	sum	+ 0 1 5.8	- 0 0 28.9
$\log \text{sum}$	2.231470	1.734800n	$\log \text{sum}$	1.818226	1.460898n
$\log n_{1,2}$	1.405693n	1.251470n	$\log n_{3,5}$	0.992449n	0.977568n

$\log p_1$	b_1	c_1	I_1	$\log p_2$	c_2	I_2
$\log \frac{a_2}{a_1} \cdot p_1$	0.873403	0.524747n	0.000000	$\log [p_2 \cdot 1]$	0.361811n	9.916470
sub	0.621057	0.020055n	9.242962	log II	9.195148	9.227936
$\log [p_2 \cdot 1]$	9.896477	9.837064		sub	0.028624	9.900441
	0.517534	0.361811n	9.916470	$\log [p_2 \cdot 2]$	0.390435n	9.816911
$\log p_4$	0.188509	0.182300	0.000000	$\log [p_4 \cdot 1]$	0.153285	0.007077
$\log \frac{a_4}{a_1} \cdot p_1$	9.593716n	8.992714	8.215621n	log II	8.964496	8.997284
sub	0.098373	9.970985		sub	9.970929	9.955317
$\log [p_4 \cdot 1]$	0.286882	0.153285	0.007077	$\log [p_4 \cdot 2]$	0.124214	9.962394
$\log p_3$	1.227045	0.701568	0.000000			
$\log \frac{a_3}{a_1} \cdot p_1$	1.151520n	0.550518	9.773425n			
sub	0.264907	9.619048				
$\log [p_3 \cdot 1]$	1.491952	0.169566	0.202354			
γ	α	δ		γ	α	δ
$\log [I_1 \cdot 2] n_{1,\gamma}$	1.368087n	0.954443n		$\log [c_2 \cdot 2] n_{1,\gamma}$	1.641905	1.368003
$\log [I_2 \cdot 2] n_{1,\gamma}$	1.068381n	0.794479n		$\log [c_4 \cdot 2] n_{1,\gamma}$	1.529907n	1.116663n
sub	9.997348	9.649968		sub	0.248631	0.193294
log numer.	1.065729n	0.444447n		log numer.	1.890536	1.561297
$\log [c_2 \cdot 2][I_1 \cdot 2]$	0.352829n					
$\log [c_4 \cdot 2][I_2 \cdot 2]$	9.941125					
sub	0.142240					
log denom.	0.495069n	0.495069n		log denom.	0.495069n	0.495069n
log sin 1"	4.685575	4.685575		log sin 1"	4.685575	4.685575
log γ	5.256235	4.634953		log z_γ	6.081042n	5.751803n
γ	α	δ		γ	α	δ
$\log \gamma_0'$	7.482973n	7.186673n		$\log \gamma_0''$	5.621814	5.749006n
$\log z_\gamma$	6.081042n	5.751803n		$\log \gamma_\gamma$	5.256235	4.634953
add	0.016881	0.015670		add	0.155622	9.965247
colog k	1.764419	1.764419		colog k^2	3.528838	3.528838
$\log \gamma'$	9.264273n	8.966762n		$\log \gamma''$	9.306274	9.243091n

With these values of the velocities and accelerations in α and δ the computation then proceeds exactly as is shown in Example No. 1.

EXAMPLE NO. 3.

DIRECT SOLUTION OF A CIRCULAR ORBIT.

To illustrate the computation for the direct solution of a circular orbit, three observations of (158) *Koronis* have been used. The computation is by R. T. CRAWFORD.

The observations were taken from *A. N.* 4127, page 373. All of them were made at the Naval Observatory, Washington; the first and third by HAMMOND, the second by FREDRICKSON.

A Ib

Reduced to the beginning of the year 1906.0, including the aberration terms, they are

1906 Gr. M. T.	α (1906.0)	δ (1906.0)
I April 23.63642	180° 12' 14".6	-1° 38' 53".4
II April 25.60822	179 58 17 .0	-1 32 7 .8
III April 27.56370	179 45 39 .6	-1 25 55 .3

The solar coördinates, reduced to 1906.0, are

X_s +0.843341	$(X)_s$ +0.824920	X_{III} +0.805728
Y_s +0.503136	$(Y)_s$ +0.529043	Y_{III} +0.554147
Z_s +0.218260	$(Z)_s$ +0.229471	Z_{III} +0.240390

NOTE.—The solar coördinates for the middle date have been corrected so as to eliminate the geocentric parallax for that place.

Further

$\log X'_s$	9.745839n	$\log R'$	0.002813	$\sin D$	9.357915
$\log Y'_s$	9.878045	$\log S$	9.991222	$\cos D$	9.988409
$\log Z'_s$	9.515359	$\log (t_s - t_i)$	0.294863	$\log (t_{III} - t_s)$	0.291254
		$\log \theta_{III}$	8.5304	$\log \theta_i$	8.5268

A IIb

ϕ	α	δ	
$\phi_s - \phi_i$	-837".6	+405".6	
$\phi_{III} - \phi_s$	-757 .4	+372 .5	
$\log (\phi_s - \phi_i)$	2.923037n	2.608098	
$\log (\phi_{III} - \phi_s)$	2.879325n	2.571126	
$\log \phi_{III}'$	2.628174n	2.313235	
$\log \phi_i'$	2.588071n	2.279872	
sub	8.98560	8.90227	
$\log (\phi_i' - \phi_{III}')$	1.57367	1.18214n	
$\log \frac{2}{k^2} \sin 1''$	8.51544		
$\text{colog } (t_{III} - t_i)$	9.40591		
$\log \phi_s''$	9.49502	9.10349n	
$\log (t_{III} - t_s) \phi_{III}'$	2.919428n	2.604489	
$\log (t_s - t_i) \phi_i'$	2.882934n	2.574735	
add	0.283166	0.286408	
sum	3.202594n	2.890897	
$\log \frac{\sin 1''}{k}$	6.449993		
$\text{colog } (t_{III} - t_i)$	9.405908		
$\log \phi_i'$	9.058495n	8.746798	
			$\sec^2 \delta_s$ 0.00031
			$\log (\tan \delta)_s$ 8.74711
			$\log \delta_s'^2$ 7.49360
			$\tan \delta_s$ 8.42823n
			$\log I$ 6.22286n
			$\log II$ 9.10349n
			add 0.00057
			$\log (I + II)$ 9.10406n
			$\log (\tan \delta)_s''$ 9.10437n

A IIIb

$\log \alpha_0'^2$	8.11699	$\log \alpha_0''$	9.49502	$\sin \alpha_0$	6.69841
$\tan \delta_0$	8.42823n	$\log n \Gamma$	9.51266n	$\cos \alpha_0$	0.00000n
$\log \alpha_0'^2 \tan \delta_0$	6.54522n	add	8.61750		
$\log (\tan \delta)_0''$	9.10437n	$\log \text{numer.}$	8.11252n	$\log \frac{\lambda}{\kappa} \cos \alpha_0$	9.36109n
add	0.00119	$\log (-2 \Phi)$	8.75143n	$\log \alpha_0' \sin \alpha_0$	5.75691n
$\log n$	9.10556n	$\log \frac{\lambda}{\kappa}$	9.36109	sub	9.99989
				$\log a_1$	9.36098n
$\cos (A - \alpha_0)$	9.92505n	$\sin \delta_0$	8.42808n		
$\tan \delta_0 \cos (A - \alpha_0)$	8.35328	$\sin D$	9.35791	$\log \frac{\lambda}{\kappa} \sin \alpha_0$	6.05950
$\tan D$	9.36951	$\log I$	7.78599n	$\log \alpha_0' \cos \alpha_0$	9.05850
sub	9.95600			add	0.00043
$\log C_1$	9.32551n	$\cos \delta_0$	9.99984	$\log a_2$	9.05893
		$\cos D$	9.98841		
$\log C_2$	9.73261n	$\cos (A - \alpha_0)$	9.92505n	$\log \frac{\lambda}{\kappa} \tan \delta_0$	7.78932n
		$\log II$	9.91330n	$\log (\tan \delta)_0'$	8.74711
$\log \Gamma$	0.40710	add	0.00323	add	9.94928
		$\log c$	9.91653n	$\log a_3$	8.69639
$\log \alpha_0'$	9.05850n	ψ	145°.6033		
$\log \Gamma (\tan \delta)_0'$	9.15421				
add	9.39190	$\log s$	9.75198		
$\log \Phi$	8.45040				

At this point we may proceed in two different ways. We may compute either a conditioned ellipse and apply the criterion $r'_0 = 0$, or a general orbit and apply the criteria $a = r_0$ and $r'_0 = 0$.

Method of Conditioned Ellipse.

A IVb

$\log a_1'^2$	8.72196	$\log N_0'^2$	9.49168	$\log a^2 R^2$	8.84259
$\log a_2'^2$	8.11786	$\log \Gamma_0'^2$	9.75609	$\log h$	1.15741
add	0.09650	add	0.18865		
sum	8.81846	sum	9.94474	$\log c$	9.91653n
$\log a_3'^2$	7.39278	$\log Z_0'^2$	9.03072	$\log p'$	0.52586
add	0.01600	add	0.04995	sub	0.09546
$\log []$	8.83446	$\log G_0'^2$	9.99469	$\log c'$	0.62132n
$\cos^2 \delta_0$	9.99969			c'	-4.1814
$\log a^2$	8.83415	$\log a^2 G_0'^2$	8.82884		
		$\log b^2$	8.72564	$\log s$	9.75198
$\log a_1 N_0'$	9.10682	sub	9.42852	$\log a^2$	8.83415
$\log a_2 \Gamma_0'$	8.93698	diff	8.15416	$\log q'^2$	0.48024
add	0.22436	$\log a^4 R^2$	7.67392	$\log (m' = \cos \vartheta_m)$	9.06637
sum	9.33118	$\log q'^2$	0.48024	S_m	83°.31
$\log a_3 Z_0'$	8.21175	q'^2	+3.02164		
add	0.03180			$\log \frac{h}{s}$	1.40543
$\log []$	9.36298			$\frac{h}{s}$	+25.435
$\cos \delta_0$	9.99984	$\log a^2 R$	8.83696	s	+0.56491
$\log b$	9.36282	$\log p'$	0.52586		

Since $p' > 0$, $c < 0$, $\psi > 125^\circ 23'$, and since the criterion for no solution expressed by means of the inequalities on page 306 is not fulfilled, the initial

velocities and accelerations can be represented by a *conditioned* ellipse ($r_0 = a$) with *one* solution.

The details for the solution of z , which are shown fully in Example No. 1, are omitted here. The solution gives $\log z = 0.27450$. To check this, the substitution of this value into the equation for z is here given.

$\log z$	0.27450	sub	0.15793	I	+ 2.1750
sub	9.89423	$\log (z - c)$	0.43243	II = q'^2	+ 3.0216
$\log (z - \phi')$	0.16873n	$\log (z - c)^2$	0.86486	- III	- 5.1966
$\log I$	0.33746	add	0.01852	I + II - III	0.0000
		$\log \mu$	0.88338	Check	
		$\log \mu^{1/2}$	0.44169		
		$\log (III = h\mu^{-1/2})$	0.71572		

A Vb

$\log R$	0.00281	$\log \frac{\lambda}{\kappa} \theta_{III}$	7.8915	$\log \frac{\lambda}{\kappa} \theta_I$	7.8879
$\log z$	0.27450	$\log (1 - \frac{\lambda}{\kappa} \theta_{III})$	9.9966	$\log (1 + \frac{\lambda}{\kappa} \theta_I)$	0.0033
$\log \rho_0$	0.27731	$\sec \delta_I$	0.0002	$\sec \delta_{III}$	0.0001
		$\log \rho_I$	0.2740	$\log \rho_{III}$	0.2806
$\log \alpha \rho_0$	8.03859n	$\log \alpha \rho_I$	8.0353n	$\log \alpha \rho_{III}$	8.0419n
$\alpha \rho_0$	- 0.01093	$\alpha \rho_I$	- 0.01085	$\alpha \rho_{III}$	- 0.01101
t_0 (obs)	25.60822	t_I (obs)	23.63642	t_{III} (obs)	27.56370
t_0	25.59729	t_I	23.62557	t_{III}	27.55269
$\cos \delta_0$	9.99984	$\log (t_0 - t_I)$	0.29485	$\log (t_{III} - t_0)$	0.29124
$\log \delta_0$	0.27715	$\log \theta_{III}$	8.53043	$\log \theta_I$	8.52682
$\log \delta_0 \cos \alpha_0$	0.27715n				
$\log (X)_0$	9.91461	$\log \alpha_0 \sigma_0$	9.63813n	$\log x_0 x'_0$	9.52182n
sub	0.15709	$\log Y'_0$	9.74584n	$\log y_0 y'_0$	9.45378
$\log x_0$	0.43424n	sub	9.44945	add	9.22943
		$\log x'_0$	9.08758	sum	8.68321n
$\log \sigma_0 \sin \alpha_0$	6.97556			$\log z_0 z'_0$	8.81582
$\log (Y)_0$	9.72349	$\log \alpha_0 \sigma_0$	9.33608	add	9.55278
sub	9.99923	$\log Y'_0$	9.87804	$\log r_0 r'_0$	8.23599
$\log y_0$	9.72272n	sub	9.85302	$\log r'_0$	7.79149
$\log \sigma_0 \tan \delta_0$	8.70538n	$\log y'_0$	9.73106n		
$\log (Z)_0$	9.36073	$\log \alpha_0 \sigma_0$	8.97354		
sub	0.08676	$\log Z'_0$	9.51536		
$\log z_0$	9.44749n	sub	9.85297		
		$\log z'_0$	9.36833n		
$\log x_0^2$	0.86848				
$\log y_0^2$	9.44544	$\log x_0'^2$	8.17516		
add	0.01610	$\log y_0'^2$	9.46212		
sum	0.88458	add	0.02187		
$\log z_0^2$	8.89498	sum	9.48399		
add	0.00442	$\log z_0'^2$	8.73666		
$\log r_0^2$	0.88900	add	0.07149		
$\log r_0$	0.44450	$\log G_0^2$	9.55548		
Check		Check			
$\log R^2$	0.00563	$\log \frac{1}{r_0}$	9.55550		
$\log \mu$	0.88338				
$\log r_0^2$	0.88901				

r'_0 is not zero since we have solved for a conditioned ellipse as pointed out on pages 305-306. Following the argument of page 306 we must now compute $\frac{1}{m}$ and trace the uncertainty in G_0^2 .

A IIIb

$\log \alpha_0'^3$	7.17548n	$\log C_1$	9.32551n	$\log R^4$	0.01125
$\tan \delta''$	8.42823n	$\log C_1 \alpha_0'$	8.38401	$\cos \delta''$	9.99984
$\log I$	5.60371			$\log R^4 \cos \delta''$	0.01109
		$\log C_2$	9.73261n	$\log m$	0.29768
$\log \alpha_0''$	9.49502	$\log C_2 (\tan \delta)''$	8.47972n	$\log \frac{1}{m}$	9.70232
$\log (\tan \delta)''$	8.74711	add	9.39190	1	
$\log II$	8.24213	sum	7.77591n	m	+0.503875
		$\log S$	9.99122		
$\log \alpha_0'$	9.05850n	$\text{colog } N$	2.54164n		
$\log (\tan \delta)''$	9.10437n	$\log \kappa$	0.30877n		
$\log III$	8.16287				
I	+0.0000402				
- II	-0.0174636				
III	+0.0145503				
N	-0.0028731				
$\log N$	7.45836n				

From Table (Part 7)

 $z = 1.89176$

From the differences of the α and δ it is seen that (assuming the seconds of arc to be without error) we know three figures in the velocities and two in the accelerations in both coördinates. In N the governing terms are the second and the third. The first of these contains α_0'' and the third $(\tan \delta)''$. In these then we know two figures, and consequently in N we know but one figure. The terms of the denominator of the expression (11) page 271 are proportional to the velocities so that we know three figures in each. But the computation shows that these terms have opposite signs, and are of such magnitude that we lose one figure in forming the denominator. It is then known to two figures, and the whole expression (11) is known to one figure. We therefore know but one figure in $\frac{1}{m}$. If we had assumed some slight errors of observation this one figure would be somewhat uncertain.

From the table it is seen that to a range of five units in the second decimal place of $\frac{1}{m}$ corresponds a range of two units in the first decimal of z . For $z = 1.9$ this uncertainty causes an uncertainty of six and one half units in the second decimal of $\log z$. Inspecting the computation of x_0 , y_0 , z_0 and r_0 it is seen that this same uncertainty extends to $\log r_0$. $\log G_0^2$ is therefore known definitely, at best, only to one decimal. As the characteristic of $\log r'^2$ is 5-10 it is seen that it cannot affect the definitely known portion of G_0^2 when subtracted to form $(G_0^2 - r'^2)$ in the expression $p_0 = r_0^2 (G_0^2 - r'^2)$. We can therefore call $r'_0 = 0$ and write $p_0 - r_0 = a$ and obtain a *circular* orbit which will represent the three observations as well as the *conditioned ellipse* will.

A VIIIb

$\log x_0 y_0'$	0.16530	$\log x_0 \cos (\Omega)$	0.43387n	$\log a$	0.44450
$\log y_0 x_0'$	8.81030n	$\log r_0 \sin (\Omega)$	8.33671	$\log a^{3/2}$	0.66675
sub	0.01877	add	9.99652	$\log \mu''$	2.88326
$\log \rho \cos (i)$	0.18407	$\log r_0 \cos (\mu)_n$	0.43039n	μ''	764"29
$\log y_0 z_0'$	9.09105	$\log z_n$	9.44749n		
$\log z_0 y_0'$	9.17855	$\sin (i)$	9.60356		
sub	9.34872	$\log r_0 \sin (\mu)_n$	9.84393n		
$\log \rho \sin (i) \sin (\Omega)$	8.43977n	$\tan (\mu)_n$	9.41354		
		$(\mu)_n$	194° 31' 42"		
$\log x_0 z_0'$	9.80257	$\sin (\mu)_n$	9.39943n		
$\log z_0 x_0'$	8.53507n	$\cos (\mu)_n$	9.98589n		
sub	0.02284	$\log r_0$	0.44450	Chc	
$\log \rho \sin (i) \cos (\Omega)$	9.82541				
$\tan (\Omega)$	8.61436n				
(Ω)	357° 38' 37"				
$\sin (\Omega)$	8.61399n				
$\cos (\Omega)$	9.99963				
$\log \rho \sin (i)$	9.82578				
$\tan (i)$	9.64171				
(i)	23° 39' 53"				

Constants for the Equator 1906.0.

$\sin (\Omega)$	8.61399n	$\cos (\Omega)$	9.99963		
$\cos (i)$	9.96185	$\cos (i)$	9.96185		
$\sin a \cos (A)$	8.57584	$\sin b \cos (B)$	9.96148		
$\sin a \sin (A)$	9.99963	$\sin b \sin (B)$	8.61399n		
$\tan (A)$	1.42379	$\tan (B)$	8.65251n		
(A)	87° 50' 30"	(B)	-2° 34' 21"		
$(\mu)_0$	194 31 42				
A'	282 22 12	B'	191 57 21	C'	194° 31' 42"
$\sin (A)$	9.99969	$\sin (B)$			
$\cos (A)$		$\cos (B)$	9.99956		
$\sin a$	9.99994	$\sin b$	9.96192	$\sin c$	9.60356
$\log a$	0.44444	$\log \beta$	0.40642	$\log \gamma$	0.04806

Representation of the First and Third Places.

	I	III		I	III
$\log (t - t_0)$	0.29485n	0.29124	$\rho \cos \delta \cos \alpha$	0.27390n	0.28054n
$\log \mu (t - t_0)$	3.17811n	3.17450	$\rho \cos \delta \sin \alpha$	7.82808n	7.90157
$\mu \Delta t$	-0° 25' 7"	+0° 24' 55"	$\tan \alpha$	7.55418	7.62103n
$A' + \mu \Delta t$	281 57 5	282 47 7	α	180° 12' 19"	179° 45' 38"
$B' + \mu \Delta t$	191 32 14	192 22 16	α (obs)	180 12 15	179 45 38
$C' + \mu \Delta t$	194 6 35	194 56 37	$(O - C) \Delta \alpha$	-4	0
$\sin [A' + \mu \Delta t]$	9.99048n	9.98910n	$\cos \alpha$	0.00000n	0.00000n
$\log x$	0.43492n	0.43354n	$\rho \cos \delta$	0.27390	0.28054
$\sin [B' + \mu \Delta t]$	9.30104n	9.33091n	$\rho \sin \delta$	8.73277n	8.67812n
$\log y$	9.70746n	9.73733n	$\tan \delta$	8.45887n	8.39758n
$\sin [C' + \mu \Delta t]$	9.38700n	9.41140n	δ	-1° 38' 52"	-1° 25' 51"
$\log z$	9.43506n	9.45946n	δ (obs)	-1 38 50	-1 25 52
x	-2.72219	-2.71356	$(O - C) \Delta \delta$	+2	-1
X	+0.84334	+0.80573	$\cos \delta$	9.99982	9.99986
ξ	-1.87885	-1.90783	$\log \rho$	0.27408	0.28068
y	-0.509867	-0.546175	$\log p_{00}$	9.567	0.344n
Y	+0.503136	+0.554147	$\log p_{01}$	0.755	0.753
η	-0.006731	+0.007972	$\log p_{10}$	9.293	0.063n
z	-0.272306	-0.288047	$\log p_{11}$	0.481	0.472
Z	+0.218260	+0.240390	p_{10}	+0".2	-1".2
ζ	-0.054046	-0.047657	p_{11}	+3.0	+3.0

A VIb

Transformation to Ecliptical Elements.

$\sin i$	9.60356	ε	23° 27' 5"	$\sin \varepsilon$	9.59985
$\cos \Omega$	9.99963	$M - \varepsilon$	0 11 44	$\sin i \sin \sigma$	8.21384n
$\log m \sin M$	9.60319	$\sin (M - \varepsilon)$	7.53315	$N - \varepsilon$	0° 13' 53"
$\log m \cos M$	9.96185	$\sin i \cos \Omega$	7.53309	$\sin (N - \varepsilon)$	7.60622
$\tan M$	9.64134	$\sin i \sin \Omega$	8.21755n	$\sin i \sin \sigma$	7.60591
M	23° 38' 49"	$\tan \Omega$	0.68446n	$\tan \sigma$	0.60793n
$\sin M$	9.60325	Ω	281° 41' 2"	σ	283° 51' 18"
$\cos M$	9.96191	$\sin \Omega$	9.99091n	$(\mu)_0$	194 31 42
$\log m$	9.99994	$\sin i$	8.22664	μ_a	270 40 24
$\cos i$	9.96185	i	0° 57' 56"		
$\log n \cos N$	9.96148	$\cos (M - \varepsilon)$	0.00000		
$\log n \sin N$	9.60356	$\cos i$	9.99994		
$\tan N$	9.64208				
N	23° 40' 58"				
$\sin N$	9.60387				
$\cos N$	9.96179				
$\log n$	9.99969				

COLLECTION OF RESULTS.

Elements.

Epoch = t_0 = 1906 April 25.5973 Gr. M. T.

μ_0	270° 40' 24"	} Ecliptic and Equinox 1906.0
Ω	281 41 2	
i	0 57 56	
$\log a$	0.44450	
μ	764".29	

Constants for the Equator 1906.0.

$$x = [0.44444] \sin (282^\circ 22' 12'' + \mu \text{ Jt})$$

$$y = [0.40642] \sin (191^\circ 57' 21'' + \mu \text{ Jt})$$

$$z = [0.04806] \sin (194^\circ 31' 42'' + \mu \text{ Jt})$$

Method of General Solution.

The value of z for the general solution taken from the table is $z = 1.89176$ (cf. the computation under A IIIb, page 404). Checking this by the method of A IVc we find the final value of $\log z = 0.27687$. With this z we find from A V

$\log x_0$	0.43589n	$\log x'_0$	9.07905	$\log a$	0.44512
$\log y_0$	9.72271n	$\log y'_0$	9.73011n	$\log r'_0$	7.89111
$\log z_0$	9.44792n	$\log z'_0$	9.36737n		
$\log r_0$	0.44608	$\log G_0^2$	9.55296		

The preceding discussion showed that the uncertainty in $\log r_0$ amounts to six units of the second decimal place. In the general solution $\log r_0$ differs from $\log a$ by only one unit in the third place. Therefore, we are justified in calling $\log a = \log r_0$. Further, as before, it is evident that r'_0 can be put equal to zero, and the two criteria for the circle hold (cf. page 306).

With the values of the heliocentric coördinates and their velocities derived from the general solution for z the elements and constants for the Equator are computed exactly as before under A VIb. They are :

Elements.

Epoch = t_0 = 1906 April 25.5972 Gr. M. T.

$$\begin{array}{lll} \mu_0 & 270^\circ 49' 59'' & \\ \Omega & 281 \ 28 \ 33 & \\ i & 0 \ 58 \ 3 & \\ \log a & 0.44608 & \\ \mu & 760''.13 & \end{array} \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \begin{array}{l} \text{Ecliptic and} \\ \text{Equinox 1906.0} \end{array}$$

Constants for the Equator 1906.0.

$$\begin{aligned} x &= [0.44602] \sin (282^\circ 19' 35'' + \mu \Delta t) \\ y &= [0.40801] \sin (191 \ 54 \ 40 + \mu \Delta t) \\ z &= [0.04959] \sin (194 \ 29 \ 27 + \mu \Delta t) \end{aligned}$$

This set of circular elements represents the first and third observations as follows :

$$(O - C) \left\{ \begin{array}{lll} \Delta \alpha & \text{I} & \text{III} \\ \Delta \delta & -2'' & +3'' \\ & +1 & -1 \end{array} \right.$$

The second method is to be preferred since no part of the computation needs to be discarded if the criteria fail and a general solution becomes necessary.

EXAMPLE NO. 4.

DIRECT SOLUTION OF A COMET ORBIT WITHOUT HYPOTHESIS
REGARDING THE ECCENTRICITY.

For an example of a direct solution of a comet orbit without hypothesis regarding the eccentricity the computation on the orbit of FAYE'S comet is given. This comet was picked up by CERULLI in November, 1910, and was announced as an unexpected comet. From the preliminary elements Professor LEUSCHNER identified it as FAYE'S comet. The identification was made certain after the derivation of elements without assumption regarding the eccentricity. The computations were made by W. F. MEYER and Miss SOPHIA LEVY.

The observations upon which the computation is based are:

1910 Gr. M. T.	α (1910.0)	δ (1910.0)	Observer.
I Nov. 9.3131	54° 38' 11".2	+8° 43' 3"	MILLOSEVICH, Rome
II Nov. 11.5801	54 35 57.4	+8 9 0	EPPES, Washington
III Nov. 13.8217	54 32 54.3	+7 36 17	YOUNG, Lick

Solar Coördinates (1910.0)

X_s	-0.681211	$(X)_s$	-0.651757	X_{III}	-0.621586
Y_s	-0.659264	$(Y)_s$	-0.683218	Y_{III}	-0.705867
Z_s	-0.285987	$(Z)_s$	-0.296406	Z_{III}	-0.306201

NOTE.—The solar coördinates for the middle date have been corrected so as to eliminate the geocentric parallax for that place.

Further

$\log X'_s$	9.885999	$\log R$	9.995487
$\log Y'_s$	9.778785n	$A - \alpha_s$	171° 45' 3"
$\log Z'_s$	9.416017n	$\log S$	9.975079
$\log G_s^2$	0.00884		

The details of the computation that are similar to those of Example No. 1 will be omitted here. The values of the principal quantities are given.

A IIc

$\log \alpha'_s$	8.29767n	$\log \delta'_s$	9.39859n	$\log (\tan \delta)_s'$	9.40741n
$\log \alpha''_s$	9.21670n	$\log \delta''_s$	9.26776	$\log (\tan \delta)_s''$	9.31676

A IIIc

$\log (\alpha'_s)^3$	4.89301n	$\log C_1$	9.23599	$\log R^4$	9.98195
$\tan \delta_s$	9.15598	$\log \alpha'_s$	8.29767n	$\cos \delta_s$	9.99559
$\log I$	4.04899n	$\log 1$	7.53366n	$\text{colog } \kappa$	0.08666n
				$\log \frac{1}{m}$	0.06420
$\log (\tan \delta)_s'$	9.40741n	$\log C_2$	9.15679	$\frac{1}{m}$	+1.15932
$\log \alpha''_s$	9.21670n	$\log (\tan \delta)_s'$	9.40741n		
$\log II$	8.62411	$\log II$	8.56420n	$\log m$	9.93580
		add	0.03870		
$\log (\tan \delta)_s''$	9.31676	$\log \text{sum}$	8.60290n		
$\log \alpha'_s$	8.29767n	$\text{colog } A$	1.33536n	ψ	167° 43' 36"
$\log III$	7.61443n	$\log S$	9.97508		
		$\log \kappa$	9.91334n		
I	-0.000001			$\log c$	9.989959
- II	-0.042083			$\log s^2$	8.655028
III	-0.004116				
N	-0.046200				
$\log N$	8.66464n				

With ψ and $\frac{1}{m}$ as arguments take a starting value of z from the table, Part 7. It is (calling it z_1)

$$\log z_1 \quad 9.830191$$

A IVc

Solution for z .

$\log z_1$	9.830191	$\log 3 \nu_1$	9.74710n
$\log c$	9.989959n	$\log II$	9.96551n
sub	0.228452	$\log I$	0.44394
$\log (z_1 - c)$	0.218411	add	9.82456
$\log (z_1 - c)^2$	0.436822	$\log []$	0.26850
$\log s^2$	8.655028	$\log \nu_1^2$	9.26998n
add	0.007119	$\log \mu_1^2$	0.88788
$\log \mu_1$	0.443941	$\log 2$	0.30103
		$\log \text{denom}$	0.72739n
$\log m$	9.935800	$\log (z_2 - z_1)$	5.7782
sub	9.439791	add	0.000039
$\log \nu_1$	9.269982n	$\log z_2 = z$	9.830230
$\log \mu_1^3$	1.331823		
$\log \nu_1^2$	8.539964		
$\log \mu_1^3 \nu_1^2$	9.871787		
$\log m^2$	9.871600		
sub	6.634		
$\log M_1$	6.5056		

A Vc

The computation from this point is exactly like that of Example No. 1, so the details will not be given. The principal quantities resulting from this value of z are:

$\log x_0$	0.01521	$\log r_0$	0.21747
$\log y_0$	0.08756	$\log (r_0^2)$	9.97171
$\log z_0$	9.59252	$\log \frac{1}{a}$	9.43966
$\log x_0'$	9.87604n	$\log r_0'$	7.46052n
$\log y_0'$	9.78011		
$\log z_0'$	8.96830		

The representation of the first and the third places gives

$$(O - C) \begin{cases} \Delta \alpha \cos \delta & \text{I} & \text{III} \\ \Delta \delta & \text{0''} & + 3'' \\ & + 2 & - 3 \end{cases}$$

Using the values of the heliocentric rectangular coördinates and their velocities given under A Vc the following elements result from A VIIIc:

T	1910 Nov. 12.4129 Gr. M. T.
ω	206° 20' 22" } 1910.0
Ω	205 29 5
i	10 14 11
e	0.54590
μ	512".34
$\log a$	0.56034
Period	6.926 years.

SUPPLEMENTARY DISCUSSION.

We will consider here the same points that were discussed under Example No. 1, viz., number of solutions, effect of neglecting third differences, parallax elimination, and character of orbit. The discussion will be condensed here; for the details of a full analysis see Example No. 1.

1. Computing the quantities ρ' and c under A IIIa we find $\rho' > 0$; $c < 0$. Then since $\psi > 125^\circ 16'$ we might have but one parabolic solution. With $\frac{1}{m}$ and ψ as arguments the table gave but one *general* solution.

2. Reduced to a $2\frac{1}{4}$ day interval we have

f_a^1	f_a^u	f_δ^1	f_δ^u
133"		2028"	
	49"		58"
184		1970	

The intervals are practically equal (2.242 and 2.267 days respectively) so that no error is introduced into the accelerations of α and δ by neglecting third differences.

f_a^u is about $\frac{1}{3} f_a^1$, and f_δ^u is about $\frac{1}{30} f_\delta^1$. Assuming f^u to bear the same ratio to f^u we have f^u to be approximately 16" and 2" in α and δ , respectively.

For the velocities, therefore, the errors would be $\frac{f^u}{6} = 3''$ in α and $1''$ in δ . These are not greater than the errors of observation, so that we can conclude that neglecting the third differences produces no greater error or uncertainty than is caused by the accidental errors of observation.

3. The discussion of general parallax (geocentric and barycentric) is made clear by aid of the table of values of the parallax factors for the three observations. (In this the letter g distinguishes the geocentric data, and m the barycentric. A detailed computation of barycentric parallax factors is given in Example No. 6.)

	I	II	III		I	II	III
$p_a^g \rho$	-5".82	-5.49	-0.97	$p_\delta^g \rho$	+5".30	+4".85	+4".35
$p_a^m \rho$	+5.95	+5.97	+4.27	$p_\delta^m \rho$	+2.28	+1.61	+0.38
$p_a \rho$	+0.13	+0.48	+3.30	$p_\delta \rho$	+7.58	+6.46	+4.73

The greatest difference between two parallax factors is 2".8, so that neglecting parallax can produce no greater error than neglecting third differences. Hence, in this example, as in Example No. 1, just the geocentric parallax was eliminated from the middle place.

4. To comply with the custom of publishing a parabola for a preliminary orbit, a parabola had been computed from these same observations, so that we have the parabolic z , viz., $\log z_p = 0.09037$.

The degree of accuracy of N is here again the degree of accuracy of α_0'' , i. e., of f_a^u the principal term upon which α_0'' depends. Since the error in this is the same as the error of observation, say about 5" or $\frac{1}{10}$ part of its value, the uncertainty of N and also of $\frac{1}{m}$ is $\frac{1}{10}$ of its value. The value of $\frac{1}{m}$ is 1.159; hence its uncertainty is 0.12. In the Tables a range of 0.01 in z corresponds to a range of 0.0115 in $\frac{1}{m}$. Then $z - z_p = 0.555$ corresponds to a range of 0.638 in $\frac{1}{m}$. This is more than five times the uncertainty in $\frac{1}{m}$ (viz. 0.12). This then leads us to conclude that the parabola can not hold and that solution must be made without assumption regarding the eccentricity.

EXAMPLE NO. 5.

DIRECT SOLUTION OF THE ORBIT OF A MINOR PLANET WITHOUT HYPOTHESIS REGARDING THE ECCENTRICITY.

For a direct solution of the orbit of a minor planet, without hypothesis regarding the eccentricity, the object chosen is 1909 HC. The details that are similar to those of previous examples are not given. The computation by R. K. YOUNG and H. C. WILSON is based upon three observations made by Mr. WILSON at the Lick Observatory.

A Ic

1910 Gr. M. T.	α (1910.0)	δ (1910.0)
I Nov. 19.6507	3° 13' 3".2	+23° 26' 40".9
II Nov. 26.7480	3 13 3.0	+22 29 31.3
III Dec. 3.7632	3 29 28.6	+21 38 48.1

Attention is directed to the fact that, in this case, the right ascensions for the first two dates are practically the same, so that a large uncertainty might be expected due to the neglected third and higher differences.

Solar Coördinates.

X_s -0.538875	X'' -0.430662	X''' -0.317074
Y_s -0.759593	Y'' -0.814331	Y''' -0.856021
Z_s -0.329503	Z'' -0.353249	Z''' -0.371338

Owing to the irregularity of motion in α the parallax was entirely neglected.

Solar Velocities.

$\log X'_s$	9.961993
$\log Y'_s$	9.599906n
$\log Z'_s$	9.237278n

A IIc

$\log \alpha'_s$	8.29905	$\log \delta'_s$	9.11121n	$\log (\tan \delta)_{s'}$	9.17994n
$\log \alpha''_s$	9.51359	$\log \delta''_s$	9.05983	$\log (\tan \delta)_{s''}$	9.17794

A IIc

$\log n$	9.17843	$\log \frac{\lambda}{\kappa}$	9.44318	$\log c$	9.76525n
$\log C_1$	9.22959	$\log \kappa$	0.36895n	$\log s$	9.91002
$\log C_2$	9.93266n	$\log \lambda$	8.71917	ψ	125° 37' 20"
$\log \Phi$	9.89418	$\frac{1}{m}$	+0.37433		

Entering the table (Part 7) with these values of ψ and $\frac{1}{m}$ the value of z was found for which $\log z = 0.41426$. Substituting this into the general equation for z , as shown in Example No. 4, the final value of z is found to give $\log z = 0.41425$, whence $\log \rho_0 = 0.40839$.

Using this value of ρ the middle date was corrected for aberration, and α'' and δ'' were corrected for parallax.

A Vc

$\log x_0$	0.44607	$\log x'_0$	9.42066n
$\log y_0$	9.97642	$\log y'_0$	9.68296
$\log z_0$	0.12481	$\log z'_0$	8.93672
$\log r_0$	0.51007	$\log r'_0$	8.70503n
$\log a$	0.51023	$\log G_0^2$	9.49009

It may be pointed out here that the application of the criteria for a circular orbit shows that these observations may be represented by a circular orbit as well as by a general orbit. Examination of the computation, exactly as in Example No. 3, shows that the uncertainty in r_0 amounts to 0.03. This is far greater than the difference between r_0 and a which is only 0.002. Further, $r_0'^2$ will not affect the definitely known part of G_0^2 in the expression $p = r_0 (G_0^2 - r_0'^2)$. Hence we could put $p = a = r_0$ and proceed with a circular orbit.

However, continuing the computation for the general orbit, the representation of the first and third places is

$$(O - C) \begin{cases} \Delta \alpha \cos \delta \\ \Delta \delta \end{cases} \begin{matrix} \text{I} \\ - 3'' \\ + 13 \end{matrix} \begin{matrix} \text{III} \\ + 1'' \\ - 12 \end{matrix}$$

The intervals being equal, and the residuals in δ being practically equal with opposite signs, it is seen at once that they can be removed by an arbitrary change of z'_0 , so that no differential correction is necessary. Changing $\log z'_0$ from 8.93672 to 8.92958 removes the residuals in declination.

The elements were not computed from the values of the heliocentric coördinates and their velocities given above as the arc was extended and a new orbit computed. This second orbit was derived by differential correction on the basis of the series for $f, g, \partial f, \partial g$ and is given as Example No. 9.

EXAMPLE NO. 6.

DIRECT SOLUTION AND COMPLETE ELIMINATION OF PARALLAX.

To illustrate the method of the *complete* elimination of the parallax in the case of a direct solution the orbit of Comet *c* 1909 (Daniel) is selected. The computations were performed by S. EINARSSON and Miss W. AITKEN.

I.

The observations, reduced to 1909.0, upon which this work is based are:

	1909 Gr. M. T.	α (1909.0)	δ (1909.0)	Observer.
I	December 11.74000	94° 23' 43".3	+ 38° 6' 15".2	Aitken
II	December 15.70788	94 29 59 .2	41 12 20 .6	Aitken
III	December 18.75589	94 30 46 .4	+ 43 25 59 .3	Aitken

The solar coördinates, reduced to (1909.0), are:

X_s	-0.1780529	X_u	-0.1093873	X_{uu}	-0.0562684
Y_s	-0.8882410	Y_u	-0.8971793	Y_{uu}	-0.9010569
Z_s	-0.3853174	Z_u	-0.3891974	Z_{uu}	-0.3908799

The velocities of the solar coördinates for the middle date are:

$\log X'_s$	0.004615
$\log Y'_s$	8.994638n
$\log Z'_s$	8.632210n

II.

$\log \alpha'_0$	8.148227	$\log \delta'_0$	9.883022	$\log (\tan \delta)_0'$	0.130182
$\log \alpha''_0$	9.568359n	$\log \delta''_0$	9.932162n	$\log (\tan \delta)_0''$	9.468466

Also:

$\log (t_0 - t_1)$	0.598559	$\log t_{uu}$	8.834140	$\log (t_{uu} - t_1)$	0.846082
$\log (t_{uu} - t_0)$	0.484016	$\log t_1$	8.719597		

Determination of $p_a \rho$ and $p_b \rho$ (general parallax factors). Footnote, page 313.

	I	II	III		I	II	III
α	94° 24'	94° 30'	94° 31'	$\cos \delta$	9.8959	9.8765	9.8610
α_0	251 14	307 33	348 59	$\sin (\alpha - \alpha_1)$	9.5945n	9.7367	9.9838
$\alpha - \alpha_0$	203 10	146 57	105 32	$\log p_a \rho$	0.4722n	0.6312	0.9243
δ_0	-22 49	23 31	10 24	$\log I$	9.4845n	9.4775n	9.1175n
$\log d_1$	5.4943			$\cos (\alpha - \alpha_2)$	9.9635n	9.9233n	9.4278n
$\cos \delta_0$	9.9646	9.9623	9.9926	$\sin \delta$	9.7903	9.8187	9.8373
$\cos \alpha_0$	9.5075n	9.7849	9.9919	$\log II$	9.7184n	9.7043n	9.2579n
$\log \Delta_1 X$	4.9664	5.2415n	5.4790n	sub	9.8535	9.8362	9.5817
$\sin \alpha_0$	9.9763n	9.8992n	9.2812n	$\log \frac{t_1}{t_0}$	9.3380n	9.3137n	8.6992n
$\log \Delta_1 Y$	5.4352	5.3558	4.7683	$\log p_b \rho$	0.1407n	0.1224n	9.5079n
$\sin \delta_0$	9.5886n	9.6010n	9.2565n				
$\log \Delta_1 Z$	5.0829	5.0953	4.7566				

The geocentric parallax factors ($p^s\rho$) were taken from *Publications of the Lick Observatory*, Volume I, all of the observations having been made at the Lick Observatory.

$\log p_a^m\rho$	5.1578n	5.3168	5.6099	$\log p_b^m\rho$	4.8323	4.8080n	4.1935n
$\log p_a^s\rho$	5.5187n	5.5810n	5.4505n	$\log p_b^s\rho$	4.8223	4.8617	
add	0.1570	9.9230	9.6470	add	8.3670	9.1193	
$\log p_a\rho$	5.6757n	5.2398n	5.0975	$\log p_b\rho$	3.1893n	3.9273	4.1935n

Formule page 313, footnote.

	I	II	III
$\cos \delta$	9.8959	9.8765	9.8610
$\sin \alpha$	9.9987	9.9987	9.9986
$\log p_a\rho$	5.6757n	5.2398n	5.0975
$\log I$	5.5703n	5.1150n	4.9571
$\sin \delta$	9.7903	9.8187	9.8373
$\cos \alpha$	8.8849n	8.8946n	8.8962n
$\log p_b\rho$	3.1893n	3.9273	4.1935n
$\log II$	1.8645n	2.6406n	2.9270
add	.0000	0.0015	0.0040
$\log \Delta_1 X$	5.5703n	5.1165n	4.9611
$\log I$	4.4565n	4.0109n	3.8547
$\log II$	2.9783n	3.7447	4.0294n
add	0.0142	9.9273	9.6950
$\log \Delta_2 Y$	4.4707n	3.6720n	3.5497n
$\log \Delta_2 Z$	3.0852	3.8038n	4.0545
$\Delta_1 X$	+ 0.00000926	- 0.00001744	- 0.00003013
$\Delta_2 X$	- 3178	- 1308	+ 914
ΔX	- 279	- 305	- 210
$\Delta_1 Y$	+ 0.00002724	+ 0.00002269	- 0.00000587
$\Delta_2 Y$	- 296	- 47	- 35
ΔY	+ 243	+ 222	+ 55
$\Delta_1 Z$	+ 0.00001210	+ 0.00001245	+ 0.00000563
$\Delta_2 Z$	+ 12	- 64	+ 113
ΔZ	+ 122	+ 118	+ 68

For Middle Date

$(R) \cos D \cos \Delta$	9.039088n
$(R) \cos D \sin \Delta$	9.952869n
$\tan \Delta$	0.913781
Δ	263° 2' 47".5
$\Delta - \alpha_s$	168 32 48.3
$\sin \Delta$	9.996794n
$\cos \Delta$	9.083012n
$(R) \cos D$	9.956075
$(R) \sin D$	9.590157n
$\tan D$	9.634082n
$\sin D$	9.597145n
$\cos D$	9.963063
$\log (R)$	9.993012
$\sin \delta_s$	9.818730
$\log I$	9.415875n
$\cos \delta_s$	9.876420
$\cos (\Delta - \alpha_s)$	9.991264n
$\log II$	9.830747n
add	0.141358
$\log c = \cos \psi$	9.972105n
ψ	159° 41' 0".0
$\log s = \sin \psi$	9.540590

	I	II	III
(X)	- 0.1780808	- 0.1094178	- 0.0562894
(Y)	- 0.8882167	- 0.8971571	- 0.9010514
(Z)	- 0.3853052	- 0.3891856	- 0.3908731

Formule page 313, footnote.

	P	X	Y	Z
$\log \Delta_2 P_{III}$		4.9611	3.5497n	4.0545
$\log I$		5.5597	4.1483n	4.6531
$\log \Delta_2 P_I$		5.5703n	4.4707n	3.0852
$\log II$		6.0543n	4.9547n	3.5692
add		9.8324	0.0630	0.0344
$\log \text{numer.}$		5.8867n	5.0177n	4.6875
$\log \Delta_2 P_{II}$		5.1165n	3.6720n	3.8038n
$\log \text{denom.}$		5.4626n	4.5181n	4.6499n
$\log d_x; d_y; d_z$		9.2051n	0.3343	0.3202n

Auxiliary Quantities, j , a , etc.

Formulæ pages 313 and 315, footnote.

$\text{colog } (R)^3$	0.0210
$\log 2 \frac{d_x}{\theta, \theta_{\infty}}$	1.9524n
add	9.9949
$\log []$	1.9473n
$\log \Delta_2 X_n$	5.1165n
$j \cos a$	7.0638
$\log 2 \frac{d_y}{\theta, \theta_{\infty}}$	3.0816
add	0.0004
$\log []$	3.0820
$\log \Delta_2 Y_n$	3.6720n
$j \sin a$	6.7540n
$\log 2 \frac{d_z}{\theta, \theta_{\infty}}$	3.0675n
add	9.9996
$\log []$	3.0671n
$\log \Delta_2 Z_n$	3.8038n
$j \tan d$	6.8709
$\tan a$	9.6902n
a	333° 54'
$\sin a$	9.6435n
$\cos a$	9.9533
$\log j$	7.1106
$\tan d$	9.7604

$\log X_n \Delta_2 X_n$	4.1556
$\log Y_n \Delta_2 Y_n$	3.6249
add	0.1121
sum	4.2677
$\log Z_n \Delta_2 Z_n$	3.3940
add	0.0545
$\log R_n \Delta_2 R_n$	4.3222
$\log \Delta_2 R_n$	4.3292

$a - \alpha$	239° 24'
$\log c_2$	9.9349n
$\cos (a - \alpha)$	9.7068
$\log c_3$	6.8174n
$\tan \delta_n$	9.9423
$\log \tan \delta_n \cos (a - \alpha)$	9.6491n
$\tan d$	9.7604
sub	0.2489
$\log c_1$	0.0093n
$\log \gamma$	9.9256

From formulæ A III we have

$\log \Gamma'$	9.666908n
$\log \Phi$	9.787231n
then	
$\log \frac{\Gamma'}{\gamma}$	9.7413n
$\log \left(1 - \frac{\Gamma'}{\gamma}\right)$	0.1907
$\log j$	7.1106
$\log c_2$	9.9349n
$\text{colog } (2 \Phi)$	9.9118n
$\log \beta$	7.1480n

Formulæ page 315, foot-note.

Computing $(\Delta X)'$ from ΔX_1 , ΔX_2 , and ΔX_3 from the same formulæ used in getting α_0' from α_1 , α_2 and α_3 , and similarly for $(\Delta Y)'$ and $(\Delta Z)'$ their values are $\log (\Delta X)' = 5.9341$, $\log (\Delta Y)' = 6.2867n$ and $\log (\Delta Z)' = 5.7519n$.

Then

P	X	Y	Z
$\log P_0'$	0.004615	8.994638n	8.632210n
$\log (\Delta P)'$	5.9341	6.2867n	5.7519n
add	0.000037	0.000850	0.000572
sum	0.004652	8.995488n	8.632782n
$\log III$	6.0426	7.1467n	7.0903n
add	0.000048	0.006109	0.012278
$\log [P]'$	0.004700	9.001597n	8.645060n

From III we have $\log N = 9.702125$, and $\log m = 9.823715$; hence

$\log c_1 \alpha_0'$	8.1575n	$\log 3 \Delta_2 R_n$	4.8063
$\log c_2 (\tan \delta)_0'$	0.0651n	$\log R_n$	9.9932
add	0.0053	$\log 3 \Delta_2 R_n$	4.8133
$\log []$	0.0704n	$\log \left(1 + 3 \frac{\Delta_2 R_n}{R_n}\right)$	0.0000
$\log j$	7.1106	$\log m$	9.823715
$\text{colog } N$	0.2979	$\log I$	9.823715
$\log \Delta H$	7.4739	$\log II = \frac{\Delta H}{R_n \cos \delta_n}$	7.6495
		add	0.002644
		$\log (m)$	9.826359

This completes the computation of the auxiliary quantities necessary for the complete elimination of the parallax. The values of both (m) and m are used in the solution of the general equation when solution is made without hypothesis

regarding the eccentricity as indicated in the *Synopsis of Formulae*, page 317. For a parabolic solution this last quantity is not needed, and the computation of the auxiliaries stops with the determination of $[X]'$, $[Y]'$ and $[Z]'$. The remaining portions of the computation are exactly like those already given in previous examples and will, therefore, be omitted here.

SUPPLEMENTARY DISCUSSION.

Following the lines of the complete discussion of Example No. 1, we have:

$$\begin{array}{rcccl} t'' - t, & 3.9679 & \alpha'' - \alpha, & 375''.9 & \delta'' - \delta, & 11165''.4 \\ t''' - t, & 3.0480 & \alpha''' - \alpha, & 47''.2 & \delta''' - \delta, & 8018''.7 \end{array}$$

On the basis of a 3.5 day interval these give:

$$\begin{array}{cccc} f'_a & f''_a & f'_\delta & f''_\delta \\ 331''.6 & 276''.8 & 9208''.6 & 638''.8 \\ 54.2 & & 9848.8 & \\ \text{Mean 193} & & \text{Mean 9529} & \end{array}$$

The means 193'' and 9529'' in f'_a and f'_δ respectively are the principal quantities upon which the velocities depend. The ratios of f'' to f' are then 1.4 and 0.07 respectively. Assuming a like ratio to connect f''' and f'' we would have $f'''_a = 393''$ and $f'''_\delta = 43''$. The velocity and acceleration are more determinate in δ than in α . The error in α' , due to the neglected third differences is $f'''_a = 66''$. Comparing this with f' it is seen that the error in α' is about $\frac{1}{8} \alpha'$. In α'' the error is

$$\frac{f'''_a - f''_a}{3} \cdot \frac{f'''_\delta}{f''_\delta} = \frac{0.92}{3} \cdot \frac{f'''_\delta}{f''_\delta} = 36''.$$

Comparing this with f'' , the principal part of α'' , it is seen that the error in α'' is about $\frac{1}{8} \alpha''$. Similarly the error in δ' is about $\frac{1}{1360} \delta'$, and in δ'' it is about $\frac{1}{160} \delta''$.

On account of the great uncertainty in α' we can conclude here that it is not worth while to take into account the *complete* elimination of the parallax. It is to be noted here, moreover, that, for the mere representation of the observations used in determining the orbit, it will be very seldom that the *complete* elimination should be used. It has been given here, however, in complete detail merely for the purpose of showing by an example how it is used. In first orbits these considerations are of more theoretical than practical value.

A glance at the parallax factors shows that even for a small ρ the p_s would not exceed the errors of observation, and could, therefore, be omitted from the computation without loss of accuracy.

Further, the differences between the successive values of $p_s \rho$ are constant, viz., $6''.2$, hence their second differences equal 0. But advantage has not been taken of these ideas here, for, as noted before, we give the complete elimination of the parallax for the sake of an example.

Let us now investigate the influence of parallax upon the accuracy of z . The computation for *N A IIIa* shows

$$N = 0.0000 + 0.4995 + 0.0041.$$

The accuracy of N and therefore of $\frac{1}{m}$ depends upon that of the second term which involves α'' . Since the second differences of $p_a \rho$ are 0, the accuracy of z will *not* be affected by the neglect of parallax. (The actual numerical value may be affected).

In order to test the workings of the complete parallax elimination numerically three orbits of this comet were computed from these same observations (those given at the beginning of this example). One was computed neglecting parallax entirely; in the second the parallax was completely eliminated as illustrated here; and in the third, the three observations were corrected for parallax at the start on the basis of the values of ρ found for the dates of observation from the second orbit.

All three will represent the observations upon which they are based, on account of the short intervals. While the accelerations are not affected by neglecting parallax, the velocities will be affected by neglecting the f^1 of the parallax factors, viz, $6''.2$. This, however, will not become appreciable until we try to represent a place with a long interval, as x'_0 , y'_0 and z'_0 are multiplied by g in such representation.

Let us call the three orbits (a), (b), and (c) respectively in the order given previously. The values of z resulting for these are:

	z	
(a)	0.4283	(Parallax neglected)
(b)	0.4364	(Parallax completely eliminated)
(c)	0.4370	(Observations corrected for parallax).

The second and third agree within the uncertainty of calculation. By neglecting parallax, however, we get a difference in z of 0.0084. (This difference would have been greater but for the fact that f^1 in the parallax factors is constant, and $p_a \rho$ is so small for all three places.)

The resulting representations of the first and the third observations for the three orbits are:

O - C	(a)	(b)	(c)
$\partial \alpha_i$	+11"	+13"	+12"
$\partial \delta_i$	-36	-38	-38
$\partial \alpha_{ii}$	-8	-7	-6
$\partial \delta_{ii}$	+17	+15	+14

On account of the short intervals (small g) the representations are practically the same for all three orbits. But compared with an observation taken 1910, March 3, two and one half months after the date of the middle observation (the epoch), the residuals are:

(O - C)	(a)	(b)	(c)
$\partial \alpha$	-9'.8	-5'.4	-5'.0
$\partial \delta$	+6.5	-2.0	-2.7

The last two are practically the same and are much better than the first. Therefore we would get a much better ephemeris by making the complete elimination of the parallax. This, however, is only of theoretical interest as we would have a second or later orbit from long intervals for an extended ephemeris.

EXAMPLE NO. 7.

DIRECT SOLUTION IN THE CASE OF MULTIPLE MATHEMATICAL SOLUTIONS, AND APPLICATION OF THE CRITERIA FOR THE DETERMINATION OF THE PHYSICAL SOLUTION.

This is an example for the direct solution of an orbit in the case of three mathematical solutions, showing the application of the criteria, page 291, for the determination of the physical solution. The object chosen is Comet *a* 1910. The orbit has been derived from short intervals by Miss SOPHIA H. LEVY.

The observations referred to 1910.0, inclusive of the aberration terms, are:

1910 Gr. M. T.	α (1910.0)	δ (1910.0)	Place of Observation
Jan. 18.1287	303° 32' 52"	- 20° 53' 27"	Rome
Jan. 19.0166	307 1 0	- 17 43 31	Algier
Jan. 20.0266	310 11 24	- 14 25 38	Algier

Exactly as in Example No. 1 the following values were found for the auxiliary quantities:

Ia		
$X_u + 0.470804$	$\log X'_0$ 9.951520	$\log R$ 9.992992
$Y_u - 0.792680$	$\log Y'_0$ 9.645695	A 300° 42' 28"
$Z_u - 0.343865$	$\log Z'_0$ 9.283051	D - 20 27 15
ψ 6° 33' 20"	$\log c$ 9.997151	$\log s$ 9.057539
IIa		
$\log \alpha'_s$ 0.557023	$\log \delta'_s$ 0.541683	$\log (\tan \delta)_s$ 0.583929
$\log \alpha''_s$ 1.670860n	$\log \delta''_s$ 1.252009n	$\log (\tan \delta)''_s$ 1.450666n
IIIa		
$\log n$ 1.51032n	$\log \Phi$ 0.60412n	$\log N$ 1.79951
$\log C_1$ 8.74264	$\log \frac{\lambda}{\kappa}$ 0.33778	$\log \kappa$ 7.51196
$\log C_2$ 9.04095n		
$\log a_s$ 0.62217	$\log a_y$ 9.63653	$\log a_z$ 0.49703
IVa		
$\log a^2$ 1.39873	q'^2 + 0.01176	c' + 0.31794
$\log b$ 0.63606	$\log p'$ 9.24434	$\frac{h}{s}$ + 0.73413
$\log G_s^2$ 0.01382	$\log h$ 8.92332	$\frac{1}{m}$ - 274.72

Since $p' > 0$ and $c > 0$, according to (49) page 288, three parabolic solutions are possible if $\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2$ is negative. The computation gives

$$\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 = -0.00006.$$

We have then a case where three mathematical parabolic solutions are

possible. The graphical solutions for z (denoting the three values of z by z_1^p , z_2^p and z_3^p) give

$$z_1^p = 1.0167 ; \quad z_2^p = 0.8643 ; \quad z_3^p = 0.6348 .$$

Owing to the difficulty of interpolation in the parts of the Table (Part 7) where $\frac{1}{m} = 274$ and ψ is less than 10° , it is more convenient to determine the values of z corresponding to the *general* solution through the solution of equation (27), page 240. From the Table, taking account of third differences the two values of z are 1.09 and 0.88. The solution of the equation gives (denoting the two solutions by z_1^g and z_2^g)

$$z_1^g = 1.0899 ; \quad z_2^g = 0.8810 .$$

We have now to determine whether one (and which one, if any) of the three parabolic solutions agrees with one of the general solutions, within the uncertainty of the solution. At first glance it appears that the best agreement is between z_2^p and z_2^g . The parabola resulting from z_2^p is therefore the preliminary, assuming that the uncertainty of z_2^g , which is to be investigated at once, is not materially less than the difference between z_2^p and z_3^p , otherwise the parabola must be completely discarded. Or, in case the next difference, that between z_1^p and z_1^g , should also come into consideration from its being smaller than the uncertainty in z_1^g , naturally, a definite decision would not, in general, be possible. *Such an uncertainty, however, is excluded in consequence of the discussion in Part 7.*

The geocentric motion being large, the uncertainty of the solution depends (*cf.* page 270) upon the errors of observation (in this case parallax has been neglected), and upon the third and higher differences of the geocentric motion, which were neglected in determining α'_0 , δ'_0 , α''_0 , δ''_0 .

On account of the great difficulty in observing this comet the resulting error, error of observation combined with the error arising from neglecting the geocentric and barycentric parallax, can be placed at about $15''$.

Reducing the differences of the observed α and δ to correspond to 1 day interval we have

f_a^I	f_a^{II}	f_δ^I	f_δ^{II}
14064".7	2860".5	12834".7	1090".3
11355 .8		11802 .2	
<hr/> 12710		<hr/> 12318	

The numbers 12710 and 12318 are the average values of f_a^I and f_δ^I , respectively, and are the principal values in determining α'_0 and δ'_0 , respectively.

Let us denote the errors of observation for the three observations by e_i , e_{ii} , and e_{iii} , respectively. Under the assumption of $e = e_i = e_{ii} = e_{iii} = 15''$, and also with a choice of sign such that the errors do not neutralize each other, but numerically augment, we have, according to page 272 (case of equal or nearly equal intervals), as the *maximum error* of α' and δ' the value e or $15''$, and for the *maximum error* of α'' and δ'' the value $4e$ or $60''$, since the constant interval i is unity.

There now remains the estimation of the error caused by the neglected third differences f_a^m and f_s^m (cf. page 274). The ratios of f^m to f^i in α and δ are 0.225 and 0.888, respectively. If we assume, further, the same ratios of f^m to f^i , we shall have $f_a^m = 644''$ and $f_s^m = 91''$, signs not being taken into consideration. The differences here being referred to a one day interval, we have, then, from (21), page 275, $\frac{f_a^m}{6} + e = 122''$ to compare with $f_a^i = 12710''$, and $\frac{f_s^m}{6} + e = 30''$ with $f_s^i = 12318''$, in order to ascertain the ratios as the error $\partial\alpha'_0$ to α'_0 , and $\partial\delta'_0$ to δ'_0 , respectively. These gave $\partial\alpha'_0 = \frac{1}{104} \alpha'_0$, and $\partial\delta'_0 = \frac{1}{411} \delta'_0$.

In order to estimate the error in α''_0 , we have to compare, (17) page 274, $\frac{\theta_m - \theta_i}{3} \cdot \frac{f_a^m}{6} + 4e = \frac{0.12 k}{3} \cdot \frac{644''}{k} + 60'' = 85''$ with $f_a^i = 2860''$. Hence the error $\partial\alpha''_0 = \frac{1}{33} \alpha''_0$. Similarly we find the error $\partial\delta''_0 = \frac{1}{17} \delta''_0$.

Further, the computation gives

$$N = \alpha'_0 \tan \delta''_0 - \alpha''_0 (\tan \delta'_0)' + \alpha'_0 (\tan \delta'_0)'' = -14.987 + 179.800 - 101.786 = 63.027.$$

It is seen that the last two terms are the decisive ones for determining the accuracy of N . And in these the most uncertain quantities are α''_0 and δ''_0 . The principal part of the uncertainty of the second term is then $\frac{1}{33}$ of its value, or a little more than 5 units. Similarly the uncertainty of the last term is $\frac{1}{17}$ of its value, or about 6 units. The maximum uncertainty in N is from this assumed to be about 11 units, or 0.17 of its value.

For $\frac{1}{m}$ the computation gives the value -274.72 , and for ψ the value $6^{\circ}.56$. Since N and $\frac{1}{m}$ are proportional to each other, the maximum uncertainty of $\frac{1}{m}$ is $\pm 0.17 (274.72) = \pm 47$ units. Since several orders of differences would have to be taken into account in the Table, it is not convenient, in this case, to draw upon it for an approximate value of the differential coefficient $\frac{\partial \frac{1}{m}}{\partial z}$. But a formula for this can be derived through the differentiation of the equation (27) page 240. Putting, for brevity, $z^2 - 2cz + 1 = \frac{r^2}{R^2} = u^2$ we get

$$\frac{\partial \frac{1}{m}}{\partial z} = \frac{1}{z} \left[\frac{3(z-c)}{u^3} - \frac{1}{m} \right].$$

For the two *general* solutions, the values of $\log r$ corresponding to z_1^g and z_2^g are given by $\log r_1^g = 9.16743$ and $\log r_2^g = 9.19766$, respectively. We have here then for $\partial z_1 = \partial z_2 = 0.01$

$$1. \quad \partial \frac{1}{m} = 38.15; \quad 2. \quad \partial \frac{1}{m} = -33.16.$$

The sign of $\partial \frac{1}{m}$ is naturally of no consequence. Since the approximate uncertainty of $\frac{1}{m}$, derived above, is equal to 47, we get for the uncertainty of z for the two general solutions

$$\Delta z_1^g = 0.0123; \quad \Delta z_2^g = 0.0142.$$

Therefore, having regard to the uncertainty of solutions, there is agreement only between the SECOND PARABOLIC solution, for which $\varepsilon_2^p = 0.8643$, with the SECOND GENERAL solution, for which $\varepsilon_2^g = 0.8810$, since the difference $\varepsilon_2^g - \varepsilon_2^p = 0.0167$ is but slightly greater than the uncertainty $\Delta\varepsilon_2 = 0.0142$. Both of the other parabolic solutions are therefore to be discarded.

It is true that the parabolic solution ε_2^p lies a little outside of the limits for agreement with the general solution ε_2^g ; but, since the uncertainty in the general solution can only be approximated, as above, there is no reason at hand for discarding the only parabolic solution that can come into consideration.

In accordance with the present custom in astronomy of considering the parabola the solution when the observations can be represented by both a parabola and a general orbit, the computation was continued with the value ε_2^p exactly as illustrated in Example No. 1.

EXAMPLE NO. 8.

EXAMPLE SHOWING CHANGE FROM PARABOLA TO GENERAL ORBIT DURING A DIFFERENTIAL CORRECTION.

In order to illustrate the ease with which the change from a parabolic hypothesis to the hypothesis without assumption regarding the eccentricity can be made, the orbit of Comet ϵ 1906 (KOPFF) is chosen. This affords also an example of what may be termed "cooking", an arbitrary process by means of which one or more differential corrections may be avoided. The computations were made by R. T. CRAWFORD and A. J. CHAMPREUX.

The observations upon which the work is based are those of 1906, August 24, September 5, and September 15 given under example No. 2, page 399. Using the velocities and accelerations found in that example by the aid of five observations a parabola was computed giving the following data:

$\log \rho_0$	0.181836		
$\log x_0$	0.374444	$\log x'_0$	9.856309
$\log y_0$	9.900377	$\log y'_0$	9.620749
$\log z_0$	9.116388	$\log z'_0$	9.519014
$\log r_0$	0.398224	$\log r'_0$	9.751731
$\log f_1$	9.999383	$\log g_1$	9.313470n
$\log f_{11}$	9.999600	$\log g_{11}$	9.238024

and the representation of the First and the Third Places

	I	III
$(O - C) \left\{ \begin{array}{l} \partial \alpha \\ \partial \delta \end{array} \right.$	$\begin{array}{l} - 4' 23''.5 \\ - 5 \quad 54.1 \end{array}$	$\begin{array}{l} + 0' 20''.0 \\ - 3 \quad 40.7 \end{array}$

The computation of the differential correction by [VII], using the series for ∂f and ∂g gave

$\log P_r$	9.01776n	$\log Q_r$	9.56035n
$\log P_y$	8.63205	$\log Q_y$	9.50890
$\log P_z$	7.93127n	$\log Q_z$	6.73217n
$\log P_{z_{11}}$	8.43132n	$\log Q_{z_{11}}$	8.49383

Using $P_{z_{11}}$ and $Q_{z_{11}}$ for P and Q the solution of the parabolic equation for $\partial \rho_0$ gives

$\log \partial \rho_0$	9.39320
$\log \partial x'_0$	8.15586n
$\log \partial y'_0$	8.56771n
$\log \partial z'_0$	8.54042n

and since

$\log \frac{\xi_0}{\rho_0}$	9.96627
$\log \frac{\eta_0}{\rho_0}$	9.53361n
$\log \frac{\zeta_0}{\rho_0}$	9.21661

we have

$\log \partial x_0$	9.35947
$\log \partial y_0$	8.92681n
$\log \partial z_0$	8.60981

Applying these corrections to get the new heliocentric coördinates and their velocities and computing the representation we find

	I	III
$(O - C) \left\{ \begin{array}{l} \partial \alpha \\ \partial \delta \end{array} \right.$	$\begin{array}{l} - 0' 39''.3 \\ - 11 \quad 49.7 \end{array}$	$\begin{array}{l} + 0' 4''.1 \\ - 0 \quad 30.7 \end{array}$

Comparing these residuals with those we tried to remove it is seen that on the whole there is no improvement. But the material difference is that in the last set most of the error has been forced into the first declination. Seeing, therefore, that we can expect no improvement in the residuals under the parabolic hypothesis, this has to be abandoned and solution made without hypothesis regarding the eccentricity. *At this point is seen one of the most important features of Leuschner's Method.* All that has to be done now is to return to the P and Q expressions and solve for $\partial\rho$ by $\partial\rho \cdot \frac{P_{z_0} - P_{z_1}}{Q_{z_0} - Q_{z_1}}$. Doing this we have

$\log \partial\rho_0$	9.76496n		
$\log \partial x_0'$	9.49924n	$\log \partial x_0$	9.73123n
$\log \partial y_0'$	9.36311	$\log \partial y_0$	9.29857
$\log \partial z_0'$	7.94697n	$\log \partial z_0$	8.98157n

Applying these we have

$\log x_0$	0.262400	$\log x_0'$	9.604906
$\log y_0$	9.775354n	$\log y_0'$	9.811791
$\log z_0$	8.542685	$\log z_0'$	9.507221
$\log r_0$	0.284378	$\log r_0'$	9.273669

which give

	I	III
$\log f$	9.998681	9.999103
$\log g$	9.313245n	9.237867
$(O - C) \left\{ \begin{array}{l} \partial\alpha \\ \partial\delta \end{array} \right.$	$\begin{array}{l} + 2' 22''.1 \\ + 2 22.5 \end{array}$	$\begin{array}{l} + 0' 2''.9 \\ + 2 1.3 \end{array}$

These residuals are due to the fact that the initial residuals were too large to be removed entirely by the linear relations.

By comparing them with those to be removed at the start, and assuming that the linear relations in the equations for $\partial\alpha$ and $\partial\delta$ are hereafter sufficient, it is seen that, if we now apply to our last values of x_0, y_0, z_0 , and their velocities corrections which come from a $\partial\rho, \partial x_0', \partial y_0'$ and $\partial z_0'$ amounting to [9.54449n] of the corrections $\partial\rho$, etc., from the foregoing solution, the residuals will be considerably reduced.

Doing this we arrive at an orbit which gives residuals

	I	III
$(O - C) \left\{ \begin{array}{l} \partial\alpha \\ \partial\delta \end{array} \right.$	$\begin{array}{l} - 0' 41''.7 \\ - 1 12.1 \end{array}$	$\begin{array}{l} + 0' 6''.7 \\ - 0 34.3 \end{array}$

These were then removed by the application of two more differential corrections. The first left residuals of

	I	III
$(O - C) \left\{ \begin{array}{l} \partial\alpha \\ \partial\delta \end{array} \right.$	$\begin{array}{l} + 3''.0 \\ + 2.5 \end{array}$	$\begin{array}{l} + 0''.1 \\ + 1.9 \end{array}$

The last differential correction removed these. This final differential correction was very simple as it was performed with the same values of the coefficients A, B , and C as were derived for the preceding differential correction.

The eccentricity of the resulting orbit is 0.52 and the period $6\frac{2}{3}$ years.

This orbit was computed before Professor LEUSCHNER had derived either his

criteria for the feasibility of a parabolic solution or the closed expressions for ∂f and ∂g so that on the one hand a direct parabolic solution was performed and on the other hand the solution of the differential corrections was made with the series forms for ∂f and ∂g . Had the criteria and the closed expressions been available, a general solution would have been made in the first place and the numerous approximations to remove the parabolic residuals would have been unnecessary.

EXAMPLE NO. 9.

DIFFERENTIAL CORRECTION OF A DIRECT SOLUTION OF THE ORBIT OF A MINOR PLANET USING THE SERIES FOR f , g , \mathcal{V} , AND \mathcal{Z} .

For an example of a differential correction of a direct solution of the orbit of a minor planet using the series for f , g , \mathcal{V} , and \mathcal{Z} the computation is here given for the orbit of 1909 HC. The direct solution for the orbit of this minor planet has been given as Example No. 5. Instead of using the same observations as for the direct solution the arc is extended so as to use observations of November 7 and December 18 for the first and third places, respectively. The middle place (November 26) of the direct solution is here retained as the middle place for the differential correction. The computations were made by R. K. YOUNG and checked by O. W. LANZENDORF.

The observations used here were made by Mr. H. C. WILSON at the Lick Observatory. Reduced to 1910.0 they are:

1910 Gr. M. T.	α (1910.0)	δ (1910.0)
I Nov. 7.8205	3° 50' 24".3	+ 25° 11' 10".5
II Nov. 26.7480	3 13 3 .0	22 29 31 .3
III Dec. 18.6262	4 54 19 .5	+ 20 14 51 .9

Solar Coördinates.

X_s -0.7000428	X_s -0.4306623	X_{sun} -0.0628037
Y_s -0.6429179	Y_s -0.08143311	Y_{sun} -0.9007035
Z_s -0.02788955	Z_s -0.3532487	Z_{sun} -0.3907160

The direct solution (Example No. 5) gave the following quantities:

$\log x_0$ 0.446074	$\log x'_0$ 9.42066n	$\log \rho_0$ 0.41425
$\log y_0$ 9.976418	$\log y'_0$ 9.68296	$\log R' \cos \psi$ 9.759411
$\log z_0$ 0.124811	$\log z'_0$ 8.93672	$\log \frac{\xi_0}{\rho_0}$ 9.9649
$\log r_0$ 0.510072	$\log r'_0$ 8.70503n	$\log \frac{\eta_0}{\rho_0}$ 8.7148
		$\log \frac{\zeta_0}{\rho_0}$ 9.5827

Computing the residuals for the first and third places (November 7 and December 18, respectively), we find:

	I	III		I	III
$\log \theta$	9.51265	9.57556	δ_s	+ 25° 9' 53".6	+ 20° 14' 49".1
$\log f$	9.999325	9.999088	δ_0	+ 25 11 11 .5	+ 20 14 52 .8
$\log g$	9.51243n	9.57526	$(O - C) \mathcal{Z} \delta$	+ 77 .9	+ 3 .7
α_c	3° 50' 23".8	4° 54' 47".9	$\log \rho$	0.381618	0.448477
α_0	3 50 26 .8	4 54 19 .8	ρ_s	+ 2".5	+ 0".3
$(O - C) \mathcal{Z} \alpha$	+ 3 .0	- 28 .1	ρ_2	+ 1 .0	+ 0 .9

The complete computation of the differential correction to remove these residuals is given in the following:

DIFFERENTIAL CORRECTION.

A VIIc

$\log \rho_0$	0.4084	$\log 4 \frac{r_0'}{r_0}$	8.7970	$\log 3 \theta_{,,,}^2$	9.5024
$\log R \cos \psi$	9.7594n			$\log 3 \theta_{,}^2$	9.6282
sub	0.0879				
$\log (\rho - R \cos \psi)$	0.4963	$\log \frac{\cos \beta}{2 r_0'}$	7.6449	$\log \theta_{,,,}^3$	8.5380
$\log r_0$	0.5101			$\log \theta_{,}^3$	8.7267
$\cos \beta$	9.9862				

ω	x	y	z	ω	x	y	z
$\log \omega_0'$	9.4207n	9.6830	8.9367	$\log II$	8.3686n	8.3526	6.1915
$\log 4 \frac{r_0'}{r_0} \omega_0$	9.2431	8.7734	8.9218	$\log (I = 3 \theta_{,}^2 \omega_0)$	0.0743	9.6046	9.7530
sub	0.2212	9.9429	8.5430	add	9.9913	0.0236	0.0001
$\log []$	9.6419n	9.6259	7.4648	$\log ()$	0.0656	9.6282	9.7531
$\log II$	8.1799n	8.1639	6.0028	$\log II$	7.7105	7.2731	7.3980
$\log (I = 3 \theta_{,}^2 \omega_0)$	9.9485	9.4788	9.6272	$\log I$	9.9640	8.7139	9.5818
sub	0.0073	9.9784	9.9999	add	0.0024	0.0155	0.0028
$\log ()$	9.9558	9.4572	9.6271	$\log f_{\omega,,,}$	9.9664	8.7294	9.5846
$\log II$	7.6007	7.1021	7.2720				
$\log I$	9.9643	8.7142	9.5820				
add	0.0019	0.0105	0.0021				
$\log f_{\omega,}$	9.9662	8.7247	9.5841				

	I	III		I	III
$\log \sin \alpha$	8.8259	8.9327	$\log f_y \sin \alpha$	7.5506	7.6621
$\log \cos \alpha$	9.9990	9.9984	$\log f_x \cos \alpha$	9.9652	9.9648
			add	0.0017	0.0022
$\log f_y \cos \alpha$	8.7237	8.7278	$\log ()$	9.9669	9.9670
$\log f_x \sin \alpha$	8.7921	8.8991	$\sin \delta$	9.6286	9.5392
sub	9.2319	9.6844	$\log () \sin \delta$	9.5955	9.5062
$\log []$	7.9556n	8.4122n	$\log f_x \cos \delta$	9.5408	9.5569
$\log \rho$	0.3816	0.4485	sub	9.1278	9.0928
$\log A$	7.5740n	7.9637n	$\log []$	8.6686	8.5990n
			$\log \rho$	0.3816	0.4485
$\log g$	9.5124n	9.5753	$\log B$	8.2870n	8.1505
$\log C$	9.1308n	9.1268			

Determination of P_z , $P_{z,,,}$, Q_z , and $Q_{z,,,}$.

$\alpha_{,,,} - \alpha_i$	+1° 4' 24".1	$\log \partial_i \alpha_i$	5.1194
$\sin (\alpha_{,,,} - \alpha_i)$	8.2691	$\log \partial_i \alpha_{,,,}$	6.1066n
$C_i C_{,,,} \sin (\alpha_{,,,} - \alpha_i)$	6.5627n		
$C_{,,,} \cos \alpha_{,,,} \partial_i \alpha_i$	4.2456	$C_{,,,} \sin \alpha_{,,,} \partial_i \alpha_i$	3.1789
$C_i \cos \alpha_i \partial_i \alpha_{,,,}$	5.2364	$C_i \sin \alpha_i \partial_i \alpha_{,,,}$	4.0633
sub	9.9532	sub	9.0393
$\log \text{numer.}$	5.1896n	$\log \text{numer.}$	4.0026n
$\log P_i$	8.6629	$\log P_y$	7.4759
$A_i C_{,,,} \cos \alpha_{,,,}$	6.692n	$A_i C_{,,,} \sin \alpha_{,,,}$	5.6335n
$A_{,,,} C_i \cos \alpha_i$	7.0935	$A_{,,,} C_i \sin \alpha_i$	5.9204
sub	0.1472	sub	0.1809
$\log \text{numer.}$	7.2407n	$\log \text{numer.}$	6.1013n
$\log Q_z$	0.7140	$\log Q_y$	9.5746

	I	III		I	III
$\log Q_x \cos \alpha$	0.7130	0.7124	$\log P_x \cos \alpha$	8.6619	8.6613
$\log Q_y \sin \alpha$	8.4005	8.5073	$\log P_y \sin \alpha$	6.3018	6.4086
add	0.0021	0.0027	add	0.0019	0.0024
$\log ()$	0.7151	0.7151	$\log ()$	8.6638	8.6637
$\log C \sin \delta$	8.7594n	8.6660	$\log C \sin \delta$	8.7594n	8.6660
$\log II$	9.4745n	9.3811	$\log II$	7.4232n	7.3297
$\log (I = B)$	8.2870n	8.1505	$\log (I = \partial \delta)$	6.5761	5.2537
add	0.0273	0.0248	add	9.9335	0.0036
$\log \text{numer.}$	9.5018n	9.4059	$\log \text{numer.}$	7.3567n	7.3333
$\log C \cos \delta$	9.0875n	6.0991	$\log C \cos \delta$	9.0875n	9.0991
$\log Q_z$	0.4143	0.3068	$\log P_z$	8.2692	8.2342

 Determination of the Correction $\partial \rho$, etc.

$\log P_{x_{III}}$	8.2342	$\log P_x$	8.6629	$\log P_y$	7.4759
$\log P_{x_I}$	8.2692	$\log Q_x \partial \rho_0$	8.1169	$\log Q_y \partial \rho_0$	6.9775
sub	8.9240	sub	9.8546	sub	9.8342
$\log (P_{x_{III}} - P_{x_I})$	7.1582n	$\log \partial x_0'$	8.5175	$\log \partial y_0'$	7.3101
$\log Q_{x_{III}}$	0.3068	$\log P_{x_I}$	8.2692	$\log P_{x_{III}}$	8.2342
$\log Q_{x_I}$	0.4143	$\log Q_{x_I} \partial \rho_0$	7.8172	$\log Q_{x_{III}} \partial \rho_0$	7.7097
sub	9.4485	sub	9.8108	sub	9.8458
$\log (Q_{x_{III}} - Q_{x_I})$	9.7553n	$\log \partial z_0'$	8.0800	$\log \partial z_0'$	8.0800
$\log \partial \rho_0$	7.4029				
$\log \partial x_0$	7.3678				
$\log \partial y_0$	6.1178				
$\log \partial z_0$	6.9856				

Application of Corrections.

ω	x	y	z	ω	x'	y'	z'
Old $\log \omega_0$	0.446074	9.976418	0.124811	Old $\log \omega_0'$	9.420660n	9.682960	8.936720
$\log \partial \omega_0$	7.3678	6.1178	6.9856	$\log \partial \omega_0'$	8.5175	7.3101	8.0800
add	0.000362	0.000060	0.000316	add	9.942018	0.001837	0.056556
$\log \omega_0$	0.446436	9.976478	0.125127	$\log \omega_0'$	9.362678n	9.684797	8.993276

With these corrected values of x_0 , y_0 , z_0 , x'_0 , y'_0 , and z'_0 the representation of the first and third places is now computed. This gives

	I	III
$\log f$	9.999323	9.999090
$\log g$	9.512425n	9.575261
α_c	3° 50' 26".3	4° 54' 20".0
δ_c	+25 11 11.7	+20 14 53.1
$(O - C) \left\{ \begin{array}{l} \partial \alpha \\ \partial \delta \end{array} \right.$	$\begin{array}{l} + 0".5 \\ - 0.2 \end{array}$	$\begin{array}{l} - 0".2 \\ - 0.3 \end{array}$

The representation being satisfactory the elements are computed from the formulæ A VIII. They are

Epoch	1910 Nov. 26.7332	Gr. M. T.
M_\odot	220° 55' 20".1	
ω	267 4 48.6	1910.0
Ω	260 40 32.2	
i	18 29 37.0	
φ	2 48 15.0	
$\log a$	0.494229	
μ	643".673	

EXAMPLE NO. 10.

DIFFERENTIAL CORRECTION OF A DIRECT SOLUTION OF A COMET ORBIT USING THE SERIES FOR f , g , ∂f , AND ∂g .

For an example of a differential correction of a direct solution of a comet orbit using the series for f , g , ∂f , and ∂g the case of Comet c 1905 (GIACOBINI) is chosen. It may be remarked here that such a differential correction is rarely necessary. The computations were made by R. T. CRAWFORD.

Using the series for f , g , ∂f , and ∂g the computation for a differential correction of a parabolic orbit is the same as for a general orbit except for the solution of $\partial \rho_0$. The details of a solution for a differential correction to a general orbit have been given in Example No. 9. Therefore, only the results of the solution are given for this example as far as the determination of $\partial \rho_0$.

The observations upon which the direct solution was based are:

1905 Gr. M. T.	α (1905.0)	δ (1905.0)	Observer.
I Dec. 6.6837	215° 24' 40"	+20° 59' 39"	GIACOBINI, Nice
II Dec. 7.9267	216 52 39	20 26 45	DUGAN, Princeton
III Dec. 8.0311	215 0 4	+20 23 55	SMITH, Lick

Solar Coördinates.

X_s	-0.264668	X_u	-0.243650	X_m	-0.241892
Y_s	-0.870396	Y_u	-0.875416	Y_m	-0.875790
Z_s	-0.377585	Z_u	-0.379770	Z_m	-0.379924

NOTE.—The solar coördinates for the middle date have been corrected so as to eliminate the geocentric parallax for that date.

The direct solution gave the following quantities:

$\log x_0$	9.93819n	$\log x'_0$	8.61736	$\log \rho_0$	0.17092
$\log y_0$	8.62269	$\log y'_0$	0.03792n	$\log R \cos \psi$	9.73417
$\log z_0$	9.95306	$\log z'_0$	9.80582n	$\log \frac{\xi_0}{\rho_0}$	9.87479n
$\log r_0$	0.09651	$\log r'_0$	9.72018n	$\log \frac{\eta_0}{\rho_0}$	9.74997n
				$\log \frac{\zeta_0}{\rho_0}$	9.54322

Computing the residuals for the first and third places we find:

	I	II		I	III
$\log \theta$	8.33006	7.25432	δ_0	+21° 0' 8"	+20° 23' 58"
$\log f$	9.99995	0.00000	δ_0	+20 59 42	+20 23' 58
$\log g$	8.33004n	7.25432	(O - C) $\partial \delta$	-26	0
α_0	215° 24' 11"	216° 59' 59"	$\log \rho$	0.17280	0.17077
α_0	215 24 36	216 59 59	ρ_a	-4"	-5"
(O - C) $\partial \alpha$	+25	0	ρ_b	+3	+3

Exactly as in Example No. 9 the following quantities are found:

A VIIa

	I	III		
$\log f_x$	9.87485n	9.87479n	$\log P_x$	9.35369
$\log f_y$	9.74984n	9.74997n	$\log P_y$	9.23080
$\log f_z$	9.54342	9.54323	$\log P_z$	8.99603n
$\log A$	8.20580	7.12955n	$\log Q_x$	9.67083n
$\log B$	7.80277n	6.73992	$\log Q_y$	0.01664
$\log C$	8.15727n	7.08355	$\log Q_z$	9.59009

In cases where the intervals are nearly equal it is better practice to use $P_{z_{III}}$ and $Q_{z_{III}}$ for P_z and Q_z , respectively, in forming P and Q so as to represent the last declination and force whatever residual, that may still remain, into the first declination. In this case, however, the second interval is very short (about 0.10 of a day), so that a better representation will be had by representing the first declination and forcing the remaining residual into the third declination. This is accomplished by calling $P_x = P_z$, and $Q_z = Q_z$. Doing this we then form P and Q as follows:

$\log \frac{\cos \beta}{r_0^2}$	9.68360	$\log x'_0 P_x$	7.97105
$\log x'_0 Q_z$	8.28819n	$\log y'_0 P_y$	9.26872n
$\log y'_0 Q_y$	0.05456n	add	9.97754
$\log z'_0 Q_z$	9.39591n	sum	9.24626n
$\frac{\cos \beta}{r_0^2}$	+0.48261	$\log z'_0 P_x$	8.80185
$-x'_0 Q_z$	+0.01942	add	9.80658
$-y'_0 Q_y$	+1.13387	$\log P$	9.05284
$-z'_0 Q_z$	+0.24884	For the first approximation	
Q	+1.88474	$\log \partial \rho$	8.77759
$\log Q$	0.27525		

Solution for $\partial \rho_0$.

	I	2	3		I	2	3
$\log \partial \rho$	8.77759	8.55757	8.56061	$\log \frac{(1-3\cos^2 \beta)}{2r_0^3}$	9.25431n	9.25431n	9.25431n
$\log P_x$	9.35369	9.35369	9.35369	$\log (\partial \rho)^2$	7.55518	7.11514	7.12122
$\log Q_x \partial \rho$	8.44842n	8.22840n	8.23144n	$\log V$	6.80949n	6.36945n	6.37553n
sub	0.05091	0.03138	0.53159	P	+0.112938	+0.112938	+0.112938
$\log \partial x'_0$	9.40460	9.38507	9.38528	$-\frac{1}{2}(\partial x'_0)^2$	- 32223	- 29451	- 29480
$\log P_y$	9.23080	9.23080	9.23080	$-\frac{1}{2}(\partial y'_0)^2$	- 5819	- 8794	- 8759
$\log Q_y \partial \rho$	8.79423	8.57421	8.57725	$-\frac{1}{2}(\partial z'_0)^2$	- 7492	- 6400	- 6411
sub	9.80212	9.89182	9.89095	$-V$	+ 645	+ 234	+ 237
$\log \partial y'_0$	9.03292	9.12262	9.12175	$Q \partial \rho$	+0.068049	+0.068527	+0.068525
$\log P_z$	8.99603n	8.99603n	8.99603n	$\log Q \partial \rho$	8.83282	8.83586	8.83585
$\log Q_z \partial \rho$	8.36768	8.14766	8.15070	$\log Q$	0.27525	0.27525	0.27525
sub	0.09177	0.05759	0.05796	$\log \partial \rho$	8.55757	8.56061	8.56060
$\log \partial z'_0$	9.08780n	9.05362n	9.05399n				

The result of the third trial being practically the same as that of the second the solution is completed and we have

$\log \partial \rho_0$	8.56060	$\log \partial x'_0$	9.38528
$\log \partial y'_0$	8.43539n	$\log \partial y'_0$	9.12175
$\log \partial z'_0$	8.10382	$\log \partial z'_0$	9.05399n

Applying these corrections to the heliocentric coördinates and their velocities given by the direct solution we have the corrected orbit.

The representation of the first and third places for the corrected orbit gives :

$$(O - C) \begin{Bmatrix} \partial\alpha & \alpha'' & . & \alpha'' \\ \partial\delta & \delta & & -2 \end{Bmatrix}$$

The elements were not computed from the data derived from this differential correction as an observation of December 9 became available. The orbit published in *Lick Observatory Bulletin No. 87* was based upon the first two observations given above and that of December 9.

EXAMPLES B. ORBITS BASED UPON PREVIOUS APPROXIMATIONS USING CLOSED EXPRESSIONS FOR f , g , \mathcal{J} , AND $\mathcal{J}g$.

EXAMPLE NO. 11.

CASE OF AN INITIAL PARABOLIC ORBIT.

For an example of the computation of an orbit based upon a previous approximation, where this previous approximation is a parabola, the case of Comet α 1910 has been selected. This example shows also the use of the closed expressions for f , g , $\mathcal{J}f$ and $\mathcal{J}g$ for a parabolic orbit. The computations were made by W. F. MEYER and Miss S. H. LEVY.

BI

The observations upon which this work is based are:

	1910 Gr. M. T.	α (1910.0)	δ (1910.0)	Observer.
I	Jan. 18.1287	303° 32' 51".9	- 20° 53' 27".0	ZAPPA (Rome)
II	Feb. 5.6211	326 41 0.4	+ 5 35 18.6	AITKEN (Lick)
III	March 13.0440	336 11 15.4	+ 15 38 53.9	AITKEN (Lick)

The solar coördinates for these dates are

X_s	+0.4570857	$(X)_s$	+0.7149449	X_{III}	+0.9847873
Y_s	-0.7993384	$(Y)_s$	-0.6232607	Y_{III}	-0.1255285
Z_s	-0.3467534	$(Z)_s$	-0.2703833	Z_{III}	-0.0544576

The coördinates for the middle date, $(X)_{II}$, etc., are those corrected for the partial elimination of the geocentric parallax.

The Elements and Constants for the Equator 1910.0 of the initial parabolic orbit which are used for the previous approximation are:

$$\begin{aligned} T &= 1910 \text{ Jan. } 16.7838 \text{ Gr. M. T.} \\ \log q &= 9.067800 \\ x &= r [9.880210] \sin (321^\circ 55' 16''.0 + v) \\ y &= r [9.985880] \sin (64^\circ 48' 26''.6 + v) \\ z &= r [9.843920] \sin (350^\circ 16' 53''.8 + v) \end{aligned}$$

From the computation of the preliminary orbit we take

$$\log \rho_s = 9.91944 \quad \log \rho_a = 0.17717 \quad \log \rho_{III} = 0.37814$$

so that the three dates corrected for aberration become

$$t_s = \text{Jan. } 18.1239 \quad t_a = \text{Feb. } 5.6124 \quad t_{III} = \text{March } 13.0302$$

Computation of the Constants of the Artificial Initial Orbit.

$t_a - T$	19.8186	$\log \sqrt{\frac{2}{q}} \cos \frac{1}{2} \tau_0$	0.225693	$\log x_0'^2$	0.104200
$\log (t_a - T)$	1.297073	$\sin a$	9.880210	$\log y_0'^2$	0.053904
$\log q^{3/2}$	8.601700	$\cos (A' + \frac{1}{2} \tau_0)$	9.946197	add	0.276610
$\log M_0$	2.695373	$\log x_0'$	0.052100	sum	0.380810
v_0	132° 1' 39".4	$\sin b$	9.985880	$\log z_0'^2$	9.627672
$\frac{1}{2} v_0$	66 0 49.7	$\cos (B' + \frac{1}{2} \tau_0)$	9.81537911	add	0.070609
$A' + \frac{1}{2} v_0$	27 56 5.7	$\log y_0'$	9.02695211	$\log G_0^2$	0.451419
$B' + \frac{1}{2} v_0$	130 49 16.3	$\sin c$	9.843920		
$C' + \frac{1}{2} v_0$	56 17 43.5	$\cos (C' + \frac{1}{2} \tau_0)$	9.744223		
		$\log z_0'$	9.813836		

Computation of the Constants of the Artificial Initial Orbit—*Continued*.

$\log \rho_0$	0.177170	$\log r_0^2$	6.457802	$\log x_0 r_0'$	9.781001
$\cos \delta_0$	9.997931	$\log y_0^2$	8.596664	$\log y_0 y_0'$	9.325284
$\log \sigma_0$	0.175101	add	0.056019	add	0.130390
		sum	9.513821	sum	9.911391
$\cos \alpha_0$	9.922024	$\log z_0^2$	9.239900	$\log z_0 z_0'$	9.433786
$\log \xi_0$	0.097125	add	0.185317	add	0.124818
$\log (X)_0$	9.854273	$\log r_0^2$	9.699138	$\log r_0 r_0'$	0.036209
sub	9.874628	$\log r_0$	9.849569	$\log r_0'$	0.186640
$\log x_0$	9.728901				
$\sin \alpha_0$	9.739782n	Test.			
$\log \eta_0$	9.914883n	$\log r_0^2$	0.451461		
$\log (Y)_0$	9.794669n	$\log G_0^2$	0.451419		
sub	9.503663	sub	5.99		
$\log y_0$	9.298332n	$\log \frac{1}{a}$	6.44		
$\tan \delta_0$	8.990553				
$\log \zeta_0$	9.165654				
$\log (Z)_0$	9.431980n				
sub	0.187970				
$\log z_0$	9.619950				

The ρ_0 , r_0' , y_0' , and z_0' for our middle date are now the same elements as in the initial orbit. But since we are here using the observed α_0 and δ_0 to get x_0 , y_0 , and z_0 these will be different from the corresponding heliocentric coördinates for this date from the preliminary orbit. The elements to be corrected, then, are the heliocentric coördinates and their velocities just found here for the middle date.

Computation of the Residuals for the First and Third Dates by Means of the Closed Expressions
for f , g , ∂f , and ∂g .

A VIIIa		B II		
Elements T , q and v_0 .		Determination of f and g for First and Third Places.		
$\log 2 r_0$	0.150599	I	III	
$\log (r_0 r_0')^2$	0.072418	$t - T$	1.31586	55 22216
sub	9.294994	$\log (t - T)$	0.119210	1.742113
$\log p$	9.367412	$\log M$	1.519637	3.142540
$\log q$	9.066382	v'	41° 58' 33".4	147° 29' 52".7
		$\frac{1}{2} v$	20 59 16.7	73 44 56.3
$\log \frac{p}{r_0}$	9.517843	$\sec \frac{1}{2} v$	0.029813	0.553080
$e \cos v_0$	9.826405n	$\sec^2 \frac{1}{2} v'$	0.059626	1.106160
		$\log r$	9.126008	0.172542
$\log \frac{1}{p}$	9.683706	sub	9.167808	9.964583
$e \sin v_0$	9.870346	$\log (r - q)$	8.234190	0.137125
$\tan v_0$	0.043941n	$\log \frac{1}{r - q}$	9.117095	0.068562
v_0	132° 6' 23".1	$\log \frac{1}{r_0 - q}$	9.885694	9.885694
		sub	9.918883	9.718991
$\sin v_0$	9.870346	$\log y_i; \log y_{iii}$	9.804577n	9.604685
$\cos v_0$	9.826405n	$\log y^2$	9.609154	9.209370
$\log e$	0.000000	$\log \frac{y^2}{r_0}$	9.759585	9.359801
$\log M_0$	2.697188	sub	9.868916	
$\log q^{\frac{3}{2}}$	8.599573	$\log f_i; \log f_{iii}$	9.628501	9.887065
$\log (t_0 - T)$	1.296761			
$t_0 - T$	19.80436	$\log 2 r r_0$	9.276607	0.323141
t_0 Jan.	36.61240	$\log p y^2$	8.976566	8.576782
T Jan.	16.80804	sub	9.998020	9.992141
		$\log []$	8.974586	0.315282
$\log r_0$	9.849569	$\log x^2$	8.583740	9.524652
sub	9.921819	$\log g_i; \log g_{iii}$	9.291870n	9.762326
$\log (r_0 - q)$	9.771388			

B II--Continued.

Representation of First and Third Places.

	I	III		I	III
$\log f x_0$	9.357402	9.615966	$\rho \cos \delta \cos \alpha$	9.666538	0.311769
$\log f y_0$	8.926833n	9.185397n	$\rho \cos \delta \sin \alpha$	9.829606n	9.951507n
$\log f z_0$	9.248451	9.507015	$\tan \alpha$	0.163068n	9.639738n
$\log g x'_0$	9.343970n	9.814426	α_c	304° 29' 15".1	336° 25' 50".3
$\log g y'_0$	9.318822	9.789278n	α_n	303 32 58 .3	336 11 12 .3
$\log g z'_0$	9.105706n	9.576162	$(O-C) \partial \alpha$	- 56 16 .8	- 14 38 .0
X	+0.4570857	+0.9847873	$\sin \alpha$	9.916059n	9.601907n
$f x_0$	+0.2277205	+0.4130150	$\cos \alpha$	9.752990	9.962169
$g x'_0$	-0.2207853	+0.6522683	$\rho \cos \delta$	9.913548	0.349600
ξ	+0.4640209	+2.0500706	$\rho \sin \delta$	9.472926n	9.808727
Y	-0.7993384	-0.1255285	$\tan \delta$	9.559378n	9.459127
$f y_0$	-0.0844954	-0.1532489	δ_c	-19° 55' 42".7	+16° 3' 25".5
$g y'_0$	+0.2083638	-0.6155712	δ_n	-20 53 19 .1	+15 38 56 .1
η	-0.6754700	-0.8943486	$(O-C) \partial \delta$	- 57 36 .4	- 24 29 .4
Z	-0.3467534	-0.0544576	$\sin \delta$	9.532561n	9.441844
$f z_0$	+0.1771948	+0.3213771	$\cos \delta$	9.973183	9.982718
$g z'_0$	-0.1275574	+0.3768442	$\log \rho$	9.940365	0.366882
ζ	-0.2971160	+0.6437637	$\log p_a'' \rho$	0.745	0.862n
			$\log p_\delta'' \rho$	0.838	0.713
			$\log p_a'$	0.805	0.495n
			$\log p_\delta'$	0.898	0.346
			p_a	+6".4	-3".1
			p_δ	+7 .9	+2 .2

B III 2

Differential Correction.

$R_0 \cos D_0 \cos A_0$	9.854273	$\log \rho_0$	0.17717
$R_0 \cos D_0 \sin A_0$	9.794669n	$\log R_0 \cos \psi$	9.95856
$\tan A_0$	9.940396n	sub	9.81577
A_0	-41° 4' 50".0	$\log r_0 \cos \beta$	9.77433
$\sin A_0$	9.817644n	$\cos \beta$	9.92476
$\cos A_0$	9.877248	$\log x'_0 \xi_0$	0.14922
$R_0 \cos D_0$	9.977025	$\log y'_0 \eta_0$	9.94184
$R_0 \sin D_0$	9.431980n	add	0.20961
$\tan D_0$	9.454955n	sum	0.35883
$\sin D_0$	9.437988n	$\log z'_0 \zeta_0$	8.97949
$\cos D_0$	9.983033	add	0.01777
$\log R_0$	9.993992	$\log \rho_0 \varphi_0$	0.37660
$\sin \delta_n$	8.988484	$\log \varphi_0$	0.19943
$\log I$	8.426472n	$\log \frac{\xi_0}{\rho_0}$	9.91996
$(A_0 - \alpha_n)$	-7° 45' 50".4	$\log \eta_0$	9.73771n
$\cos (A_0 - \alpha_n)$	9.996001	$\log \rho_0$	
$\cos \delta_n$	9.997931	$\log \frac{r_0}{\rho_0}$	8.98848
$\log II$	9.976965		
add	9.987598		
$\cos \psi$	9.964563		

	I	III		I	III
$\log \varphi_0$	0.19943	0.19943	$\log f_z \cos \alpha$	9.08415n	9.76815n
$\log \sqrt{2} \frac{\cos \beta}{y}$	0.27070n	0.47059	$\log f_z \sin \alpha$	9.87633n	9.63941n
add	9.25127	0.18628	sub	9.92357	9.53788
$\log \Phi$	9.45070n	0.65687	$\log []$	9.79990	9.17729n
$\log x_0' r_0 y$	9.70625n	9.50635	$\log \rho$	9.94036	0.36688
$\log \sqrt{2} x_0$	9.87942	9.87942	$\log I$	9.85954	8.81041n
add	9.69015	0.15338	$\log (p_a \rho)$	5.431	5.548n
$\log []$	9.39640	0.03280	$\cos \delta$	9.973	9.983
$\log \frac{y^3}{r r_0}$	0.43815n	8.79194	$\csc \rho^2$	0.119	9.266
$\log g_z$	9.83455n	8.82474	$\log II$	5.523	4.797n
$\log y_0' r_0 y$	9.68110	9.48120n	add	0.00002	0.00004
$\log \sqrt{2} y_0$	9.44885n	9.44885n	$\log A_{f_i}; \log A_{f_{iii}}$	9.85956	8.81045n
add	9.84946	0.28515	$\log f_y \sin \alpha$	9.24722	9.40789
$\log []$	9.29831	9.76635n	$\log f_z \cos \alpha$	9.71326	9.99967
$\log g_y$	9.73646n	8.55829n	add	0.12774	0.09898
$\log z_0' r_0 y$	9.46799n	9.26809	$\log ()$	9.84100	0.09865
$\log \sqrt{2} z_0$	9.77046	9.77046	$\sin \delta$	9.53256n	9.44184
add	0.00288	0.11876	$\log \sin \delta ()$	9.37356n	9.54049
$\log []$	9.47087	9.88922	$\log f_z \cos \delta$	9.71768	9.59162
$\log g_z$	9.90902n	8.68116	sub	0.16220	9.09673
$\log I = f \frac{\xi_0}{\rho_0}$	9.54846	9.80702	$\log []$	9.87988n	8.63722n
$\log y^2 x_0$	9.33805	8.93827	$\log I$	9.93952	8.27034
$\log II$	8.56367	9.16389	$\log (p_a \rho)$	5.524	5.399
$\log III = g_z \Phi$	9.28525	9.48161	$\log II$	5.643	4.665
I	+0.35356	+0.64124	add	0.00002	0.00011
II	+0.36616	+0.14584	$\log B_{f_i}; \log B_{f_{iii}}$	9.93954	8.27045
III	+0.19286	+0.30311	$\log g_y \cos \alpha$	9.48945n	8.52046n
f_z	+0.91258	+1.09019	$\log g_z \sin \alpha$	9.75061	8.42665n
$\log f_{z_i}; \log f_{z_{iii}}$	9.96027	0.03750	sub	0.18979	9.38221
$\log I = f \frac{\eta_0}{\rho_0}$	9.36621n	9.62477n	$\log []$	9.94040n	7.80886n
$\log y^2 y_0$	8.90748n	8.50770n	$\log A_{g_i}; \log A_{g_{iii}}$	0.00004n	7.44198n
$\log II$	9.13310n	8.73332n	$\log g_y \sin \alpha$	9.65252	8.16020
$\log III = g_y \Phi$	9.18716	9.21516n	$\log g_z \cos \alpha$	9.58754n	8.78691
I	-0.23238	-0.42147	add	9.20786	0.09210
II	-0.13586	-0.05411	$\log ()$	8.79540	8.87901
III	+0.15387	-0.16412	$\log \sin \delta ()$	8.32796n	8.32085
f_y	-0.21437	-0.63970	$\log g_z \cos \delta$	9.88220n	8.66388
$\log f_{y_i}; \log f_{y_{iii}}$	9.33116n	9.80598n	sub	9.98771	0.08029
$\log I = f \frac{\xi_0}{\rho_0}$	8.61698	8.87554	$\log []$	9.86991	8.40114n
$\log y^2 z_0$	9.22910	8.82932	$\log B_{g_i}; \log B_{g_{iii}}$	9.92955n	8.03426
$\log II$	9.45472	9.05494	$\log g$	9.29187n	9.76233
$\log III = g_z \Phi$	9.35972	9.33803	$\log C_i; \log C_{iii}$	9.35151n	9.39545
I	+0.04140	+0.07508	$\log A_{f_i} A_{g_{iii}}$	7.30154n	
II	+0.28492	+0.11348	$\log A_{f_{iii}} A_{g_i}$	8.81049	
III	+0.22894	+0.21779	sub	0.01325	
f_z	+0.55526	+0.40635	$\log A$	8.82374n	
$\log f_{z_i}; \log f_{z_{iii}}$	9.74450	9.60890			

	For a_1	For b_1		For c_1		For e_1
$\log C, A_{g,m}$	6.79349		$\log c_1$	8.44369n	$\log \partial \alpha,$	3.52850n
$\log I$	6.70955n	6.54648			$\sin 1''$	4.68558
$\log C_m A_{g,}$	9.39549n		$\log C, \cos \delta,$	9.32469n	$\cos \delta,$	9.97318
$\log II$	8.99740	9.35766n	$\log z_0 B_{g,}$	9.54950n	$\log \partial, \alpha,$	8.18726n
sub	0.00223	0.00067	add	0.20301		
$\log a_1; \log b_1$	8.99963n	9.35833n	$\log c_1$	9.75251	$\log \partial \alpha_{m,}$	2.94350n
					$\cos \delta_{m,}$	9.98272
$\log C, A_{f,m}$	8.16196		$\log C_m \cos \delta_{m,}$	9.37817	$\log \partial, \alpha_{m,}$	7.61180n
$\log I$	8.07802n	7.91495	$\log z_0 B_{g,m}$	7.65421		
$\log C_m A_{f,}$	9.25501		add	0.00812	$\log A_{g,} \partial, \alpha_{m,}$	7.61184
$\log II$	8.85692n	9.21718	$\log c_1$	9.38629n	$\log A_{g,m} \partial, \alpha,$	5.62924
$\log III$	8.55264n	8.12207			sub	9.99546
I	-0.011968	+0.008221			$\log e,$	7.60730
-II	+0.071932	-0.164885	$\log d_1$	8.82374n	$\log A_{f,} \partial, \alpha_{m,}$	7.47136n
$\pm III$	-0.035698	-0.013245	$\log d_3$	9.93954	$\log A_{f,m} \partial, \alpha,$	6.99771
$a_2; -b_2$	+0.024266	-0.169909	$\log d_1$	8.27045	sub	0.12581
$\log a_2; \log b_2$	8.38500	9.23021			$\log e_2$	7.59717n
$\log C, \sin \delta,$	8.88407				$\log \partial \delta,$	3.53862n
$\log I$	8.63706	8.80013n			$\log e_3$	8.22420
$\log II$	9.65845n	9.22788				
sub	0.03949	0.13782			$\log \partial \delta_{m,}$	3.16714n
$\log a_3; \log b_3$	9.69794	9.36570n			$\log e_4$	7.85272
$\log C_m \sin \delta_{m,}$	8.83729					
$\log I$	8.79946	8.43920n				
$\log II$	7.76316	7.33259n				
sub	9.95809	9.96462				
$\log a_4; \log b_4$	8.75755	8.40382n				

 For a, b, c .

	$\log \frac{c_3}{c_2}$	1.30882n		$\log \frac{d_3}{d_1}$	1.11580n
$\log II$	9.69382n	$\log II$	0.53903n	$\log II$	8.90599
$\log III$	0.11543	$\log III$	0.47413	$\log III$	8.72310n
$I = a_3$	+0.49881	$I = b_3$	-0.23211	$I = e_3$	+0.016757
-II	+0.49411	-II	+3.45962	-II	-0.080536
-III	-1.30445	-III	-2.97940	-III	+0.052857
a	-0.31153	b	+0.24811	e	-0.010922
$\log a$	9.49350n	$\log b$	9.39465	$\log e$	8.03830n

As the residuals of the first place are much greater than those of the third we shall take, in formulæ A VIIa, $P_z = P_{z_1}$ and $Q_z = Q_{z_1}$, so that $P_{z_{111}}$ and $Q_{z_{111}}$ are not needed.

$\log a_1 b$	8.39428n	$\log a_3 P_z$	7.88878	A VII a	
$\log a b_1$	8.85183	$\log b_3 P_y$	7.75548		
sub	0.12992	ϵ_3	+0.0167573	$\log \frac{1}{2} a_0$	6.14
$\log \text{denom.}$	8.98175n	$-a_3 P_x$	-0.0077407	$\log x_0' P_z$	8.24294
$\log b \epsilon_1$	7.00195	$-b_3 P_y$	-0.0056949	$\log y_0' P_y$	8.41673
$\log b_1 \epsilon$	7.39663	$\epsilon_3 P_{z_1}$	+0.0033217	$\log z_0' P_z$	7.58269
sub	0.17094	$\log \epsilon_3 P_{z_1}$	7.52136	$\log P$	+0.000138
$\log \text{numer.}$	7.17259n	$\log P_{z_1}$	7.76885	$2a_0$	-0.017496
$\log P_x$	8.19084			$-x_0' P_x$	-0.026105
				$-y_0' P_y$	-0.003825
				$-z_0' P_z$	-0.047288
				$\log P$	8.67475n
$\log a_1 \epsilon$	7.03793	$\log a_3 Q_z$	8.93458n	$\log \frac{\cos \beta}{r_0^2}$	0.22562
$\log a \epsilon_1$	7.10080n	$\log b_3 Q_y$	8.70119	$\log x_0' Q_z$	9.28874n
sub	0.27073	d_3	+0.870040	$\log y_0' Q_y$	9.36244
$\log \text{numer.}$	7.37153	$a_3 Q_z$	-0.086016	$\log z_0' Q_z$	9.98264n
$\log P_y$	8.38978n	$b_3 Q_y$	+0.050256	$\cos \beta$	+1.68119
		$-c_3 Q_{z_1}$	+0.83428	r_0^2	-0.19442
$\log b d_1$	8.21839n	$\log c_3 Q_{z_1}$	9.92131n	$-x_0' Q_x$	-0.23038
$\log Q_z$	9.23664n	$\log Q_{z_1}$	0.16880	$-y_0' Q_y$	+0.96082
				$-z_0' Q_z$	+2.60605
$\log a d_1$	8.31724			Q	0.41598
$\log Q_y$	9.33549n			$\log Q$	

First approximation for $\partial \rho$,

viz, $\frac{P}{Q}$ gives

$$\log \partial \rho_1 = 8.25877n$$

Solution for $\partial \rho_1, \partial x_0', \partial y_0', \partial z_0'$

Trials.					
	1	2		1	2
$\log \partial \rho_1$	8.25877n	8.26036n	$\log \text{coef. of } (\partial \rho)^2$	0.20005n	0.20005n
$\log P_x$	8.19084	8.19084	$\log (\partial \rho)^2$	6.51754	6.52072
$\log Q_z \partial \rho$	7.49541	7.49700	$\log V$	6.71759n	6.72077n
sub	9.90221	9.90180	$\log \text{coef. of } (\partial \rho)^3$	9.9545n	9.9545n
$\log \partial x_0'$	8.09305	8.09264	$\log (\partial \rho)^3$	4.7763n	4.7811n
$\log P_y$	8.38978n	8.38978n	$\log VI$	4.7308	4.7356
$\log Q_y \partial \rho$	7.59426	7.59585	P	-0.047288	-0.047288
sub	0.06451	0.06473	$- \frac{1}{2} (\partial x_0')^2$	-77	-77
$\log \partial y_0'$	8.45429n	8.45451n	$- \frac{1}{2} (\partial y_0')^2$	-405	-405
$\log P_z$	7.76885	7.76555	$- \frac{1}{2} (\partial z_0')^2$	-218	-220
$\log Q_z \partial \rho$	8.42757	8.42916	$-V$	+522	+526
sub	9.89242	9.89287	$+VI$	+5	+5
$\log \partial z_0'$	8.31999n	8.32203n	$Q \partial \rho$	-0.047461	-0.047459
$\log (\partial x_0')^2$	6.18610	6.18528	$\log Q \partial \rho$	8.67634n	8.67632n
$\log (\partial y_0')^2$	6.90858	6.90902	$\log Q$	0.41598	0.41598
$\log (\partial z_0')^2$	6.63998	6.64406	$\log \partial \rho_f$	8.26036n	8.26034n

In spite of the large starting residuals the first approximation for $\partial\rho$ is much less than $\frac{r_0}{1.2}$, hence the solution is convergent.

Application of Corrections.

w	ρ	x	y	z	
Old log w_0	0.177170	9.728901	9.298332n	9.619950	
log ∂w_0	8.26034n	8.18030n	7.99805	7.24882n	
add	9.994708	9.987543	9.977684	9.998149	
log w_0	0.171878	9.716444	9.276016n	9.618099	
w'	.	x'	y'	z'	
Old log w'_0		0.052100	0.026952n	9.813836	
log $\partial w'_0$		8.09264	8.45451n	8.32203n	
add		0.004742	0.011471	9.985774	
log w'_0		0.056842	0.038423n	9.799610	
log x_0^2	9.432888	log $x_0'^2$	0.113684	log $x_0 x_0'$	9.773286
log y_0^2	8.552032	log $y_0'^2$	0.076846	log $y_0 y_0'$	9.314439
add	0.053680	add	0.283002	add	0.129580
sum	9.486568	sum	0.396686	sum	9.902866
log z_0^2	9.236198	log $z_0'^2$	9.599220	log $z_0 z_0'$	9.417709
add	0.193643	add	0.064240	add	0.122944
log r_0^2	9.680211	log G_0^2	0.460926	log $r_0 r_0'$	0.025810
log r_0	9.840105			log r_0'	0.185705
		log $\frac{2}{r_0}$	0.460925 check.		

 A VIII a, Elements T, q, v_0

log $2 r_0$	0.141135	sin v_0	9.891334
log $(r_0 r_0')^2$	0.051620	cos v_0	9.797599n
sub	9.359637	log e	9.999999
log ϕ	9.411257 -		
log q	9.110227 -	log M_0	2.625391
		log $q^{3/2}$	8.665340
log $\frac{\phi}{r_0}$	9.571152	log $(t_0 - T)$	1.290731
$e \cos v_0$	9.797598n	$t_0 - T$	19.53130
		t_0	Jan. 36.61253
log $\frac{\phi}{r_0}$	9.705628	T	Jan. 17.08123
$e \sin v_0$	9.891333		
tan v_0	0.093735n	log r_0	9.840105
v_0	128° 51' 51".4	log q	9.110227
		sub	9.910485
		log $(r_0 - q)$	9.750590

	I	II	III
Old log ρ	9.9404		0.3669
log $\partial\rho$	8.2603n		8.2603n
add	9.9908		9.9966
log ρ	9.9312	0.1719	0.3635
log $\alpha\rho$	7.6925n	7.9332n	8.1248n
$\alpha\rho$	-0.00493	-0.00857	-0.01333
t	Jan. 18.1287	36.6211	72.0440
cor. t	Jan. 18.12377	36.61253	72.03067

Determination of f and g for First and Third Places.

	I	III		I	III
$t - T$	1.04254	54.94944	$\log \gamma^2$	9.631370	9.222658
$\log (t - T)$	0.018093	1.739964	$\log \frac{\gamma^2}{r_0}$	9.791465	9.382553
$\log M$	1.352753	3.074624	sub	9.789826	
v	29° 58' 5".7	145° 34' 43".8	$\log f_i; \log f_{iii}$	9.581291	9.880072
$\frac{1}{2} v$	14 59 2.8	72 47 21.9			
$\sec \frac{1}{2} v$	0.015024	0.528878	$\log 2 r r_0$	9.281410	0.309118
$\sec^2 \frac{1}{2} v$	0.030048	1.057756	$\log p \gamma^2$	9.042827	8.633915
$\log r$	9.140275	0.167983	sub	9.864595	9.990727
			$\log []$	8.907422	0.299845
sub	8.855143	9.960210	$\log g^2$	8.538992	9.522503
$\log (r - q)$	7.965370	0.128193	$\log g_i; \log g_{iii}$	9.269496n	9.761252
$\log \sqrt{r - q}$	8.982685	0.064096			
$\log \sqrt{r_0 - q}$	9.875295	9.875295			
sub	9.940490	9.736034			
$\log \gamma_i; \log \gamma_{iii}$	9.815785n	9.611329			

Representation of First and Third Places.

	I	III		I	III
$\log f x_0$	9.297735	9.596516	$\rho \cos \delta \cos \alpha$	9.646965	0.309101
$\log f y_0$	8.857307n	9.156088n	$\rho \cos \delta \sin \alpha$	9.824865n	9.953886n
$\log f z_0$	9.199390	9.498171	$\tan \alpha$	0.177900n	9.644785n
$\log g x_0'$	9.326338n	9.818094	α_0	303° 34' 48".0	336° 11' 8".2
$\log g y_0'$	9.307919	9.799675n	α_0	303 32 58.4	336 11 12.3
$\log g z_0'$	9.069106n	9.560862	(O - C)	- 1 49.6	+ 0 4.1
X	+0.4570857	+0.9847873	$\sin \alpha$	9.920705n	9.606140n
$f x_0$	+0.1984882	+0.3949264	$\cos \alpha$	9.742805	9.961354
$g x_0'$	-0.2120010	+0.6578000	$\rho \cos \delta$	9.904160	0.347747
ξ	+0.4435729	+2.0375137	$\rho \sin \delta$	9.485345n	9.795352
Y	-0.7993384	-0.1255285	$\tan \delta$	9.581185n	9.447605
$f y_0$	-0.0719958	-0.1432477	δ_0	- 20° 52' 5".7	+ 15° 39' 27".3
$g y_0'$	+0.2031977	-0.6304857	δ_0	- 20 53 19.0	+ 15 38 56.1
η	-0.6681365	-0.8992619	(O - C) $\delta \delta$	- 1 13.3	- 0 31.2
Z	-0.3467534	-0.0544576	$\sin \delta$	9.551718n	9.431183
$f z_0$	+0.1582668	+0.3148986	$\cos \delta$	9.970533	9.983578
$g z_0'$	-0.1172481	+0.3637992	$\log \rho$	9.933627	0.364169
ζ	-0.3057347	+0.6242402	$\log p_a'' \rho$	0.745	0.862n
			$\log p_\delta'' \rho$	0.838	0.713
			$\log p_a''$	0.811	0.498n
			$\log p_\delta''$	0.904	0.349
			p_a	+6".5	-3".1
			p_δ	+8.0	+2.2

The computations given illustrate in detail all of the work necessary for a differential correction. Generally, one can substitute the new residuals into the expressions e_1, e_2, e_3 and e_4 , keeping all of the other quantities the same and resolving for a further $\partial \rho_0, \partial x'_0, \partial y'_0$ and $\partial z'_0$. This process will generally give the final ρ_0, x'_0, y'_0, z'_0 which will remove the residuals and thus save the computation of all the quantities for a second differential correction. Such a procedure is hardly sufficient in this problem. Since the original residuals which were to be removed were unusually large and combined with unequal intervals with small heliocentric

distance, it is necessary to recompute the differential coefficients based on the corrected values of the geocentric distance and of the heliocentric velocities at the middle date and the last set of residuals.

This second differential correction produced the following results :

$\log \rho_0$	0.171757				
$\log x_0$	9.716156	$\log x'_0$	0.057116	$\log \frac{2}{r_0}$	0.461142
$\log y_0$	9.275494n	$\log y'_0$	0.038560n	$\log G_0^2$	0.461142
$\log z_0$	9.618056	$\log z'_0$	9.799089		
$\log r_0$	9.839888	$\log r'_0$	0.185700		

For the representation

	I	III
$\log f$	9.579983	9.879909
$\log g$	9.268867n	9.761224
(O - C) $\left\{ \begin{array}{l} \partial \alpha \\ \partial \delta \end{array} \right.$	$\left\{ \begin{array}{l} - 0''.9 \\ - 1.0 \end{array} \right.$	$\left\{ \begin{array}{l} - 0''.3 \\ + 19.7 \end{array} \right.$

The residual in the third declination is due to the use of $P_z = P_{z'}$ and $Q_z = Q_{z'}$ for the determination of $\partial \rho_0$, and also contains any errors in the observations and any slight deviation from a parabola. An observation taken by Professor BARNARD June 7th was represented with the foregoing quantities. This observation gave for residuals $\partial \alpha = 0''$, $\partial \delta + 1' 3''$. This may be considered satisfactory for an observation taken three months after our third date on which the orbit is based.

The resulting Elements and Constants for the Equator are :

Elements.

T	1910 Jan. 17.08880 Gr. M. T.
ω	320° 57' 51".4
Ω	88 49 28.8
i	138 46 42.5
$\log q$	9.111002

Constants for the Equator 1910.0.

$$\begin{aligned} x &= r [9.876385] \sin (322^\circ 31' 36''.1 + v) \\ y &= r [9.981342] \sin (67^\circ 44' 3''.9 + v) \\ z &= r [9.856490] \sin (354^\circ 35' 3''.1 + v) \end{aligned}$$

EXAMPLE NO. 12.

CASE OF AN INITIAL ELLIPTIC ORBIT OF A COMET OR A MINOR PLANET.

For an example of the computation of an orbit based upon a previous approximation, where this previous approximation is an ellipse, the case of Comet *c* 1909 (DANIEL) has been selected. This example shows also the use of the closed expression for f , g , ∂f , and ∂g for an elliptic orbit. The computations were made by S. EINARSSON and R. K. YOUNG.

I.

The observations upon which this work is based are:

	Date Gr. M. T.	α (1909.0)	δ (1009.0)	Observer
I	1909 Dec. 7.66050	94° 09' 31".4	+34° 44' 21".5	BARNARD
II	1909 Dec. 18.75589	90 30 46 .4	+43 25 59 .3	AITKEN
III	1910 Mar. 3.47726	111 14 26 .1	+52 54 37 .2	RAMBAUD

The preliminary orbit upon which this computation is based is one that had been computed here. Its elements and the corresponding Constants for the Equator are:

Elements.

Epoch	1909 Dec. 18.75330 Gr. M. T.
M_0	2° 50' 0".9
ω	3 2 1 .7
Ω	71 9 32 .0
i	19 54 39 .0
e	0.625247
$\log a$	0.569603
μ	496."1411

Constants for the Equator 1909.0.

$$\begin{aligned} x &= r [9.976182] \sin (163^\circ 5' 12".6 + v) \\ y &= r [9.944456] \sin (83^\circ 40' 45".4 + v) \\ z &= r [9.758980] \sin (44^\circ 2' 4".4 + v) \end{aligned}$$

From the data of the computation in the preliminary orbit we take the following:

$\log \mu$	9.651336	E_0	7° 31' 30".6
v_0	15° 35' 48".2		
X_0	-0.2477581	X''_0	-0.0562888
Y_0	-0.8745050	Y''_0	-0.9010553
Z_0	-0.3793684	Z''_0	-0.3908799
		X'''_0	+0.9458926
		Y'''_0	-0.2734097
		Z'''_0	-0.1186062

These solar coördinates have been corrected for the elimination of the geocentric parallax, and referred to (1909.0).

Determination of the Constants of the Artificial Initial Orbit.

$A' + v_0$	178° 41' 0".8	$\log \rho_0$	9.651336	$\log x_0'^2$	0.005216
$B' + v_0$	99 16 33.6	$\cos \delta_0$	9.861042	$\log y_0'^2$	7.465562
$C' + v_0$	59 37 52.6	$\log \sigma_0$	9.512378	add	0.001252
				sum	0.006468
$\cos A'$	9.980797n	$\cos \alpha_0$	8.895883n	$\log z_0'^2$	9.123756
$\cos B'$	9.041762	$\log \tilde{z}_0$	8.408261n	add	0.053465
$\cos C'$	9.856681	$\log (N)_u$	8.750422n	$\log G_0^2$	0.059933
		sub	0.078702		
		$\log x_0$	8.486963	$\log \frac{2}{r_0}$	0.151486
$\cos (A' + v_0)$	9.999885n			sub	9.370468
$\log e \cos A'$	9.776847n	$\sin \alpha_0$	9.998652	$\log \frac{1}{a}$	9.430401
add	0.203675	$\log \eta_0$	9.511030	$\log a$	0.569599
sum	0.203560n	$\log (Y)_u$	9.954752n		
$\log \frac{\sin a}{V \dot{p}}$	9.299048	sub	0.133532		
$\log x_0'$	0.002608n	$\log y_0$	0.088284		
		$\tan \delta_0$	9.976235	$\log x_0 x_0'$	8.489571n
$\cos (B' + v_0)$	9.207339n	$\log \tilde{w}_0$	9.488613	$\log y_0 y_0'$	8.821065n
$\log e \cos B'$	8.837812	$\log (\tilde{Z})_0$	9.592043n	add	0.166172
add	0.127647	sub	0.252387	sum	8.987237n
sum	8.965459n	$\log z_0$	9.844430	$\log z_0 z_0'$	9.406308
$\log \frac{\sin b}{V \dot{p}}$	9.767322			add	0.210760
$\log y_0'$	8.732781n	$\log x_0'^2$	6.973926	$\log r_0 r_0'$	9.197997
		$\log y_0'^2$	0.176568	$\log r_0'$	9.048453
$\cos (C' + v_0)$	9.703775	add	0.000272		
$\log e \cos C'$	9.652731	sum	0.176840		
add	0.276257	$\log z_0'^2$	9.688860		
sum	9.980032	add	0.122249		
$\log \frac{\sin c}{V \dot{p}}$	9.581846	$\log r_0'^2$	0.299089		
$\log z_0'$	9.561878	$\log r_0$	0.149544		

II.

Determination of the Residuals of the First and Third Places (using Closed Expressions for f and g).

I		III		I		III	
$(t - E\phi)$	-11.09529	74.71784		$\log fx_0$	8.484104	8.367181	
$\log(t - E\phi)$	1.045138n	1.873424		$\log fy_0$	0.085425	9.968502	
$\log \mu(t - E\phi)$	3.740740n	4.569026		$\log fz_0$	9.841571	9.724648	
$\mu(t - E\phi)$	-1° 31' 44".8	10° 17' 50".3		$\log gx'_0$	9.282372	0.078330n	
M	1 18 16 .1	13 7 51 .2		$\log gy'_0$	8.012545	8.808503n	
E_i	3 28 38 .9	32 14 36 .0		$\log gz'_0$	8.841642n	9.637600	
$\log \sin E_i$	8.782874	9.727148		(X)	-0.2477581	+0.9458926	
$\log e'' \sin E_i$	3.893351	4.837625		fx_0	+0.0304863	+0.0232907	
$e'' \sin E_i$	2° 10' 22".6	19° 6' 45".8		gx'_0	+0.1915896	-1.1976500	
M_i	1 18 16 .3	13 7 50 .2		ξ	-0.0256822	-0.228467	
$M - M_i$	-0.2	+1.0		(Y)	-0.8745050	-0.2734097	
$E - E_i$	-0.5	+2.1		fy_0	+1.217377	+0.9300400	
E	3 28 38 .4	32 14 38 .1		gy'_0	+0.0102931	-0.0643433	
$\log \cos E$	9.999200	9.927260		η	+0.3531651	+0.5922870	
$\log e \cos E$	9.795252	9.723312		(Z)	-0.3793684	-0.1186062	
sub	9.779822	9.949870		fz_0	+0.6943383	+0.5304544	
$\log(1 - e \cos E)$	9.575074	9.673182		gz'_0	-0.0694452	+0.4341100	
$\log r$	0.144677	0.242785		ζ	+0.2455247	+0.8459582	
$E - E_0$	-4° 2' 52".2	+24° 43' 7".5		$\rho \cos \delta \cos \alpha$	8.409632n	9.358823n	
g	-2 1 26 .1	-12 21 33 .8		$\rho \cos \delta \sin \alpha$	9.547978	9.772532	
$\sin g$	8.547980n	9.330501		$\tan \alpha$	1.138346n	0.413709n	
$\log \sqrt{2a}$	0.435316	0.435316		α_c	94° 9' 33".2	111° 5' 36".5	
$\log r_i; \log r_{iii}$	8.983296n	9.765817		α_0	94 9 31 .4	111 14 26 .1	
$\log r^2$	7.966592	9.531634		(O - C) $\partial \alpha$	-1.8	+8.49.6	
$\log \frac{r^2}{r_0}$	7.817048	9.382090		$\sin \alpha$	9.998855	9.969881	
sub				$\cos \alpha$	8.860509n	9.556170n	
$\log f_i; \log f_{iii}$	9.997141	9.880218		$\rho \cos \delta$	9.549123	9.802653	
$\log 2rr_0$	0.595251	0.693359		$\rho \sin \delta$	9.390095	9.927349	
$\log p r^2$	8.320859	9.885901		$\tan \delta$	9.840972	0.124696	
sub	9.997685	9.926450		δ_c	34° 44' 12".2	53° 6' 53".0	
$\log []$	0.592936	0.619809		δ_0	34 44 21 .5	52 54 37 .2	
$\log g^2$	8.559528	0.151443		(O - C) $\partial \delta$	+9.3	-12 15 .8	
$\log g_i; \log g_{iii}$	9.279764n	0.075722		$\sin \delta$	9.755728	9.903003	
				$\cos \delta$	9.914755	9.778307	
				$\log \rho$	9.634368	0.024346	

III.

$R_0 \cos D \cos A$	8.750422n	$\sin \delta_u$	9.837277	$\log x'_0 \xi_0$	8.41087
$R_0 \cos D \sin A$	9.954752n	$\sin D$	9.599137n	$\log y'_0 \eta_0$	8.24381n
$\tan A$	1.204330	$\log I$	9.436414n	add	9.67128
A	266° 25' 31".4			sum	7.91509
$A - \alpha_u$	171 54 45 .0	$\cos \delta_u$	9.861042	$\log z'_0 \zeta_0$	9.05049
$\sin A$	9.999154n	$\cos D$	9.962692	add	0.03069
$\cos A$	8.794823n	$\cos(A - \alpha_u)$	9.995660n	$\log \rho_0 \varphi_0$	9.08118
$R_0 \cos D$	9.955598	$\log II$	9.819394n	$\log \varphi_0$	9.42984
$R_0 \sin D$	9.592043n	add	0.150455		
$\tan D$	9.636445n	$\cos \psi$	9.969849n		
$\sin D$	9.599137n			$\log \sqrt{2} \cos \beta$	0.13638
$\cos D$	9.962692	$\log \rho_0$	9.651336	$\log \frac{\cos \beta}{r_0^2}$	9.68678
$\log R_0$	9.992906	$\log R_0 \cos \psi$	9.962755n		
		sub	0.172656		
		$\log r_0 \cos \beta$	0.135411		
		$\cos \beta$	9.985867		

	I	III		I	III
$\log (t - t_0)$	1.04514	1.87342	$\log \frac{y^2}{r}$	7.82191	9.28885
$\log \theta_{III}; \log \theta$	9.28072	0.10900	$\log \left[1 - \frac{y^2}{r} \right]$	9.99711	9.90609
$\log \frac{y^2}{2a}$	7.09596	8.66100	$\log \theta \left[1 - \frac{y^2}{r} \right]$	9.27783n	0.01509
$\log (y_e)^2$	9.99946	9.97963	$\log g$	9.27976n	0.07572
$\log (y_e)$	9.99973	9.98982	sub	7.64800	9.17554
$\log \sqrt{2} \cos \beta (y_e)$	0.13610	0.12620	$\log ()$	6.92583	9.19063
$\log II$	1.15281n	0.36038	$\log I$	7.10192	9.36672n
$\log (I = \varphi_0)$	9.42984		$\log \sqrt{2} r_0 (y_e)$	0.29979	0.28388
add	9.99170	0.04819	$\log \frac{1}{2} r_0 r_0' y$	7.88027n	8.66279
$\log \Phi$	1.14451n	0.40857	add	9.99835	0.01013
$I = \log 2 r_0 y (y_e)$	9.43360n	0.20621	$\log ()$	0.29814	0.30001
$\log y^2$	7.96659	9.53163	$\log y^3$	6.94990n	9.29746
$II = \log \frac{r_0 r_0'}{\sqrt{2}} y^2$	7.01407	8.57911	colog r	9.85532	9.75722
add	9.99835	0.01013	$\log II$	7.10336n	9.35469
$\log \text{sum}$	9.43195n	0.21634	add	7.52000	8.44850
$III = \log \frac{3}{\sqrt{2}} \theta$	9.60732	0.43560	$\log []$	4.62192n	7.80319n
add - sub	9.69680	9.81741	$\log N$	5.19152	8.37279
$\log ()$	9.12874	0.03375n			
$\log y_e ()$	9.12847	0.02357n	$\log \frac{\xi_0}{\rho_0}$	8.75792n	
$\log r y$	9.12798n	0.00860	$\log \frac{\eta_0}{\rho_0}$	9.85969	
add	7.05000	8.54500	$\log \frac{\zeta_0}{\rho_0}$	9.83727	
$\log []$	6.17798	8.55360n			
$\log y$	9.98330n	9.76582			
colog r	9.85532	9.75722			
$\log M$	5.43666	8.49670			
$\log I \quad g_{\omega_i}$		g_{ω_i}	g_{ω_i}	$g_{\omega_{III}}$	$g_{\omega_{III}}$
$\log II$	9.13545	7.86562	8.69472n	9.91797n	8.64814n
add	8.63721	0.23853	9.99468	8.62730	0.22862
$\log []$	0.11975	0.00184	9.97767	9.97717	9.98843
$\log \frac{y_i^3}{r_i r_0}$	9.25520	0.24037	9.97235	9.89514n	0.21705
$\log g_{\omega_i}$	6.65568n			8.90513	0.10230
	5.91088n	6.89605n	6.62803n	8.80027n	9.12218
$\log M_i \quad m_{\omega_i}$		m_{ω_i}	m_{ω_i}	$m_{\omega_{III}}$	$m_{\omega_{III}}$
$\log N_i \omega_0'$	3.92362	5.52494	5.28109	6.98366	8.58498
add	5.19413n	3.92430n	4.75340	8.37540n	7.10557n
$\log m_{\omega_i}$	9.97606	9.98897	0.11284	9.98201	9.98536
	5.17019n	5.51391	5.39393	8.35741n	8.57034
$\log I \quad f_{\omega_i}$		f_{ω_i}	f_{ω_i}	$f_{\omega_{III}}$	$f_{\omega_{III}}$
$\log 2 m_{\omega_i}$	8.75406n	9.85683	9.83441	8.63714n	9.73991
$\log \omega_0 y_i^2$	5.47122n	5.81494	5.69496	8.65844n	8.87137
add	6.45356	8.05488	7.81103	8.01860	9.61992
$\log []$	9.95224	0.00249	0.00331	9.88696	0.07130
$\log II$	6.40580	8.05737	7.81434	8.54540n	9.69122
$\log III$	6.09258	7.74415	7.50112	8.23218n	9.37800
I	7.05539	8.04056	7.77254	9.20884n	9.53075
II	-0.0567625	+0.719167	+0.682983	-0.0433650	+0.549425
III	+0.0001237	+0.005548	+0.003170	-0.0170680	+0.238784
f_{ω_i}	+0.0011360	+0.010979	+0.005923	-0.161748	+0.339431
$\log f_{\omega_i}$	-0.0555028	+0.735694	+0.692076	-0.222181	+1.127640
	8.74431n	9.86670	9.84016	9.34670n	0.05217

A_{p_i}	A_{f_i}	A_{g_i}	A_{m_i}	$A_{f_{III}}$	$A_{g_{III}}$	$A_{m_{III}}$
$\log p_i \cos \alpha_i$	8.72721n	5.75656	4.37442n	9.61122n	8.68123n	8.12939n
$\log p_i \sin \alpha_i$	8.74317n	5.90974n	5.16905n	9.31615n	8.76972n	8.32686n
sub	8.57325	0.23116	9.92404	9.98800	9.35411	9.76019
$\log [\quad]$	7.30046	6.14090	5.09309	9.30415n	8.03534	7.88958
$\log \rho_i$	9.63437			0.02435		
$\log I$	7.66609			9.27980n		
$\log \left(II = \frac{(p_i \rho_i)}{\rho_i^2} \cos \delta_i \right)$	6.10492n			5.33067		
add	9.98791			9.99995		
$\log A_{p_i}$	7.65400	6.50653	5.45872	9.27975n	8.01099	7.86523

B_{p_i}	B_{f_i}	B_{g_i}	B_{m_i}	$B_{f_{III}}$	$B_{g_{III}}$	$B_{m_{III}}$
$\log p_i \sin \alpha_i$	9.86556	6.89491n	5.51277	0.02162	9.09163	8.53979
$\log p_i \cos \alpha_i$	7.60482	4.77139	4.03070	8.90575	8.35932	7.91646
add	0.00238	9.99672	0.01408	0.03205	0.07380	0.09274
$\log (\quad)$	9.86794	6.89163n	5.52685	0.05367	9.16543	8.63253
$\sin \delta_i$	9.75573			9.90184		
$\log \sin \delta_i (\quad)$	9.62367	6.64736n	5.28258	9.95551	9.06728	8.53437
$\log p_i \cos \delta_i$	9.75492	6.54279n	5.30869	9.74776	8.78780	8.26521
sub	9.54758	9.43495	8.79217	9.78776	9.95575	9.93374
$\log [\quad]$	9.17125n	5.97774n	4.07475n	9.53552	8.74356	8.19895
$\log I$	9.53688			9.51117n		
$\log \left(II = \frac{(p_i \rho_i)}{\rho_i^2} \right)$	5.80176			4.56970n		
add	0.00008			0.00001		
$\log B_{p_i}$	9.53696	6.34337	4.44038	9.51118n	8.71921n	8.17460n

	I	III
$\log K$	9.27976n	0.07572
$\log \rho$	9.63437	0.02435
$\log C_i; \log C_{III}$	9.64539n	0.05137

Computation of $\beta_i, \gamma_i, \delta_i, \nu_i$.

	I	III		I	III
$\log (I = C \sin \alpha)$	9.64425n	0.02082	$\log (I = C \cos \alpha)$	8.50590	9.61042n
$\log II$	4.99349	6.49795	$\log II$	6.59481	8.09927
$\log III$	5.76236n	8.16887n	$\log III$	4.49253n	6.89904n
I	-0.440810	+1.049098	I	+0.032055	-0.407773
- II	-0.0000099	-0.0003147	II	+0.000394	+0.0125680
- III	+0.0000579	+0.0147527	III	-0.000003	-0.0007926
$-\beta_1; -\beta_3$	-0.440762	+1.063536	$\gamma_1; \gamma_3$	+0.032446	-0.395998
$\log \beta_1; \log \beta_3$	9.64421	0.02675n	$\log \gamma_1; \log \gamma_3$	8.51116	9.59770n
$\log I$	8.26163	9.51256n	$\log I$	9.39998n	9.92266
$\log II$	4.83033	7.20618n	$\log II$	6.43165	8.80750n
$\log III$	4.74402n	8.47824	$\log III$	3.47419n	7.20841
I	+0.0182654	-0.325285	I	-0.2511765	+0.836880
- II	-0.0000068	+0.0016076	- II	-0.0002702	+0.0641943
- III	+0.0000055	-0.0300773	- III	+0.0000003	-0.0016159
$-\beta_2; -\beta_4$	+0.018264	-0.353755	$-\gamma_2; -\gamma_4$	-0.251446	+0.899458
$\log \beta_2; \log \beta_4$	8.26160n	9.54871	$\log \gamma_2; \log \gamma_4$	9.40045	9.95398n

	I	III		I	III
log I	6.35096	7.85543	log $\partial \alpha$	0.25527n	2.72395
log II	5.32163	7.72814	cos δ	9.91476	9.78037
add	0.03880	0.24204	log sin $1''$	4.68557	4.68557
log δ_1 ; log δ_3	6.38976	8.09747	log r_1 ; log r_3	4.85560n	7.18989
log I	9.56015n	9.83174	log $\partial \delta$	0.96848	2.86676n
log II	6.18780	8.56365n	log r_2 ; log r_4	5.65405	7.55233n
log III	4.30329	8.03751n			
I	-0.363200	+0.678800			
II	+0.000154	-0.036614			
III	+0.000002	-0.010902			
δ_2 ; δ_4	-0.363044	+0.631284			
log δ_2 ; log δ_4	9.55997n	9.80023			

Equations for Solution (Coefficients Logarithmic).

$$\begin{aligned}
 (7.65400) \partial \rho_0 + (9.64421) \partial x_0' + (8.51116) \partial y_0' + (6.38976) \partial z_0' &= (4.85560n) \\
 (9.53696) &+ (8.26160n) &+ (9.40045) &+ (9.55997n) &= (5.65405) \\
 (9.27975n) &+ (0.02675n) &+ (9.59770n) &+ (8.09747) &= (7.18989) \\
 (9.51118n) &+ (9.54871) &+ (9.95398n) &+ (9.80023) &= (7.55233n)
 \end{aligned}$$

The solution of these equations gives

$$\begin{aligned}
 \log \partial \rho_0 &8.04315n \\
 \log \partial x_0' &5.44027 \\
 \log \partial y_0' &7.00605 \\
 \log \partial z_0' &7.99555n
 \end{aligned}$$

From the value of $\log \partial \rho_0$ we have

$$\begin{aligned}
 \log \partial x_0 &6.80007 \\
 \log \partial y_0 &7.90284n \\
 \log \partial z_0 &7.88042n
 \end{aligned}$$

Applying these values of ∂x_0 , ∂y_0 , ∂z_0 , $\partial x_0'$, $\partial y_0'$, and $\partial z_0'$ to the old values of x_0 , y_0 , z_0 , x_0' , y_0' , and z_0' respectively, we have the new or corrected values

$$\begin{aligned}
 \log x_0 &8.495803 & \log x_0' &0.002596n \\
 \log y_0 &0.085441 & \log y_0' &8.724556n \\
 \log z_0 &9.839687 & \log z_0' &9.549926 \\
 \log r_0 &0.146242 & \log r_0' &9.027470 \\
 \log \rho_0 &9.640497 & \log G_0^2 &0.057168
 \end{aligned}$$

For the representation

$$\begin{aligned}
 & & \text{I} & & \text{III} \\
 \log f & & 9.997076 & & 9.876614 \\
 \log g & & 9.279742n & & 0.074726 \\
 (O - C) \left\{ \begin{array}{l} \partial \alpha \\ \partial \delta \end{array} \right. & & \begin{array}{l} -0''.9 \\ -1''.2 \end{array} & & \begin{array}{l} +2''.4 \\ -12''.6 \end{array}
 \end{aligned}$$

Then substituting these residuals to form new absolute terms in our four equations for $\partial \rho_0$, etc., keeping the other coefficients the same and resolving for further corrections $\partial \rho_0$, etc., we remove the residuals and have finally:

$$\begin{aligned}
 \log x_0 &8.495919 & \log x_0' &0.002600 \\
 \log y_0 &0.085404 & \log y_0' &8.724205n \\
 \log z_0 &9.839623 & \log z_0' &9.549805 \\
 \log r_0 &0.146200 & \log r_0' &9.027352 \\
 & & \log G_0^2 &0.057147
 \end{aligned}$$

whence

IV. (A VIIIc)

Elements.

$\log \frac{2}{r_0}$	0.154830	$\log 1/\bar{p}$	0.172602	$\frac{1}{2} v_0$	7° 37' 30".1
$\log G_0^2$	0.057147	$\log e \sin v_0$	9.199954	$\tan \frac{1}{2} v_0$	9.126694
sub	9.401790	$\tan v_0$	9.435578	$\log \sqrt{\frac{1-e}{1+e}}$	9.697282
$\log \frac{1}{a}$	9.458937	v_0	15° 15' 0".2	$\tan \frac{1}{2} E_0$	8.823976
$\log a$	0.541063	$\sin v_0$	9.420009	$\frac{1}{2} E_0$	3° 48' 52".8
$\log r_0'^2$	8.054704	$\cos v_0$	9.984432	E_0	7 37 45 .6
sub	9.995659	$\log e$	9.779944	$\sin E_0$	9.123080
$\log (G_0^2 - r_0'^2)$	0.052806	$\log e^2$	9.559888	$\log e''$	5.094369
$\log r_0'^2$	0.292399	sub		$\log e'' \sin E_0$	4.217449
$\log \bar{p}$	0.345205	$\log (1 - e^2)$	9.804150	$e'' \sin E_0$	16498".7
		$\log a$	0.541055 check	M_0	3° 2' 46".9
$\log \frac{\bar{p}}{r_0}$	0.199005	$\log a^3$	1.623189	Epoch	1909 Dec. 18.75330
sub		$\log a^{\frac{1}{2}}$	0.811594		Gr. M. T.
$\log e \cos v_0$	9.764376	Period	6.48029 years		
		$\log \kappa''$	3.550007		
		$\log \mu''$	2.738413		
		μ''	547".5362		

Constants for the Equator.

$d \ w \ D'$	$a \ x \ A'$	$b \ y \ B'$	$c \ z \ C'$
$\log r_0 w_0'$	0.148800	8.870405	9.696005
$\log r_0' w_0$	9.523271	9.112756	8.866975
sub	0.001027	0.196545	9.930316
$\log [\]$	0.149827	9.309301n	9.626321
$\sin d \cos (D' + v_0)$	9.977225n	9.136699n	9.453719
$\sin d \sin (D' + v_0)$	8.349719n	9.939204	9.693423
$\tan (D' + v_0)$	8.372494n	0.802505n	0.239704
$D' + v_0$	178° 38' 57".7	98° 57' 17".7	60° 3' 55".0
v_0	15 15 0 .2		
D'	163 23 57 .5	83 42 17 .5	44 48 54 .8
$\sin (D' + v_0)$	8.372373	9.994674	9.937816
$\cos (D' + v_0)$	9.999879n	9.192169n	9.698112
$\sin d$	9.977346	9.944530	9.755607

The details of the computation for the elements ω , Ω and i are not given here as they are exactly the same as shown in Example No. 1, page 395. They are given in the collection of elements.

The representation of the First and Third Places for the corrected orbit is computed exactly as shown on page 442 of this example, so the details are omitted. The principal quantities are

$\log f_1$	9.997075	$\log g_1$	9.279743n
$\log f_{111}$	9.876577	$\log g_{111}$	0.074712

	I	III
$(O - C) \left\{ \begin{array}{l} \partial_1 \alpha \\ \partial_1 \delta \end{array} \right.$	+1".0 +0.2	-0".4 +0.5

The corrected orbit then is given by the following collected

<i>Elements.</i>				
Epoch	1909 Dec. 18.75330 Gr. M. T.			
M_{\odot}	3°	2'	46".9	
ω	3	28	42 .6	} (1909.0)
Ω	70	58	54 .4	
i	19	26	48 .0	
e	0.602481			
$\log a$	0.541063			
μ	547".5362			

and

Constants for the Equator 1909.0.

$$\begin{aligned} x &= r [9.977346] \sin (163^{\circ} 23' 57".5 + v) \\ y &= r [9.944530] \sin (83^{\circ} 42' 17".5 + v) \\ z &= r [9.755607] \sin (44^{\circ} 48' 54".8 + v) \end{aligned}$$

EXAMPLE NO. 13.

CASE OF AN INITIAL NEARLY PARABOLIC ORBIT.

A portion of the work on HALLEY'S comet is given in this example to illustrate, in detail, the computation of a second orbit based upon a preliminary orbit. The observations used cover a range of 164 days, giving intervals of 90 and 74 days respectively. These long intervals necessitate the use of the closed expressions in f , g , ∂f , and ∂g , so that this example will show not only how a second orbit is computed but also the use of these closed expressions. Furthermore, the orbit is of the nearly parabolic type. Since the eccentricity of this orbit is well defined the formulæ 3 (Part 7, page 333) are used here instead of formulæ 2 (Part 7, page 332) which would ordinarily be used for a *very* nearly parabolic orbit.

It must be borne in mind that the computation given here is merely to show the applications of the formulæ of LEUSCHNER'S Method and is not to be taken as a final orbit for the comet as it is based upon only three observations at one apparition. The computations were made in duplicate by R. T. CRAWFORD and W. F. MEYER.

I.

The first observation is by BARNARD. The second place is a normal based upon four observations by AITKEN. The third is an observation by AITKEN.

They are:

	Date	Gr. M. T. 1909	α (1909.0)	δ (1909.0)
I	January	260.85720	94° 44' 51".6	+17° 8' 59".2
II	January	350.77347	49 56 33.7	+13 59 51.8
III	January	424.63910	8 32 55.9	+ 7 53 50.0

The dates of these observations, as given, have been corrected for aberration and the coördinates have been corrected for parallax, and referred to 1909.0.

The preliminary orbit upon which this computation is based is one that had been computed here. Its elements and the corresponding constants for the equator are:

Elements.

T = 1910	April 19.71304	Gr. M. T.
ω	111° 46' 31".70	} 1909.0
Ω	57 15 36.95	
i	162 12 32.88	
e	0.9667007	
log a	1.2458385	
log q	9.7682756	

Constants for the Equator 1909.0.

$$\begin{aligned} x &= r [9.9851611] \sin (145^\circ 48' 18".47 + v) \\ y &= r [9.9884770] \sin (239 21 52.17 + v) \\ z &= r [9.5354488] \sin (189 5 11.47 + v) \end{aligned}$$

From the data of the computation in the preliminary orbit we take for the middle date,

$\log \rho_0$	0.1324984	$\log X_0$	8.9582938n
v_0	239° 50' 22".75	$\log Y_0$	9.9536780
		$\log Z_0$	9.5909691n

Determination of Heliocentric Coördinates and Their Velocities for the Middle Date in the Artificial Orbit.

$A' + v_0$	25° 38' 41".2	$\log \rho_0$	0.1324984	$\log x_0'^2$	7.9247134
$B' + v_0$	110 12 14.9	$\cos \delta_0$	9.9869084	$\log y_0'^2$	9.8978608
$C' + v_0$	68 55 34.2	$\log \sigma_0$	0.1194068	add	0.0045955
				sum	9.9024563
$\cos A'$	9.9175742n	$\cos \alpha_0$	9.8085846	$\log z_0'^2$	8.5579156
$\cos B'$	9.7072079	$\log \xi_0$	9.9279914	add	0.0192132
$\cos C'$	9.9945155n	$\log X_0$	8.9582938n	$\log G_0^2$	9.9216695
		sub	0.0442365		
$\cos (A' + v_0)$	9.9549631	$\log x_0$	9.9722279	$\log \frac{2}{r_0}$	9.9502398
$\log e \cos A'$	9.9028662n			sub	1.1960764
add	0.9467605	$\sin \alpha_0$	9.8838891		
sum	9.0082026	$\log \eta_0$	0.0032959	$\log \frac{1}{a}$	8.7541634
$\log \frac{\sin \alpha}{V/p}$	9.9541541	$\log Y_0$	9.9536780n	$\log a$	1.2458366
$\log x_0'$	8.9623567	sub	0.2769293		
		$\log y_0$	0.2802252		
$\cos (B' + v_0)$	9.6883510n				
$\log e \cos B'$	9.6924999n	$\tan \delta_0$	9.3966975	$\log x_0 x_0'$	8.9345846
add	0.2989605	$\log \zeta_0$	9.5161043	$\log y_0 y_0'$	0.2291556n
sum	9.9914604n	$\log Z_0$	9.5909691n	add	0.0226190
$\log \frac{\sin b}{V/p}$	9.9574700	sub	0.2652088	sum	0.2065366n
$\log y_0'$	9.9489304n	$\log z_0$	9.8561779	$\log z_0 z_0'$	9.1351357n
				add	0.0353655
$\cos (C' + v_0)$	9.5557843	$\log x_0'^2$	9.9444558	$\log r_0 r_0'$	0.2419021n
$\log e \cos C'$	9.9798075n	$\log y_0'^2$	0.5604504	$\log r_0'$	9.8911119n
add	0.2052915	add	0.0941586		
sum	9.7745160n	sum	0.6546090		
$\log \frac{\sin c}{V/p}$	9.5044418	$\log z_0'^2$	9.7123558		
$\log z_0'$	9.2789578n	add	0.0469714		
		$\log r_0'^2$	0.7015804		
		$\log r_0$	0.3507902		

II.

Representation of First and Third Places.

(f, f_{III}, g, g_{III} determined by closed form using the preliminary elements to determine r, r_{III}, v , and v_{III} by OPPOLZER's nearly parabolic method).

	I	III		I	III
$t - T$	- 213.85584	- 50.07394	$\log f_{x0}$	9.9410772	9.9118070
$\log(t - T)$	2.3301212n	1.6996118n	$\log f_{y0}$	0.2490745	0.2198043
$\log M$	2.6710984n	2.0405890n	$\log f_{z0}$	9.8250272	9.7957570
w	- 130° 57' 29".60	- 90° 3' 46".87	$\log g_{x0'}$	9.1435949n	9.0402135
$\frac{1}{2} w$	- 65 28 44 .80	- 45 1 53 .43	$\log g_{y0'}$	0.1301686	0.0267872n
$\tan \frac{1}{2} w$	0.3408765n	0.0004776n	$\log g_{z0'}$	9.4601960	9.3568146n
$\log x$	0.3438401n	0.0034412n	N	- 1.0003516	+ 0.9293000
$\log x^2$	0.6876802	0.0068824	f_{x0}	+ 0.8731266	+ 0.8162196
$\log \eta$	8.9161658	8.2353680	$g_{x0'}$	- 0.1391858	+ 0.1097017
η	0.0824453	0.0171936	ξ	- 0.2664108	+ 1.8552213
$\log G$	0.0147238	0.0030038	Y	+ 0.0839090	- 0.3157650
$\log H$	- 2	0	f_{y0}	+ 1.7744939	+ 1.6588393
$\tan \frac{1}{2} v$	0.3585637n	0.0064450n	$g_{y0'}$	+ 1.3494866	- 1.0636218
$\frac{1}{2} v$	- 66° 20' 54".29	- 45° 25' 30".45	η	+ 3.2078895	+ 0.2794525
v, v_{III}	- 132 41 48 .59	- 90 51 0 .90	Z	+ 0.0364017	- 0.1369780
$\tan \frac{1}{2} v$	0.7171274	0.0128900	f_{z0}	+ 0.6683858	+ 0.6248230
$\log \theta$	8.9458252	8.2415878	$g_{z0'}$	+ 0.2885333	- 0.2274126
$\log(1 + \theta)$	0.0367376	0.0075095	ζ	+ 0.9933208	+ 0.2604324
add	0.0762073	0.2946328	$\rho \cos \delta \cos \alpha$	9.4255518n	0.2683957
$\log(1 + \tan^2 \frac{1}{2} v)$	0.7933347	0.3075228	$\rho \cos \delta \sin \alpha$	0.5062194	9.4463080
$\text{colog}(1 + \theta)$	9.9632624	9.9924905	$\tan \alpha$	1.0806676n	9.1779123
$\log r, \log r_{III}$	0.5248727	0.0682889	α_c	94° 44' 50".79	8° 33' 57".89
$v - v_0$	- 12° 32' 11".34	+ 29° 18' 36".35	α_0	94 44 51 .60	8 32 55 .90
$\frac{1}{2}(v - v_0)$	- 6 16 5 .67	+ 14 39 18 .18	$(O - C) \Delta \alpha$	+ 0 .81	- 61.99
$\log r r_0$	0.8756629	0.4190791	$\sin \alpha$	9.9985075	9.1730404
$\log \sqrt{r r_0}$	0.4378314	0.2095396	$\cos \alpha$	8.9178399n	9.9951281
$\sin \frac{1}{2}(v - v_0)$	9.0381564n	9.4031188	$\rho \cos \delta$	0.5077119	0.2732676
$\log \gamma, \log \gamma_{III}$	9.5954958n	9.7321664	$\rho \sin \delta$	9.9970896	9.4156950
$\log \gamma^2$	9.1909916	9.4643328	$\tan \delta$	9.4893777	9.1424274
$\log \frac{\gamma^2}{r_0}$	8.8402014	9.1135426	δ_c	+ 17° 8' 58".38	+ 7° 54' 10".25
sub	0.0311507	0.0604209	δ_0	+ 17 8 59 .20	+ 7 53 50 .00
$\log f, \log f_{III}$	9.9688493	9.9395791	$(O - C) \Delta \delta$	+ .0 .82	- 20 .25
$\sin(v - v_0)$	9.3365823n	9.6897847	$\sin \delta$	9.4696259	9.1382830
$\log g, \log g_{III}$	0.1812382n	0.0778568	$\cos \delta$	9.9802482	9.9958556
			$\log \rho$	0.5274637	0.2774120

The solar coördinates for these places were derived from the data of the American Ephemeris, and reduced to 1909.0.

III.

Differential Correction to Remove Residuals.

$R_0 \cos D \cos A$	8.95829n	$\sin \delta_{\alpha}$	9.38361	$\log r_0' \xi_0$	8.89035
$R_0 \cos D \sin A$	9.95368n	$\sin D$	9.59799n	$\log r_0' \eta_0$	9.95223n
$\tan A$	0.99539	$\log I$	8.98160n	add	9.96061
A	264°.2288	$\cos \delta_{\alpha}$	9.98691	sum	9.91284n
α_{α}	49.9427	$\cos D$	9.96291	$\log r_0' \zeta_0$	8.79506n
$A - \alpha_{\alpha}$	214.2861	$\cos (A - \alpha_{\alpha})$	9.91710n	add	0.03192
$\sin A$	9.99779n	$\log II$	9.86692n	$\log \rho_0 \varphi_0$	9.94476n
$\cos A$	9.00240n	add	0.05316	$\log \varphi_0$	9.81226n
$R_0 \cos D$	9.95589	$\cos \psi$	9.92008n		
$R_0 \sin D$	9.59097n	$\log \rho_0$	0.13250	$\log \sqrt{2} \cos \beta$	0.13725
$\tan D$	9.63508n	$\log R_0 \cos \psi$	9.91306n	$\log \frac{\cos \beta}{r_0^2}$	9.28515
$\sin D$	9.59799n	sub	0.20502		
$\cos D$	9.96291	$\log r_0 \cos \beta$	0.33752		
$\log R_0$	9.99298	$\cos \beta$	9.98673		

	I	III		I	III
$\log (t - t_0)$	1.95384n	1.86844	$\log \frac{\gamma^2}{r}$	8.66612	9.39604
$\log \theta_{\alpha}; \log \theta$	0.18942	0.10402	$\log \left[1 - \frac{\gamma^2}{r} \right]$	9.97938	9.87570
$\log \frac{\gamma^2}{2a}$	7.64412	7.91747	$\log \theta \left[1 - \frac{\gamma^2}{r} \right]$	0.16880n	9.97972
$\log (\gamma_c)^2$	9.99808	9.99639	$\log g$	0.18124n	0.07786
$\log (\gamma_c)$	9.99904	9.99820	sub	8.46333	9.40405
$\log \sqrt{2} \frac{\cos \beta}{\gamma}$	0.54175n	0.40508	$\log ()$	8.63213	9.38377n
$\log II$	0.54079n	0.40328	$\log I$	8.80822	9.55986n
$\log (I = \varphi_0)$	9.81226n	9.81226n	$\log \sqrt{2} r_0 (\gamma_c)$	0.50035	0.49951
add	0.07440	9.87132	$\log \frac{1}{2} r_0 r_0' \gamma$	9.53637	8.67304n
$\log \Phi$	0.61519n	0.27460	add	0.04479	9.92987
$\log I$	0.12037n	9.80046	$\log ()$	0.54514	0.42938
$\log 2 r_0 \gamma$	0.24732n	0.38399	$\log \gamma^3$	8.78649n	9.19650
$\log II$	0.24540n	0.38039	colog r	9.47513	9.93171
$\log \gamma^2$	9.19099	9.46433	$\log II$	8.86676n	9.55759
$\log \frac{r_0 r_0'}{\sqrt{2}}$	0.09139n	0.09139n	add	7.52667	7.72000
$\log III$	9.28142n	9.55392n	$\log []$	6.33343	7.27759n
$\log IV$	0.51507	0.42883	$\log N$	7.57927n	8.52343
I	-1.31939	+0.63163	$\log \frac{\xi_0}{\rho_0}$	9.79549	
II	-1.75954	+2.40100	$\log \frac{\eta_0}{\rho_0}$	9.87080	
III	-0.19117	-0.35803	$\log \frac{\zeta_0}{\rho_0}$	9.38361	
$\pm IV$	+3.27392	-2.68431			
[]	+0.00382	-0.00971			
$\log []$	7.58206	7.98722n			
$\log \gamma$	9.59550n	9.73217			
colog r	9.47513	9.93171			
$\log M$	7.54774	8.54615			

g_{ω_i}	g_{x_i}	g_{y_i}	g_{z_i}	$g_{x_{III}}$	$g_{y_{III}}$	$g_{z_{III}}$
log I	8.90865n	9.89522	9.22525	9.04532	0.03189n	9.36192n
log II	0.12179	0.42979	0.00574	0.12095	0.42895	0.00490
add	9.97257	0.11127	0.06661	0.03504	0.17462	9.88788
log []	0.09436	0.54106	0.07235	0.15599	0.20651	9.89278
log $\frac{\gamma_i^3}{r_i r_0}$	7.91083n			8.77742		
log g_{ω_i}	8.00519n	8.45189n	7.98318n	8.93341	8.98393	8.67020

m_{ω_i}	m_{x_i}	m_{y_i}	m_{z_i}	$m_{x_{III}}$	$m_{y_{III}}$	$m_{z_{III}}$
log $M_i \omega_0$	7.51997	7.82797	7.40392	8.51838	8.82638	8.40233
log $N_i \omega_0'$	6.54163n	7.52820	6.85823	7.48579	8.47236n	7.80239n
add	9.95177	0.17651	0.10879	0.03853	0.10021	9.87435
log m_{ω_i}	7.47174	8.00448	7.51271	8.55691	8.57257	8.27668

f_{ω_i}	f_{x_i}	f_{y_i}	f_{z_i}	$f_{x_{III}}$	$f_{y_{III}}$	$f_{z_{III}}$
log I	9.76434	9.83965	9.35246	9.73507	9.81038	9.32319
log $2m_{\omega_i}$	7.77277	8.30551	7.81374	8.85794	8.87360	8.57771
log $\omega_0 \gamma_i^2$	9.16322	9.47122	9.04717	9.43656	9.74456	9.32051
add	0.01732	0.02869	0.02466	0.10170	0.05484	0.07218
log []	9.18054	9.49991	9.07183	9.53826	9.79940	9.39269
log II	8.46569	8.78506	8.35698	8.82341	9.08455	8.67784
log III	8.62038	9.06708	8.59837	9.20801	9.25853	8.94480
I	+0.58121	+0.69127	+0.225142	+0.54334	+0.64622	+0.210470
II	+0.02922	+0.06096	+0.022750	+0.06659	+0.12149	+0.047626
III	+0.04172	+0.11670	+0.039662	+0.16144	+0.18135	+0.088064
f_{ω_i}	+0.65215	+0.86893	+0.287554	+0.77137	+0.94906	+0.346160
log f_{ω_i}	9.81434	9.93898	9.45872	9.88726	9.97729	9.53928

A_{p_i}	A_{x_i}	A_{y_i}	A_{z_i}	$A_{x_{III}}$	$A_{y_{III}}$	$A_{z_{III}}$
log $p_{y_i} \cos \alpha_i$	8.85682n	7.36973	6.92232n	9.97242	8.97906	8.56770
log $p_{x_i} \sin \alpha_i$	9.81285	8.00370n	7.47025	9.06030	8.10645	7.72995
sub	0.04553	0.09072	0.10829	9.94329	9.93747	9.93182
log []	9.85843n	8.09442	7.57854n	9.91571	8.91653	8.49952
log ρ_i	0.52746			0.27741		
log A_{p_i}	9.33097n	7.56696	7.05108n	9.63830	8.63912	8.22211

B_{p_i}	B_{x_i}	B_{y_i}	B_{z_i}	$B_{x_{III}}$	$B_{y_{III}}$	$B_{z_{III}}$
log $p_{y_i} \sin \alpha_i$	9.93749	8.45040n	8.00299	9.15033	8.15697	7.74561
log $p_{x_i} \cos \alpha_i$	8.73218n	6.92303	6.38958n	9.88239	8.92854	8.55204
add	9.97205	9.98691	9.98929	0.07384	0.06789	0.06302
log ()	9.90954	8.43731n	7.99228	6.95623	8.99643	8.61506
sin δ_i	9.46963			9.13828		
log I	9.37917	7.90694n	7.46191	9.09451	8.13471	7.75334
log II	9.43897	7.96343n	7.49296	9.53514	8.66606	8.27254
sub	9.16915	9.14275	8.86986	9.80445	9.84868	9.84352
log []	8.54832n	7.04969	6.33177n	9.33959	8.51474n	8.11606n
log B_{p_i}	8.02086	6.52223n	5.80431	9.06218	8.23733	7.83865

	I	II
log g	0.18124n	0.07786
log ρ	0.52746	0.27741
log C_i ; log C_{III}	9.65378n	9.80045

For $\beta_i, \gamma_i, \delta_i, r_i$.

	I	III		I	III
$\log (I = C \sin \alpha)$	9.65229n	8.97349	$\log (I = C \cos \alpha)$	8.57162	9.79558
$\log II$	7.53919	8.61135	$\log II$	7.84719	8.91935
$\log III$	6.31447n	7.48550	$\log III$	7.30104	8.47207n
I	-0.44904	+0.094078	I	+0.0372925	+0.62457
-II	- 346	- 40865	II	+ 70338	+ 8305
-III	+ 21	- 3058	III	+ 20000	- 2965
$-\beta_1; -\beta_2$	-0.45229	+0.050155	$\gamma_1; \gamma_2$	+0.0463263	+0.67797
$\log \beta_1; \log \beta_2$	9.65542	8.70032n	$\log \gamma_1; \log \gamma_2$	8.66583	9.83121
$\log I$	8.04125	8.93386	$\log I$	9.12192n	8.11177
$\log II$	6.49446n	8.20956	$\log II$	6.80246n	8.51756
$\log III$	5.06770	7.10204	$\log III$	6.05427n	8.08861n
I	+0.0109964	+0.085874	I	-0.132409	+0.0129351
-II	+ 3122	- 16202	-II	+ 635	- 329277
-III	- 117	- 1265	-III	+ 113	+ 122634
$-\beta_2; -\beta_3$	+0.0112969	+0.068407	$-\gamma_2; -\gamma_3$	-0.131661	-0.0077292
$\log \beta_2; \log \beta_3$	8.05296n	8.83510n	$\log \gamma_2; \log \gamma_3$	9.11946	7.88813
$\log I$	7.42314	8.49530	$\log \partial \alpha$	9.90849	1.79232n
$\log II$	6.63107	7.80210n	$\cos \delta$	9.98025	9.99586
add	0.06499	9.90163	$\log \sin 1''$	4.68558	
$\log \delta_1; \log \delta_2$	7.48813	8.39693	$\log r_1; \log r_2$	4.57432	6.47376n
$\log I$	9.63403n	9.79631	$\log \partial \delta$	9.92428	1.30643n
$\log II$	6.37841n	8.09351	$\log r_2; \log r_3$	4.60986	5.99201n
$\log III$	5.38430n	7.41864n			
I	-0.43056	+0.62561			
II	- 24	+ 1240			
III	- 2	- 262			
$\delta_2; \delta_3$	-0.43082	+0.63539			
$\log \delta_2; \log \delta_3$	9.63430n	9.80304			

Equations for Solution (Coefficients Logarithmic).

$$\begin{aligned}
 (9.33097n) \partial \rho_0 + (9.65542) \partial x'_0 + (8.66583) \partial y'_0 + (7.48813) \partial z_0 &= (4.57432) \\
 (8.02086) &+ (8.05296n) &+ (9.11946) &+ (9.63430n) &= (4.60986) \\
 (9.63830) &+ (8.70032n) &+ (9.83121) &+ (8.39693) &= (6.47376n) \\
 (9.06218) &+ (8.83510n) &+ (7.88813) &+ (9.80304) &= (5.99201n)
 \end{aligned}$$

The solution of these equations gives

$$\begin{aligned}
 \log \partial \rho_0 &5.37967 \\
 \log \partial x'_0 &5.82046 \\
 \log \partial y'_0 &6.64756n \\
 \log \partial z_0 &6.16537n
 \end{aligned}$$

From the value of $\log \partial \rho_0$ we have

$$\begin{aligned}
 \log \partial x_0 &5.17516 \\
 \log \partial y_0 &5.25047 \\
 \log \partial z_0 &4.76328
 \end{aligned}$$

Applying these values of $\partial x_0, \partial y_0, \partial z_0, \partial x'_0, \partial y'_0$, and $\partial z'_0$ to the old values of $x_0, y_0, z_0, x'_0, y'_0$, and z'_0 , respectively, we have the new values

$$\begin{aligned}
 \log x_0 &9.9722348 & \log x'_0 &8.9626699 \\
 \log y_0 &0.2802293 & \log y'_0 &9.9491473n \\
 \log z_0 &9.8561814 & \log z'_0 &9.2792920n \\
 \log r_0 &0.3507948 & \log r'_0 &9.8913326n \\
 & & \log G_0^2 &9.9221155
 \end{aligned}$$

IV.

From these the following quantities are derived by A VIII (the details of the computation are not given):

τ_0	239° 53' 36".95
$\log e$	9.9855082
$\log a$	1.2525224
$\log q$	9.7686346
T	1910 April 19.67760 Gr. M. T.

Using these quantities the new values of f , f''' , g , and g''' were derived as on page 450 of this example. They are:

$\log f$	9.9688565
$\log f'''$	9.9395596
$\log g$	0.1812403n
$\log g'''$	0.0778450

With these and the corrected heliocentric coördinates and their velocities the residuals for the first and the third places were computed. They are:

	I	III
$(O - C) \left\{ \begin{array}{l} \Delta \alpha \\ \Delta \delta \end{array} \right.$	$\begin{array}{l} 0''.0 \\ 0.0 \end{array}$	$\begin{array}{l} -0''.1 \\ 0.0 \end{array}$

SHORT METHODS OF DETERMINING ORBITS.

THIRD PAPER.

(No. 1000.)

THE DIRECT SOLUTION OF ORBITS OF DISTURBED BODIES.

BY A. O. LEUSCHNER,

DIRECTOR OF THE STUDENTS' OBSERVATORY,

BERKELEY ASTRONOMICAL DEPARTMENT.

1891

SHORT METHODS OF DETERMINING ORBITS.

THIRD PAPER.*

THE DIRECT SOLUTION OF ORBITS OF DISTURBED BODIES.

By A. O. LEUSCHNER.

CONTENTS.

	PAGE.
INTRODUCTION - - - - -	459
DERIVATION OF THE VELOCITIES AND ACCELERATIONS α' , δ' ; α'' , δ'' AT THE EPOCH - - - - -	467
DERIVATION OF THE FORMULE FOR THE DISTANCES OF THE DISTURBED BODY FROM THE EARTH AND FROM THE ATTRACTING BODIES -	468
DERIVATION OF THE ELEMENTS AND COMPUTATION OF THE EPHEMERIS	479
SYNOPSIS OF FORMULE - - - - -	482

*This paper, with the exception of the references to the orbit of the Eighth Satellite of Jupiter, was written in 1905 and read in abstract under the title of *An Analytical Method of Determining the Orbits of New Satellites* at the seventh meeting of the *Astronomical and Astrophysical Society of America*, New York, December, 1905. *Publications*, Vol. 1, Page 249.

INTRODUCTION.

The recent¹ discoveries by W. H. PICKERING and C. D. PERRINE of distant satellites of Saturn and Jupiter, and the fact that the powerful photographic methods of search now in use undoubtedly will lead to the further discovery of satellites, make it imperative that an adequate method should be established for the determination of the orbits of new satellites from a limited number of observations. Any method proposed for this purpose should be simple in its application and should permit of the ready correction of the preliminary elements on the basis of additional observations. It is also of the greatest importance that the principal perturbations of the satellite, during the period covering the observations on which the orbit is based, should be taken into account in the first approximation of the elements. The graphical methods, which are patterned in the main after the methods in use for the determination of double star orbits, seem to be wholly inadequate for the solution of an orbit to the degree of accuracy which the observations are capable of yielding even in the case of a short arc. Solutions which are partially analytic but involve arbitrary assumptions or variations of certain elements, as the period, position of the orbit plane, or eccentricity, also should be avoided. In the case of satellites which are subject to large perturbations, the essential object to be aimed at is the immediate determination of a set of osculating elements of the greatest possible accuracy. On the basis of such a set of elements together with the perturbations, comparison may be made with observations and the elements improved. A graphical construction of what may be termed an average orbit through the given observations, or a partially analytical solution on the basis of certain arbitrary assumptions, is very likely to result in a set of elements which differ to such an extent from the osculating elements that any perturbations based on the same will be greatly in error, nor can they be confounded with mean elements owing to the shortness of the arc. But, if the perturbations are greatly in error, the comparison of theory with observation will lead to residuals which are also erroneous, and the improvement of the elements on the basis of such residuals may be questioned. In fact, the whole process may turn out to be non-convergent.

My attention was first directed to this problem through W. H. PICKERING'S valuable memoir *On the Ninth Satellite of Saturn*, *Annals of Harvard College Observatory*, LIII, No. III. A direct analytical method of solution of the general problem involved in this case seemed to be eminently desirable, particularly as it

¹This paper was read at the Seventh Meeting of the Astronomical and Astrophysical Society of America, Columbia University, New York, December, 1905; cf. *Publications of the Astronomical and Astrophysical Society of America*, Volume I, page 249.

might be expected that such a method would readily reveal all the solutions possible on the basis of the given series of observations; but no serious thought was given to the matter until after the appearance in September of this year¹ of *Lick Observatory Bulletins* 78 and 82 giving PERRINE's account of his discovery of the sixth and seventh satellites of Jupiter, the first series of observations secured at the Lick Observatory, and the results of the investigation of the orbits by PERRINE and ROSS. On page 130 of *Lick Observatory Bulletin* No. 78, PERRINE calls attention to the fact that no method of deriving the elements of a satellite directly from three or more observations has been published, and that consequently his solutions were derived by graphical methods and on the basis of arbitrary assumptions. The method used by ROSS in the case of the seventh satellite was that of arbitrary variation of the position of the plane of the orbit. No reference is given, however, as to the method used in the case of his determination of the orbit of the sixth satellite of Jupiter. The method used by PICKERING in the case of the ninth satellite of Saturn is entirely graphical and based upon the assumption that the plane of the orbit is inclined to the ecliptic at an angle of about five degrees, which assumption was obtained by inspection.

From correspondence with Professor NEWCOMB and Dr. ROSS during the summer, it appears that considerable difficulty was encountered in determining the orbits of the sixth and seventh satellites of Jupiter. In all of these determinations, the object of the investigators was to satisfy the observations by an *elliptic* orbit, *irrespective of any perturbations* which might have prevailed during the period of observation, although ROSS fully realized that the perturbations of the Sun were quite appreciable. The *preliminary* orbits obtained by PICKERING, PERRINE and ROSS, therefore, leave residuals which considerably exceed the possible errors of observation, although ROSS's results are the most satisfactory in this respect.

My own *original* intention, also, was to set up a method which would satisfy the observations by *elliptic orbits irrespective of perturbations*, but by *direct analytical solution without any arbitrary assumption*, so that the numerical values of none of the constants would be prejudiced beforehand. My further aim was to arrange the formulæ in such a manner that all possible solutions would reveal themselves at one and the same time within the degree of certainty that a limited arc would permit. In deriving my solution of the problem, I had the able assistance of Dr. RUSSELL TRACY CRAWFORD and Mr. A. J. CHAMPREUX, who participated in a seminar which was formed for the purpose of considering this problem. They also volunteered to perform the necessary numerical work under my immediate direction. Their results are given in Part 10 of this volume and will be freely referred to by me in this discussion. For a first application of the analytical method of solution, *irrespective of the perturbations*, the seventh satellite of Jupiter

¹1905.

was selected, partly because ROSS encountered greater difficulties in this case than with the sixth satellite, in spite of the experience gained with the latter and Phoebe, and partly because PERRINE had settled on the retrograde motion as being more probable than the direct motion. In order to make the analytical results immediately comparable with those obtained by ROSS, it was decided to base the determination of the orbit on the same observations that ROSS had used, namely, on the observations of January 3, the first of the two secured on February 8, and March 6. The observations were taken from *Lick Observatory Bulletin* No. 78, page 133. CRAWFORD and CHAMPREUX found by direct computation that these observations could be satisfied by two and only two distinct and well-determined orbits,¹ one placing the satellite at a nearer distance from the Earth than Jupiter's distance, with a retrograde motion; and the other at a farther stance with direct motion. A comparison of these orbits with those obtained by PERRINE and ROSS is printed on page 498 of this volume. In the mean time, Mr. ALBRECHT had secured positions of the satellite with the Crossley reflector on August 8, 9, and 10. A comparison with ROSS's ephemeris in *Lick Observatory Bulletin* No. 82 showed the following approximate residuals, O — C: in position angle, — 4°.6; in distance, 29'.6, the computed position angle and distance being respectively, 294° and 25'. CRAWFORD and CHAMPREUX found the following residuals from their *first approximation irrespective of the perturbations*, to the two possible solutions in the sense O — C:

Date, August 9	<i>p</i>	<i>s</i>
Direct motion:	— 5°.6	+ 33'.6
Retrograde motion	— 270	+ 22'.8

An improvement of the first approximation, irrespective of perturbations, was not attempted, as it was decided to compute a new orbit by taking immediate account of the solar perturbations. The agreement of the residuals for the first approximation to direct motion with those of ROSS's orbit is very marked. But the representation of either orbit is by no means satisfactory. The lack of representation for retrograde motion does not necessarily imply at this stage of the investigation that the motion is direct rather than retrograde, although further investigation shows this to be the case. The magnitude of the residuals which the August observations leave for all the orbits derived on the basis of undisturbed motion, plainly indicates that no satisfactory orbit can be obtained without taking into immediate account the principal perturbations to which the satellite was subjected during the period covered by the observations on which the orbits were based. Fortunately only a simple generalization of the original analytical method for determining an orbit irrespective of perturbations was necessary for the derivation of formulæ for the *direct computation of osculating elements* from three

¹Cf. the footnote on page 501.

or more observations in cases where the satellite is strongly disturbed by the attraction of one or more bodies. The modifications arising from the oblateness of the central body, etc., as yet have not been formulated in this connection.

In the case of comets and asteroids, it has been found possible to approximate an osculating orbit closely by neglecting the perturbations in the first solution. Such elements are used for the determination of special perturbations on the basis of which the original elements are changed into real osculating elements. At most, the process of determining the special perturbations and correcting the elements is repeated until the perturbations no longer change and the residuals of the normal places are satisfactory. But let us consider the case of an asteroid suffering large periodic perturbations during the discovery opposition. The process referred to above, if applied properly, would be very laborious. It may even happen in extreme cases that the successive corrections to the special perturbations and the osculating elements do not approach zero. I am not aware that such a case has hitherto actually been encountered. But it does occur that the normal places of the discovery opposition can be satisfactorily represented by an orbit in which the perturbations are neglected, whereas, if taken account of, there result osculating elements which differ considerably from the adopted elements. If, then, the adopted elements are made the basis of an ephemeris for the next opposition, with or without the inclusion of special perturbations, the asteroid may be very far from the predicted place and even lost. Such has been the case with (132) *Æthra*, discovered by WATSON at Ann Arbor on June 13, 1873. The planet was observed from June 13 to July 5. During this time and shortly after it was disturbed by Mars, but WATSON computed two sets of elements irrespective of the perturbations. WILHELM LUTHER also has published, particularly in the *Astronomische Nachrichten*, 3101, several sets of elements and opposition ephemerides up to 1896. But in all these calculations, the perturbations during the period covered by the observations, as well as the subsequent ones, have been neglected. The planet was of the eleventh magnitude at the time of discovery and, in spite of LUTHER's extensive calculations and a careful photographic search instituted by various observers, has never again been identified. It is extremely desirable that the orbit should be investigated again by including the perturbations in the computation of an orbit from the available observations by the method outlined in these pages. The computation of such an orbit already has been undertaken at this Observatory.

In cases of new satellites, asteroids and comets which are strongly disturbed at the time of the observations, which must form the basis of the computation, it is, therefore, essential that the disturbing forces be taken into account in the original determination of the orbit from the observations. This applies even to arcs covering only a few days.

It is proposed to accomplish this by an extension of the *Short Method of Deter-*

mining Orbits from Three Observations, given in Parts I and VII of this volume, to the case of more than one attracting body. *The analytical method of determining the orbit of a satellite* from three or more observations *irrespective of perturbations*, referred to above, consists in an adaptation of the *Short Method* to the solution of satellite orbits. The most general solution is that which gives an osculating orbit of an asteroid, comet, or satellite, considered as material points, from three or more observed positions by taking immediate account of the appreciable attraction of other bodies and of their figures besides the attraction of the supposed central body. The ordinary cases of an asteroid or comet moving solely under the Sun's attraction, or of a satellite moving solely under the attraction of its primary are merely special cases of the former.

In applying the general solution to the seventh satellite of Jupiter only the attraction of Jupiter and the Sun have been considered by CRAWFORD and CHAMPREUX, and this has been found entirely sufficient for a representation of the available observations. The resulting elements osculating for February 8 are printed on page 498 of this volume and those derived from the solution *irrespective of the perturbations*, are printed alongside for comparison. The differences between the two sets of elements are due solely to solar perturbations. The osculating elements represent the actual Jovicentric position and velocity of the satellite at the epoch on the assumption that the satellite was subject only to perturbations by the Sun from the first to the last dates covered by the geocentric arc on which the elements are based. *These elements are elements that would have resulted from a solution irrespective of the perturbations, if the observed right ascensions and declinations could have been corrected in advance for the solar perturbations, starting with February 8 as epoch of osculation.* The *first approximation of such osculating elements* gave as residuals for the direct orbit for August 9 in position angle $+2^{\circ}.7$, in distance, $+5'.2$, all perturbations being neglected in the computation of the position, as compared to $-5^{\circ}.6$ and $+33'.6$ for the solution *irrespective of the perturbations*. The fact that the *preliminary osculating elements* represent the August observation so much better than the elements computed *irrespective of perturbations* clearly shows that shortly after discovery the satellite was subjected to periodic perturbations of considerable magnitude and that, therefore, the latter solution is insufficient. CRAWFORD and CHAMPREUX have also computed the residuals on the basis of the *final osculating elements and of the perturbations* resulting therefrom for all of the observations from January 3 to March 6 printed in *Lick Observatory Bulletin* No. 82. These residuals are entirely comparable with the errors of observation, so that the result must be considered eminently satisfactory.

¹The discovery of the eighth satellite of Jupiter has led Messrs. COWELL and

¹These concluding paragraphs of the *Introduction* were added in June, 1911.

CROMMELIN to derive a method of dealing with this problem which has given eminently satisfactory results as indicated by the smallness of the residuals of the observations taken during the discovery opposition and by the accuracy of their predictions of the positions for subsequent oppositions. The process adopted, however, by Messrs. COWELL and CROMMELIN appears to be very laborious in application. It is not possible, at this writing, to enter upon a full discussion of the practical and theoretical merits of their method. A brief statement of the principal features of the process adopted by them in dealing with the orbit of the eighth satellite of Jupiter, as compared with the method developed in these pages, must therefore suffice.

In my *Direct Solution of the Orbits of Disturbed Bodies*, the first step consists in computing the velocities and accelerations α', δ' ; α'', δ'' at the adopted epoch. The number of observations to be utilized for this purpose is readily determined by inspection on the basis of the run of the differences of the observed α, δ , the given intervals, and the available arc.

The next step consists in the direct solution of the distances of the disturbed body from the Earth and from the attracting bodies, together with the velocity and acceleration in the line of sight.

The final step consists in the computation of the osculating elements. But evidently, after the coördinates of the disturbed body and their velocities become known, osculating elements may be computed with reference to either attracting mass, *e. g.*, if there be two such masses. In the case of two solutions for the geocentric distance in a problem where two attracting masses are being considered, it is therefore possible to compute simultaneously four sets of osculating elements on the basis of the initial values of $\alpha', \delta', \alpha'', \delta''$. The selection of the physical solution in the usual sense is then accomplished by the representation of the observations. In some cases it may be necessary to base the selection on observations subsequent to the observations utilized in the solution of the problem.

The computation of an ephemeris is accomplished in the usual way by applying the special perturbations to the undisturbed positions computed from the osculating elements. The improvement of the orbit on the basis of subsequent observations is accomplished by differential correction of the osculating elements on the basis of the differences between theory and observation.

The first step in COWELL and CROMMELIN'S procedure¹ consists in interpolating three places for equidistant dates from the observed places. The next step consists in the determination of the geocentric distances by *trial and error* and the subsequent correction of these distances on account of the previously neglected terms of the fourth order with respect to the time. Unfortunately, the details of this solution by trial and error, etc., have not been published, and it

¹*Monthly Notices of the Royal Astronomical Society*, Vol. LXVIII, No. 8, page 576.

is, therefore, not possible to make an immediate comparison of the numerical processes involved with those fully printed for my method in Part 10 of this volume, but the description given of the procedure leaves little doubt that the numerical operations are exceedingly laborious.

The next step in COWELL and CROMMELIN'S procedure consists in the application of mechanical quadratures to the *disturbed* coördinates. This process is a modification of the process of special perturbations in rectangular coördinates. The subsequent disturbed coördinates depend upon the initial coördinates. Any inaccuracy in these initial coördinates, which may reveal itself later by comparison with subsequent observations, appears to require for the improvement of the orbit a complete revision of all the preceding work. Furthermore, according to the theory of interpolation, the irregularities in the differences of the undisturbed rectangular coördinates on the one hand and of the perturbations in these coördinates on the other, will combine in the differences of the disturbed coördinates, so that, in general, not only much smaller intervals of time are required in applying the process of mechanical quadratures to the *disturbed* coördinates than would be required, if it were applied to the *perturbations alone*, but the process must also be continued over periods for which the perturbations are insignificant.

It would seem, therefore, that the direct solution, as proposed in these pages, combined with the process of special perturbations after preliminary osculating elements have been obtained, presents many points of advantage as compared with the process adopted by COWELL and CROMMELIN.

At the time when my direct method was applied by CRAWFORD and MEYER¹ to the solution of the preliminary osculating orbit of the eighth satellite of Jupiter, I had not as yet derived the closed expressions for ∂f and ∂g (cf. page 247), so that the removal of the original residuals was attempted with the aid of the series for ∂f and ∂g . These series had proved entirely sufficient in the case of the seventh satellite of Jupiter, but were insufficient in the case of the eighth satellite. Final adjustment of the orbit was accomplished by the laborious process of variation of constants. Since then MEYER has applied my closed expressions for ∂f and ∂g to the improvement of the orbit immediately after the direct solution and thereby has been enabled to correct the orbit as indicated in Part 10 in a remarkably short time.

In the next section the formulæ will be developed for the determination of the distances of the disturbed body from the Earth and the attracting bodies, the derivation being limited to two attracting mass-points, for simplicity's sake. The formulæ may be readily extended, if the occasion demands, for more than two attracting bodies and for taking account of their figures.

The derivation of the formulæ may be accomplished in various ways accord-

¹L. O. Bulletin 137.

ing to the choice of coördinates. They may be derived very conveniently by considering the problem the converse of ENCKE'S method of special perturbations. The developments are, therefore, based on the equations of motion in rectangular coördinates. In the problem of special perturbations, the object is to determine disturbed coördinates on the basis of osculating elements. In the problem to be considered here the object is to determine osculating elements or equivalent constants, on the basis of disturbed positions furnished by observation.

Only the case of a disturbed satellite has been fully developed, the solution of the osculating orbit of a disturbed comet or planet being merely indicated. But the computer need experience no difficulty in applying the proposed methods to the latter cases. Two such applications, one to the determination of the osculating orbit of a comet, the other to that of a disturbed planet, are now under way at the Students' Observatory.

THE VELOCITIES AND ACCELERATIONS α' , δ' , α'' , δ'' , AT THE EPOCH.

The computation of the initial velocities and accelerations at the epoch from five or more observations is accomplished as in Part 7, page 229, *et seq.* The procedure is fully illustrated by Example 2 in Part 8. The velocities and accelerations are most conveniently determined in mean solar days as the unit of time. In case of disturbed comets or minor planets the unit is then changed to $\frac{1}{k}$ mean solar days by dividing the velocities by k and the accelerations by k^2 , where k^2 is the solar constant of attraction. In case of a disturbed satellite or also in case of the determination of the orbit of a satellite irrespective of the perturbations, it is most convenient to change the unit to $\frac{1}{(k)}$ mean solar days, where $(k)^2$ is the constant of attraction of the primary. If the mass of the planet in terms of the Sun's mass be (m) , then $(k)^2 = (m)k^2$, the numerical ratio of the constants of attraction of the bodies being the same as the numerical ratio of their masses, if the same units of time, mass, and length be chosen. Thus, since according to NEWCOMB, Jupiter's mass $(m) = \frac{1}{1047.355}$, we have for the logarithm of Jupiter's constant of attraction $\log (k)^2 = 3.4510689 - 10$.

If in the case of a newly discovered body, it can not be decided beforehand whether the Sun or the major planet is the primary, then the velocities and accelerations may be expressed either in $\frac{1}{k}$ or $\frac{1}{(k)}$ mean solar days as the unit of time.

If the geocentric motion of the disturbed body be very irregular, then it may be advantageous to depart from the analytical or graphical methods of determining the velocities and accelerations, outlined in Part 7, page 229, *et seq.*, by deriving the velocities and accelerations in position angle and distance and then properly transforming them into velocities and accelerations in the geocentric coördinates.

In this connection it is to be observed that a complete elimination of parallax may be performed in the solution of the orbits of disturbed bodies in exactly the same manner as in Part 7, page 233, *et seq.* But as any minor inaccuracies committed in the derivation of the preliminary osculating orbit are removed by differential correction in the final representation of the observations, the parallax may be either entirely neglected or, in case of a disturbed satellite, the observations may be corrected for parallax on the basis of the distance of the primary.

The planetary aberration is accounted for by a procedure similar to that adopted in Part 7 in the determination of the orbits of undisturbed bodies. This procedure will be considered in detail after the derivation of the fundamental formulæ, which follows.

DERIVATION OF THE FORMULÆ FOR THE DISTANCES OF THE DISTURBED BODY FROM THE EARTH AND FROM THE ATTRACTING BODIES.

OSCULATING SATELLITE ORBITS.

The velocities and accelerations in right ascension and declination of the body of which the orbit is sought having been determined in the preceding section, expressions shall next be developed for the geocentric distance, its velocity, and acceleration at the time $t=0$. No restriction need be placed on the number of attracting masses. For the sake of simplicity, however, it will be assumed that the body is attracted by only two masses. The terms depending on additional masses may then be written by analogy. If it be possible to judge in advance which of the attracting bodies is the central body, it is convenient to place the origin of coördinates in the center of that body. Thus, in the case of a suspected satellite, subject to perturbations by the Sun, the center of the primary should be chosen as origin so as to avoid a subsequent transformation. Similarly we should choose the center of the Sun as origin in cases of asteroids or comets which are greatly disturbed by one or more planets. This choice, however, is not essential to the solution of the problem, as the same results will ultimately be obtained, no matter which mass be chosen as origin of coördinates.

If it can not be decided in advance which is the central body, the center of either mass may be chosen as origin, as would necessarily be the case with equal attractions. We shall first consider the case of a satellite, which is disturbed by the Sun. Taking the Earth's equator as fundamental plane, we may denote by

$\alpha, \delta, \rho, \sigma = \rho \cos \delta, \xi, \eta, \zeta$	the coördinates of the satellite referred to the Earth,
$a, d, r, s = r \cos d, x, y, z$	the coördinates of the satellite referred to the primary,
$[a], [d], [r], [s] = [r] \cos [d], [x], [y], [z]$	the coördinates of the satellite referred to the Sun,
$(\alpha), (\delta), (\rho), (\sigma) = (\rho) \cos (\delta), (\xi), (\eta), (\zeta)$	the coördinates of the primary referred to the Earth,
$(a), (d), (r), (s) = (r) \cos (d), (x), (y), (z)$	the coördinates of the primary referred to the Sun,
$A, D, R, S = R \cos D, X, Y, Z$	the coördinates of the Sun referred to the Earth.

Then the equations of motion of the satellite referred to the center of mass of the primary are:

$$\begin{aligned}
 \frac{d^2 x}{dt^2} + mk^2 \frac{x}{r^3} &= k^2 \left\{ \frac{-(x) - x}{[r]^3} - \frac{-(x)}{(r)^3} \right\} \\
 \frac{d^2 y}{dt^2} + mk^2 \frac{y}{r^3} &= k^2 \left\{ \frac{-(y) - y}{[r]^3} - \frac{-(y)}{(r)^3} \right\} \\
 \frac{d^2 z}{dt^2} + mk^2 \frac{z}{r^3} &= k^2 \left\{ \frac{-(z) - z}{[r]^3} - \frac{-(z)}{(r)^3} \right\}.
 \end{aligned} \tag{1}$$

Introducing

$$\begin{aligned} mk^2 &= (k)^2 ; & \frac{k^2}{(k)^2} &= \frac{1}{m} = \gamma \\ \frac{1}{(k)^2} \frac{d^2 x}{dt^2} &= x'' , & \frac{1}{(k)^2} \frac{d^2 y}{dt^2} &= y'' , & \frac{1}{(k)^2} \frac{d^2 z}{dt^2} &= z'' , \end{aligned} \quad (2)$$

so that the unit of time is $\frac{1}{(k)}$ mean solar days, we obtain after dividing equations (1) throughout by $(k)^2$:

$$\begin{aligned} x'' &= -x \left[\frac{1}{r^3} + \frac{\gamma}{[r]^3} \right] - \gamma \frac{(x)}{[r]^3} + \gamma \frac{(x)}{(r)^3} \\ y'' &= -y \left[\frac{1}{r^3} + \frac{\gamma}{[r]^3} \right] - \gamma \frac{(y)}{[r]^3} + \gamma \frac{(y)}{(r)^3} \\ z'' &= -z \left[\frac{1}{r^3} + \frac{\gamma}{[r]^3} \right] - \gamma \frac{(z)}{[r]^3} + \gamma \frac{(z)}{(r)^3} . \end{aligned} \quad (3)$$

After substituting

$$\begin{aligned} x &= \xi - (\xi), & y &= \eta - (\eta), & z &= \zeta - (\zeta) \\ x'' &= \xi'' - (\xi)'', & y &= \eta'' - (\eta)'', & z &= \zeta'' - (\zeta)'', \end{aligned} \quad (4)$$

and remembering that

$$(\xi) - (x) = X, \quad (\eta) - (y) = Y, \quad (\zeta) - (z) = Z, \quad (5)$$

equations (3) become

$$\begin{aligned} \xi'' + \xi \left[\frac{1}{r^3} + \frac{\gamma}{[r]^3} \right] &= \frac{(\xi)}{r^3} + \gamma \frac{X}{[r]^3} + \gamma \frac{(x)}{(r)^3} + (\xi)'' \\ \eta'' + \eta \left[\frac{1}{r^3} + \frac{\gamma}{[r]^3} \right] &= \frac{(\eta)}{r^3} + \gamma \frac{Y}{[r]^3} + \gamma \frac{(y)}{(r)^3} + (\eta)'' \\ \zeta'' + \zeta \left[\frac{1}{r^3} + \frac{\gamma}{[r]^3} \right] &= \frac{(\zeta)}{r^3} + \gamma \frac{Z}{[r]^3} + \gamma \frac{(z)}{(r)^3} + (\zeta)'' . \end{aligned} \quad (6)$$

$(\xi)'', (\eta)'', (\zeta)''$ are the accelerations of the primary in the geocentric coördinates. If we neglect the perturbations of the primary, then $(\xi)'', (\eta)'', (\zeta)''$ may be expressed as follows:

$$(\xi)'' = (x)'' + X'', \quad (\eta)'' = (y)'' + Y'', \quad (\zeta)'' = (z)'' + Z'' \quad (7)$$

$$\begin{aligned} (x)'' &= -\gamma \frac{(x)(1+m)}{(r)^3}, & (y)'' &= -\gamma \frac{(y)(1+m)}{(r)^3}, & (z)'' &= -\gamma \frac{(z)(1+m)}{(r)^3} \\ (X)'' &= -\gamma \frac{X}{R^3}, & Y'' &= -\gamma \frac{Y}{R^3}, & Z'' &= -\gamma \frac{Z}{R^3}, \end{aligned} \quad (8)$$

where the mass of the Earth is neglected because it would affect only the seventh place of decimals. When the approximations (8) are not sufficient $(\xi)'', (\eta)'', (\zeta)''$ may be accurately computed as indicated later (page 473).

From (7) and (8) we obtain for the sum of the last three terms on the right hand side of the first of equations (6)

$$\gamma \frac{X}{[r]^3} + \gamma \frac{(x)}{(r)^3} + \xi'' = \gamma X \left(\frac{1}{[r]^3} - \frac{1}{R^3} \right) - \frac{(x)}{(r)^3} \quad (9)$$

and similar expressions for the last two equations (6).

In order that equations (6), after introducing expressions (9), may be made available for the solution of ρ , ρ' , ρ'' , or σ , σ' , σ'' , in terms of the observed right ascension and declination at the time $t=0$, their velocities, and accelerations, it will be necessary to introduce polar coördinates throughout. We have

$$\xi = \sigma \cos \alpha, \quad \eta = \sigma \sin \alpha, \quad \zeta = \sigma \tan \delta, \quad (10)$$

and after differentiating twice with respect to t and dividing by $(k)^2$

$$\begin{aligned} \xi'' &= \sigma'' \cos \alpha - 2 \sigma' \sin \alpha \alpha' - \sigma \sin \alpha \alpha'' - \sigma \cos \alpha \alpha'^2 \\ \eta'' &= \sigma'' \sin \alpha + 2 \sigma' \cos \alpha \alpha' + \sigma \cos \alpha \alpha'' - \sigma \sin \alpha \alpha'^2 \\ \zeta'' &= \sigma'' \tan \delta + 2 \sigma' (\tan \delta)' + \sigma (\tan \delta)'' . \end{aligned} \quad (11)$$

We also have

$$\begin{aligned} (\xi) &= (\sigma) \cos (\alpha), \quad (\eta) = (\sigma) \sin (\alpha), \quad (\zeta) = (\sigma) \tan (\delta) \\ (x) &= (s) \cos (a), \quad (y) = (s) \sin (a), \quad (z) = (s) \tan (d) \\ X &= S \cos A, \quad Y = S \sin A, \quad Z = S \tan D. \end{aligned} \quad (12)$$

On substituting, as indicated above, expressions (9) in (6) and then introducing polar coördinates in accordance with (10) to (12), we obtain the following equations for the determination of σ , σ' , σ'' :

$$\begin{aligned} \sigma'' \cos \alpha - 2 \sigma' \sin \alpha \alpha' - \sigma \sin \alpha \alpha'' - \sigma \cos \alpha \alpha'^2 + \sigma \cos \alpha \left[\frac{1}{r^3} + \frac{\gamma}{[r]^3} \right] = \\ \frac{(\sigma) \cos (\alpha)}{r^3} + \gamma S \cos A \left[\frac{1}{[r]^3} - \frac{1}{R^3} \right] - \frac{(s) \cos (a)}{(r)^3} \end{aligned} \quad (a)$$

$$\begin{aligned} \sigma'' \sin \alpha + 2 \sigma' \cos \alpha \alpha' + \sigma \cos \alpha \alpha'' - \sigma \sin \alpha \alpha'^2 + \sigma \sin \alpha \left[\frac{1}{r^3} + \frac{\gamma}{[r]^3} \right] = \\ \frac{(\sigma) \sin (\alpha)}{r^3} + \gamma S \sin A \left[\frac{1}{[r]^3} - \frac{1}{R^3} \right] - \frac{(s) \sin (a)}{(r)^3} \end{aligned} \quad (b) \quad (13)$$

$$\begin{aligned} \sigma'' \tan \delta + 2 \sigma' (\tan \delta)' + \sigma (\tan \delta)'' + \sigma \tan \delta \left[\frac{1}{r^3} + \frac{\gamma}{[r]^3} \right] = \\ \frac{(\sigma) \tan (\delta)}{r^3} + \gamma S \tan D \left[\frac{1}{[r]^3} - \frac{1}{R^3} \right] - \frac{(s) \tan (d)}{(r)^3} . \end{aligned} \quad (c)$$

Multiply (a) by $\cos \alpha$ and (b) by $\sin \alpha$ and add. Then multiply (a) by $-\sin \alpha$ and (b) by $\cos \alpha$ and add, and copy (c). We then obtain the following equations in σ , σ' , and σ'' :

$$\begin{aligned} \sigma'' + \sigma \left[\frac{1}{r^3} - \alpha'^2 + \frac{\gamma}{[r]^3} \right] &= \frac{(\sigma)}{r^3} \cos [(\alpha) - \alpha] + \gamma S \left[\frac{1}{[r]^3} - \frac{1}{R^3} \right] \cos [A - \alpha] - \frac{(s)}{(r)^3} \cos [(a) - \alpha] \quad (a) \\ 2 \sigma' \alpha' + \sigma \alpha'' &= \frac{(\sigma)}{r^3} \sin [(\alpha) - \alpha] + \gamma S \left[\frac{1}{[r]^3} - \frac{1}{R^3} \right] \sin [A - \alpha] - \frac{(s)}{(r)^3} \sin [(a) - \alpha] \quad (b) \\ \sigma'' \tan \delta + 2 \sigma' (\tan \delta)' + \sigma \left[\frac{\tan \delta}{r^3} + (\tan \delta)'' + \gamma \frac{\tan \delta}{[r]^3} \right] &= \\ \frac{(\sigma)}{r^3} \tan (\delta) + \gamma S \left[\frac{1}{[r]^3} - \frac{1}{R^3} \right] \tan D - \frac{(s)}{(r)^3} \tan (d) . \end{aligned} \quad (14) \quad (c)$$

In these equations $\gamma, \alpha, \delta, \alpha', \delta', \alpha'', \delta'', (\sigma), (\alpha), (\delta), (r), (a), (d), (s), R, A, D, S$, are known and $(\tan \delta)'$ and $(\tan \delta)''$ are given by

$$(\tan \delta)' = \sec^2 \delta_{\alpha} \delta'; \quad (\tan \delta)'' = \sec^2 \delta_{\alpha} [2 \tan \delta_{\alpha} \delta'^2 + \delta''] . \quad (15)$$

The unknowns involved are $\sigma, \sigma', \sigma'', r$ and $[r]$.

The equations may be conveniently solved for $\sigma, \sigma', \sigma''$ by determinants. The coefficients of σ in the first and third equations contain the unknowns r and $[r]$. But the determinant of the coefficients of $\sigma, \sigma', \sigma''$ is independent of r and $[r]$. This determinant shall be denoted by 2Δ . On the right-hand side $\frac{1}{r^3}$ is a factor of the first term, $[\frac{1}{[r]^3} - \frac{1}{R^3}]$ of the second term, and $\frac{1}{(r)^3}$ of the third term in each of the three equations. If, then, in solving for σ , we replace in 2Δ the coefficients of σ by the corresponding absolute terms, the new determinant may be written as the algebraical sum of three determinants, of which the first has the factor $\frac{1}{r^3}$, the second the factor $[\frac{1}{[r]^3} - \frac{1}{R^3}]$ and the third the factor $\frac{1}{(r)^3}$. Hence, if we write

$$\begin{aligned} \Delta &= \alpha'^3 \tan \delta - \alpha'' (\tan \delta)' + \alpha''' (\tan \delta)'' \\ \Delta \kappa &= -S \{ (\tan \delta \cos [A - \alpha] - \tan D) \alpha' + \sin [A - \alpha] (\tan \delta)' \} \\ \Delta (\kappa) &= -(\sigma) \{ (\tan \delta \cos [(\alpha) - \alpha] - \tan (\delta)) \alpha' + \sin [(\alpha) - \alpha] (\tan \delta)' \} \\ \Delta [\kappa] &= -(\kappa) \{ (\tan \delta \cos [(a) - \alpha] - \tan (d)) \alpha' + \sin [(a) - \alpha] (\tan \delta)' \} , \end{aligned} \quad (16)$$

then

$$\sigma = \frac{(\kappa)}{r^3} + \gamma \kappa \left[\frac{1}{[r]^3} - \frac{1}{R^3} \right] - \frac{[\kappa]}{(r)^3} . \quad (17)$$

Let

$$E = \frac{(\kappa)}{\cos \delta}; \quad F = -\frac{\gamma \kappa}{\cos \delta}; \quad G = -\frac{\gamma}{\cos \delta} \left[\frac{\kappa}{R^3} + \frac{m [\kappa]}{(r)^3} \right] - F \left[\frac{1}{R^3} + \frac{m [\kappa]}{\kappa (r)^3} \right], \quad (18)$$

then

$$\rho = \frac{E}{r^3} + \frac{F}{[r]^3} + G, \quad (19)$$

where E, F , and G are completely known.

As a check we evidently have $(\kappa) = \kappa + [\kappa]$.

To obtain two other equations for the complete determination of the unknowns ρ, r , and $[r]$, let ψ be the angle subtended at the Earth by the satellite and the center of its primary and $[\psi]$ the angle subtended at the Earth by the satellite and the Sun. Then

$$\begin{aligned} r^2 &= (\rho)^2 + \rho^2 - 2 (\rho) \rho \cos \psi \\ [r]^2 &= R^2 + \rho^2 - 2 R \rho \cos [\psi], \end{aligned} \quad (20)$$

where ψ and $[\psi]$ depend on known quantities. Equations (19) and (20) may be solved conveniently by approximations in several ways, according to the conditions of the problem. In the case of a satellite, for example, let

$$\frac{F}{[r]^3} + G = H, \quad (21)$$

where in the first approximation we can put $[r] = (r)$. Then

$$\rho = \frac{E}{r^3} + H. \quad (22)$$

Eliminating ρ from (22) and the first of (20), we have

$$r^3 - [(\rho)^2 - 2H(\rho) \cos \psi + H^2] r^6 + 2E[(\rho) \cos \psi - H] r^3 - E^2 = 0, \quad (23)$$

or

$$r^3 - \{[(\rho) \cos \psi - H]^2 + (\rho)^2 \sin^2 \psi\} r^6 + 2E[(\rho) \cos \psi - H] r^3 - E^2 = 0,$$

or

$$[(\rho) \cos \psi - H]^2 r^6 - 2E[(\rho) \cos \psi - H] r^3 + E^2 = r^6 - (\rho)^2 \sin^2 \psi r^6,$$

or

$$\{[(\rho) \cos \psi - H] r^3 - E\}^2 = r^6 - (\rho)^2 \sin^2 \psi r^6,$$

or

$$[(\rho) \cos \psi - H] r^3 - E = \pm r^3 \sqrt{r^6 - (\rho)^2 \sin^2 \psi}, \quad (24)$$

or

$$r^3 = \frac{E \pm r^3 \sqrt{r^6 - (\rho)^2 \sin^2 \psi}}{(\rho) \cos \psi - H}. \quad (25)$$

As a first approximation we have

$$r^3 = \frac{E}{(\rho) \cos \psi - H}. \quad (26)$$

With the value of r found from (26) we obtain ρ from (22) and then $[r]$ from the second of (20). H may now be more accurately computed from (21). A closer value of r will then result from (25), if at the same time the first approximation of r obtained from (26) is introduced on the right hand side of (25). This method of approximation has proved thoroughly satisfactory in the case of the seventh satellite of Jupiter, as shown by Dr. CRAWFORD'S and Mr. CHAMPREUX'S investigation in Part 10, page 492.

Instead of substituting successively for a given value of H a previous approximation of r in the second term of the numerator of the right hand member of (25), it is advantageous to apply the method of differential correction. Let

$$f(r) = r^3 - \frac{E \pm r^3 \sqrt{r^6 - (\rho)^2 \sin^2 \psi}}{K}, \quad (27)$$

where

$$K = (\rho) \cos \psi - H. \quad (28)$$

It is required to find the values of r such that $f(r) = 0$. Let r_i be any approximation, *e. g.*, the first, if obtained from (26), and let $f(r_i) = M_i$. Then the correction to be applied to r_i , so that $f(r) = f(r_i + \partial r_i) = 0$, is

$$\partial r = \frac{-M_i}{\left(\frac{\partial f(r)}{\partial r}\right)_{r=r_i}}. \quad (29)$$

By differentiating (27) with respect to r and substituting the differential coefficient for $r = r_i$ in (29), we readily derive:

$$\partial r = 3 r_i^2 \left[1 \mp \frac{4 r_i^2 - 3 (\rho)^2 \sin^2 \psi}{3 K^2 r_i^2 - (\rho)^2 \sin^2 \psi} \right]. \quad (30)$$

Two solutions, r_1 and r_2 , result after the first approximation and these are then continued separately until $f(r_1) = 0$ and $f(r_2) = 0$. The values of H and K may then be improved for both solutions by the second equation of (20), by (21), and (28). The approximations for r are then repeated with the new value of K , etc. This method of approximation was successfully applied by Messrs. CRAWFORD and MEYER in the case of the eighth satellite of Jupiter, as shown in Part 10, page 500.

It should be observed, however, that by proper transformation of the second of equations (20) and of (23), tables similar to those given at the end of Part 7, might be constructed so as to serve for interpolating the values of the geocentric distances corresponding to the two solutions, for successive values of H or K .

The foregoing solution is based on approximate values of $(\xi)''$, $(\eta)''$, $(\zeta)''$, the perturbations of the primary being neglected. More accurate values of these accelerations may be obtained by numerical differentiation from the data given in the *Astronomisches Jahrbuch*, the *American Ephemeris and Nautical Almanac*, or other similar publications. Let us assume for the present that accurate numerical values of $(\xi)''$, $(\eta)''$, $(\zeta)''$, are available. Then, after introducing polar coördinates in accordance with (10) to (12), equations (6) become:

$$\begin{aligned} \sigma'' \cos \alpha - 2 \sigma' \sin \alpha \alpha' - \sigma \sin \alpha \alpha'' - \sigma \cos \alpha \alpha'^2 + \sigma \cos \alpha \left[\frac{1}{r^3} + \frac{\gamma}{[r]^3} \right] = \\ \frac{(\sigma) \cos(\alpha)}{r^3} + \gamma S \cos \alpha \frac{1}{[r]^3} + \gamma \frac{(\kappa) \cos(\alpha)}{(r)^3} + (\xi)'' \quad (a) \\ \sigma'' \sin \alpha + 2 \sigma' \cos \alpha \alpha' + \sigma \cos \alpha \alpha'' - \sigma \sin \alpha \alpha'^2 + \sigma \sin \alpha \left[\frac{1}{r^3} + \frac{\gamma}{[r]^3} \right] = \\ \frac{(\sigma) \sin(\alpha)}{r^3} + \gamma S \sin \alpha \frac{1}{[r]^3} + \gamma \frac{(\kappa) \sin(\alpha)}{(r)^3} + (\eta)'' \quad (b) \quad (31) \\ \sigma'' \tan \delta + 2 \sigma' (\tan \delta)' + \sigma (\tan \delta)'' - \sigma \tan \delta \left[\frac{1}{r^3} + \frac{\gamma}{[r]^3} \right] = \\ \frac{(\sigma) \tan(\delta)}{r^3} + \gamma S \tan \delta \frac{1}{[r]^3} + \gamma \frac{(\kappa) \tan(\delta)}{(r)^3} + (\zeta)'' \quad (c) \end{aligned}$$

It will be noted that the terms divided by $\frac{1}{(r)^3}$ and $\frac{1}{[r]^3}$ appear in consequence of the solar perturbations.

As before, multiply (a) by $\cos \alpha$ and (b) by $\sin \alpha$ and add. Then multiply (a) by $-\sin \alpha$ and (b) by $\cos \alpha$ and add, and copy (c). We then obtain the following new equations in σ , σ' , σ'' :

$$\sigma'' + \sigma \left[\frac{1}{r^3} - \alpha'^2 + \gamma \frac{1}{[r]^3} \right] = \quad (a)$$

$$\frac{(\sigma)}{r^3} \cos[(\alpha) - \alpha] + \gamma \frac{1}{[r]^3} \cos[(\alpha) - \alpha] + \gamma \frac{(\kappa)}{(r)^3} \cos[(\alpha) - \alpha] + (\xi)'' \cos \alpha + (\eta)'' \sin \alpha$$

$$2\sigma' \alpha' + \sigma \alpha'' \quad (32)$$

$$\frac{(\sigma)}{r^3} \sin[(\alpha) - \alpha] + \gamma \frac{1}{[r]^3} \sin[(\alpha) - \alpha] + \gamma \frac{(\kappa)}{(r)^3} \sin[(\alpha) - \alpha] + (\xi)'' \sin \alpha + (\eta)'' \cos \alpha \quad (b)$$

$$\sigma'' \tan \delta + 2\sigma' (\tan \delta)' + \sigma \left[\frac{\tan \delta}{r^3} + (\tan \delta)'' + \gamma \frac{\tan \delta}{[r]^3} \right] = \quad (c)$$

$$\frac{(\sigma)}{r^3} \tan(\delta) + \gamma \frac{1}{[r]^3} \tan \delta + \gamma \frac{(\kappa)}{(r)^3} \tan(\delta) + (\zeta)''.$$

The left-hand sides of these equations are the same as those of (14).

On the right-hand side, however, the first three terms have the common factors $\frac{1}{r^3}$, $\frac{1}{[r]^3}$ and $\frac{1}{(r)^3}$, respectively, while the remaining terms are multiplied by the geocentric accelerations of the primary. In solving for σ by determinants, we have, therefore, an additional determinant depending upon these accelerations.

Let this be denoted by $2(\Delta)$, so that

$$2(\Delta) = \begin{vmatrix} 1 & 0 & +(\xi)'' \cos \alpha + (\eta)'' \sin \alpha \\ 0 & 2\alpha' & -(\xi)'' \sin \alpha + (\eta)'' \cos \alpha \\ \tan \delta & 2(\tan \delta)' & +(\zeta)'' \end{vmatrix}, \quad (33)$$

or

$$2(\Delta) = 2\alpha' (\zeta)'' - 2\alpha' \tan \delta (\xi)'' \cos \alpha - 2\alpha' \tan \delta (\eta)'' \sin \alpha + 2(\tan \delta)' (\xi)'' \sin \alpha - 2(\tan \delta)' (\eta)'' \cos \alpha,$$

or

$$(\Delta) = (\zeta)'' \alpha' + (\xi)'' [\sin \alpha (\tan \delta)' - \cos \alpha \tan \delta \alpha'] - (\eta)'' [\cos \alpha (\tan \delta)' + \sin \alpha \tan \delta \alpha']. \quad (34)$$

Then performing the solution by determinants exactly as in the case of equations (14) and letting

$$\Delta \kappa_0 = (\Delta), \quad (35)$$

we obtain

$$\sigma = \frac{(\kappa)}{r^3} + \gamma \frac{\kappa}{[r]^3} + \gamma \frac{[\kappa]}{(r)^3} + (\kappa)_0. \quad (36)$$

Then letting E and F represent the same expressions as in (18) and putting

$$G = -\frac{\gamma [\kappa]}{\cos \delta (r)^3} + (\kappa)_0, \quad (37)$$

we have as before in (19)

$$\rho = \frac{E}{r^3} + \frac{F}{[r]^3} + G. \quad (38)$$

This equation together with equations (20) gives the values of ρ , r , and $[r]$, the method of solution being exactly the same as in the former case.

After ρ and therefore σ have been determined by equation (17), σ' may be computed from equation (14b) and σ'' from equation (14a) or (14c).

ORBITS OF SATELLITES IRRESPECTIVE OF PERTURBATIONS.

If it be intended to compute the orbit of the satellite without regard to any perturbations whatsoever, formulæ (32) to (36) in connection with the first of equations (20), are at once available, if all terms involving γ be omitted, these terms appearing in consequence of the solar perturbations. The equations for σ , σ' , σ'' may also be written out directly by analogy, as follows:

$$\begin{aligned}\sigma'' &= \sigma \left[\frac{1}{r^3} - \alpha'^2 \right] = \frac{(\sigma)}{r^3} \cos [(\alpha) - \alpha] + (\xi)'' \cos \alpha + (\eta)'' \sin \alpha \quad (a) \\ 2 \sigma' \alpha' + \sigma \alpha'' &= \frac{(\sigma)}{r^3} \sin [(\alpha) - \alpha] - (\xi)'' \sin \alpha + (\eta)'' \cos \alpha \quad (b) \quad (39) \\ \sigma'' \tan \delta + 2 \sigma' (\tan \delta)' + \sigma \left[\frac{\tan \delta}{r^3} + (\tan \delta)'' \right] &= \frac{(\sigma)}{r^3} \tan (\delta) + (\zeta)'' \quad (c)\end{aligned}$$

From these equations we obtain, by the first and third of (16), by the first of (18), by (34) and (35)

$$\rho = \frac{E}{r^3} + \kappa_0. \quad (40)$$

This equation, together with the first of (20), gives the various solutions for r or ρ in the usual manner.

The geocentric acceleration $(\xi)''$, $(\eta)''$, $(\zeta)''$ in (39) are supposed to have been accurately computed from the data of the astronomical ephemerides. Approximate values, however, for $(\xi)''$, $(\eta)''$, $(\zeta)''$ may be obtained as in the general case from (7) and (8). We shall then have in (39):

$$\begin{aligned}(\xi)'' \cos \alpha + (\eta)'' \sin \alpha &= \gamma \frac{(1+m)}{(r)^3} (S) \cos [(\alpha) - \alpha] - \gamma \frac{S}{R^3} \cos (A - \alpha) \\ - (\xi)'' \sin \alpha + (\eta)'' \cos \alpha &= \gamma \frac{(1+m)}{(r)^3} (S) \sin [(\alpha) - \alpha] - \gamma \frac{S}{R^3} \sin (A - \alpha) \quad (41) \\ (\zeta)'' &= \gamma \frac{(1+m)}{(r)^3} (S) \tan d - \gamma \frac{S}{R^3} \tan D.\end{aligned}$$

Solving (39) by determinants as before, we have by (16),

$$\sigma = \frac{(\kappa)}{r^3} - \gamma (1+m) \frac{[\kappa]}{(r)^3} - \gamma \frac{\kappa}{R^3}. \quad (42)$$

Now, by the first of (18) and letting

$$G = \gamma \frac{\kappa}{\cos \delta} \left[\frac{1}{R^3} - \frac{(1+m)}{\kappa (r)^3} [\kappa] \right], \quad (43)$$

we obtain, as before

$$\rho = \frac{E}{r^3} + G. \quad (44)$$

As before, σ' is given by equation (39b) and σ'' either by (39a) or (39c), after ρ or σ have been found.

OSCULATING ORBITS OF COMETS OR MINOR PLANETS.

The formulæ for the determination of the distances of a disturbed comet or asteroid from the Earth, Sun, and disturbing planet may be derived either directly from the equations of motion in rectangular coördinates by proceeding in a similar manner as above in the case of a disturbed satellite, or they may be written by analogy from the formulæ for a disturbed satellite, or they may be deduced from the latter by change of notation and transformation of the origin from the center of the planet to that of the Sun.

If the last mentioned method be adopted, we would first substitute in equation (3) $x = [x] - (x)$; $x'' = [x]'' - (x)''$, etc.; then we would multiply both sides of the equations by $\frac{(k)^2}{k^2} = m$, to change the unit of time from $\frac{1}{(k)}$ to $\frac{1}{k}$ mean solar days. Then we would substitute $(x)'' = -\frac{(1+m)(x)}{(r)^3}$, where $(x)''$ now represents the acceleration of the planet in $\frac{1}{k}$ mean solar days. Finally, we would write x, y, z, r in place of $[x], [y], [z], [r]$, and *vice versa*, so that, as in the case of disturbed satellites, the coördinates of the disturbed body with reference to the central and disturbing body, may be designated without parentheses and in square brackets, respectively.

After the introduction of geocentric coördinates, the only change in equations (6) and following would then be that m would appear in place of γ , that in place of x, y, z, r we would have $[x], [y], [z], [r]$, and *vice versa*, that in place of $(\xi), (\eta), (\zeta)$ we would have X, Y, Z , and *vice versa*, and that in place of $(x), (y), (z)$ we would have $-(x), -(y), -(z)$. The right hand side of the first equation (6), for example, would then contain terms multiplied by $(\xi), (x)$ and X . Since any one of these coördinates may be expressed in terms of the other two, the equation may be reduced so as to contain only two of these coördinates. After the coefficients of these coördinates have been written down, only two expressions (16) remain, and the final formulæ (17) and following may be written out at once.

ORBITS OF COMETS AND PLANETS IRRESPECTIVE OF PERTURBATIONS.

Formulæ for the determination of the heliocentric and geocentric distances may be written at once by merely omitting in the preceding case all terms multiplied by m , the mass of the disturbing body. The resulting formulæ will then be those of the *Short Method* as derived in Parts 1 and 7.

In case no assumption can be made regarding the nature of the newly discovered body, the decision whether the body is a disturbed satellite or disturbed minor planet, or whether it is moving under nearly equal attractions, can be made readily after the solution of the orbit has been accomplished either by the formulæ for a disturbed satellite or a disturbed planet, by representing the observations on the basis of the various resulting orbits. This is due to the fact that either set of formulæ takes rigid account of the attractions of the bodies involved, the difference being merely in the choice of the unit of time and of the origin to which the accelerations are referred.

As stated on page 467 the planetary aberration is accounted for in a similar manner as in Part 7. The aim of the preliminary computation must be to lead to the solution of the true rectangular coördinates x, y, z , and velocities x', y', z' of the disturbed body at the (reduced) epoch with respect to the primary, which may be either the Sun or a major planet. The observed (apparent) coördinates α, δ of the disturbed body, therefore, are reduced to the beginning of the year inclusive of the aberration of the fixed stars and the geocentric rectangular coördinates X, Y, Z of the Sun are interpolated from an ephemeris for the observed (uncorrected) time.

After the solution of the geocentric distance ρ of the disturbed body and the computation of $\xi = \rho \cos \alpha \cos \delta$, etc., we have $\xi - X$, etc., for the *true* heliocentric coördinates of the disturbed body at its (reduced) epoch t . It is evident that the geocentric distance ρ just referred to is the distance of the true position of the disturbed body (at the reduced epoch t) from the position of the Earth at the observed epoch. The true heliocentric coördinates of the disturbed body are designated by x, y, z , or by $[x], [y], [z]$, according to whether the Sun is regarded as the primary or as the disturbing body.

Similarly, if the apparent geocentric coördinates, $(\alpha), (\delta)$, of the major planet be interpolated from an ephemeris with the observed time and reduced to the beginning of the year, inclusive of the aberration terms, then, with its *proper* geocentric distance (ρ) , we have $(x) = (\rho) \cos(\alpha) \cos(\delta) - X = (\xi) - X$, etc., for the true heliocentric place of the major planet at its reduced time, t' .

It is evident that the reduced time t of the disturbed body, and the reduced time t' of the major planet are not identical, so that the combination of the true heliocentric coördinates of the disturbed body and of the major planet produces the coördinates of the disturbed body at the time t with respect to the major planet at the time t' . The difference between the times t and t' can not be ascertained until the solution has progressed to the determination of the geocentric distance of the disturbed body. In the direct solution of the osculating orbit of a disturbed satellite this difference in general is entirely negligible, but it should be considered later if necessary, when the residuals $O - C$ are being determined. In

the case of a disturbed minor planet or of a comet in which the position of the major planet need not be known with absolute rigidity, the difference between t and t' in general may also be disregarded in the course of the direct solution.

The *proper* geocentric distance (ρ) of the major body referred to above is the distance of its true position at its *reduced* time t' from the position of the Earth at the observed time, but for practical purposes (ρ) may be directly interpolated with the observed time.

As a check on (x) , (y) , (z) or on the corresponding polar coördinates (a) , (d') , (r) , the true heliocentric coördinates (l) , (b) , (r) of the major planet, referred to the mean equinox and ecliptic of date, may be interpolated with *its* reduced time t' directly from the *American Ephemeris and Nautical Almanac* and then reduced to the beginning of the year and transformed to the equator. Furthermore, the sum of these two coördinates (x) , (y) , (z) of the major body and of the solar coördinates X , Y , Z at the observed time will furnish a check on the apparent coördinates (α) , (δ) of the major body and at the same time will produce the *proper* geocentric distance (ρ) referred to above.

Reference to these points is made for the purpose of preparing for a rigid procedure in computing the *true* rectangular coördinates x , y , z and velocities x' , y' , z' of the disturbed body at the (reduced) epoch with reference to its primary after the geocentric distance and its velocity have become known in the course of the solution. In the case of a disturbed satellite where a major planet is the primary, the simplest procedure appears to be to re-determine the *true* heliocentric rectangular coördinates (x) , (y) , (z) of the major planet, but this time with the reduced time t of the disturbed satellite. The subtraction of (x) , (y) , (z) from the true heliocentric rectangular coördinates $[x]$, $[y]$, $[z]$ of the disturbed satellite at the time t will then produce the true rectangular coördinates x , y , z of the disturbed satellite at the reduced time t with reference to the major planet. The true heliocentric coördinates, $[x] = \xi - X$, etc., of the disturbed satellite at its reduced time t evidently are computed exactly as are the heliocentric coördinates x , y , z of an undisturbed comet or planet in Part 7. In the case of a disturbed planet or comet the heliocentric coördinates are designated here by x , y , z , just as in Part 7 and are the only heliocentric coördinates to be computed.

A similar procedure must be adopted in the comparison between theory and observation for the first and third places.

The geocentric distances necessary for correcting the intervals for planetary aberration may be derived from the geocentric distance at the epoch and from its velocity, its acceleration, in general, being neglected. But if the geocentric distances resulting from the representation of the places does not agree within the required accuracy with the geocentric distances adopted for the determination of the reduced times t , and t''' for the first and third places, then the representation should be repeated with improved aberration times.

DERIVATION OF THE ELEMENTS AND COMPUTATION OF THE EPHEMERIS.

The elements may now be determined for each solution of σ , σ' , and σ'' . With the aid of formulæ VIII of the *Synopsis of Formulæ* in Part 7, we first compute the geocentric coördinates ξ , η , z and the velocities ξ' , η' , z' of the disturbed body. From these we may then compute the coördinates with reference to the primary, x , y , z , and their velocities, x' , y' , z' , or with reference to the disturbing body, $[x]$, $[y]$, $[z]$, and their velocities $[x]'$, $[y]'$, $[z]'$. In the case of the direct solution of the disturbed orbit of a satellite the planet is the primary and the Sun the disturbing body. In the case of the disturbed orbit of a minor planet, or of a comet, the Sun is the primary and the planet is the disturbing body. But if no decision can be made beforehand regarding the primary and disturbing body, then the coördinates or velocities should be determined with reference to both the Sun and the planet. In order to compute the velocities referred to, it is necessary to derive by numerical differentiation the geocentric velocities X' , Y' , Z' of the Sun, or $(\xi)'$, $(\eta)'$, $(z)'$ of the planet, or both, in the doubtful case, from the data of an ephemeris.

For each set of rectangular coördinates and velocities, an osculating orbit may now be computed by the formulæ VIIIc of the *Synopsis of Formulæ*, Part 7. The computation of the constants to the equator corresponding to the osculating elements and of the corresponding ephemeris is then readily accomplished in the usual way.

In the case of the orbit of a disturbed comet or minor planet, the special perturbations are most conveniently computed by ENCKE'S method of special perturbations for the purpose of comparing the computed positions with future observations.

In the case of a disturbed satellite, ENCKE'S method of special perturbations requires slight modifications on account of the change in the constant of attraction. The necessary changes are indicated in the *Synopsis of Formulæ* which follows.

If the computations have yielded more than one solution, the physical solution is readily determined by the representation of the observations on the basis of the various orbits. For this purpose it is not necessary to compute the special perturbations for all mathematical data. The perturbations need to be computed only for the physical orbit after it has been determined.

If the representation of the observations on the basis of the physical elements and the special perturbations is not entirely satisfactory, the perturbations may be considered constant, and the osculating elements may be corrected in exactly the same manner as in Part 7, *Synopsis of Formulæ*, B.

In case of very large corrections to the osculating elements, the special perturbations may require recomputation. The necessary formulæ for the determination of the orbit of the disturbed body and for its differential correction are given in proper order, and in detail, in the *Synopsis of Formulæ* which follows.

SYNOPSIS OF FORMULÆ.

The directions for computing the orbit of a disturbed satellite are given in detail. For the computation of the orbit of a satellite irrespective of perturbations, see page 475. For the computation of the orbit of a disturbed comet or minor planet, see page 476. For the computation of the orbit of a comet or minor planet irrespective of perturbations, apply the "Short Method," Part 7.

In case no assumption has been possible regarding the nature of the newly discovered body, the orbit may be computed either by the formulæ for a disturbed satellite or by those for a disturbed minor planet.

I.

Express the observed places in terms of α and δ ,¹ reduced to the beginning of the year, inclusively of the aberration of the fixed stars. Compute α' , δ' , α'' , δ'' by the formulæ A II, footnote 2, *Synopsis of Formulæ*, Part 7.

The parallax may, in general, be neglected. Its complete elimination as set forth in the footnotes of the *Synopsis of Formulæ*, Part 7, is unwarranted on account of the additional computation involved as the solution, in general, is based on more than three observations, and also because any inaccuracies committed in the direct solution may later be removed by differential correction. In case of a suspected satellite, however, the parallax corrections may be applied on the basis of the primary's distance.

Compute with (m) the mass of the attracting planet,

$$\gamma = \frac{1}{(m)}, \quad (k)^2 = (m)k^2, \quad (k), \quad \frac{\sin 1''}{(k)}, \quad \frac{2}{(k)},$$

and if account is to be taken of the Earth's mass,

$$\frac{k^2}{(k)^2} (1 + m_{\oplus}); \quad \log (1 + m_{\oplus}) = 0.000001.$$

Interpolate, for the beginning of the year, from an ephemeris, the solar coördinates X , Y , Z for the observed date of the epoch (for which it is most convenient to choose the date of an observation near the middle of the available arc). Then interpolate also the apparent geocentric coördinates of the attracting body at the observed epoch (ρ) , (α) , (δ) and reduce these to the beginning of the year, including the aberration terms. Then $(\sigma) = (\rho) \cos (\delta)$.

¹In exceptional cases it is more advantageous to express the observed places in terms of position angle p and distance s , to derive the velocities p' , s' , and accelerations p'' , s'' , in the same manner as α' , δ' , α'' , δ'' are derived, and then to transform p' , s' into α' , δ' , and p'' , s'' into α'' , δ'' .

²Cf. page 477.

Similarly, the velocities¹ $(\xi)', (\eta)', (z)'$ may be derived, either here or under IV, in terms of $\frac{1}{(k)}$ mean solar days by numerical differentiation from the data of an ephemeris, exactly as in Part 7, *Synopsis of Formulæ* A I. For this purpose the $(\xi), (\eta), (z)$ may be computed from the corresponding $(\rho), (\alpha), (\delta)$.

Correct the observed date of the middle place for planetary aberration on the basis of the primary's distance. With the corrected date, take from an ephemeris the heliocentric coördinates of the planet (primary) $(r), (l), (b)$, from which $(a), (d)$ may be computed. Then $(s) = (r) \cos (d)$.

$$R \cos D \cos A = X, \quad R \cos D \sin A = Y, \quad R \sin D = Z, \quad S = R \cos D$$

$$\cos \left\{ \begin{smallmatrix} \psi \\ [\psi] \end{smallmatrix} \right\} = \sin \omega \sin \delta + \cos \omega \cos \delta \cos (\Omega - \alpha),$$

where ψ is computed from $\omega = (\delta)$, $\Omega = (\alpha)$ and $[\psi]$ from $\omega = D$ and $\Omega = A$.

For small values of ψ (or $[\psi]$) use

$$\cos \psi = \cos \delta \cos (\alpha - \Omega), \quad \sin \psi \cos P = \cos \delta \sin (\alpha - \Omega), \quad \sin \psi \sin P = \sin \delta.$$

II.

$$A = \alpha' \tan \delta - \alpha'' (\tan \delta)' + \alpha' (\tan \delta)'',$$

$$\kappa = -\frac{S}{A} \{ (\tan \delta \cos [A - \alpha] - \tan D) \alpha' + \sin [A - \alpha] (\tan \delta)' \},$$

$$(\kappa) = -\frac{(\sigma)}{A} \{ (\tan \delta \cos [(\alpha) - \alpha] - \tan (\delta)) \alpha' + \sin [(\alpha) - \alpha] (\tan \delta)' \},$$

$$[\kappa] = -\frac{s}{A} \{ (\tan \delta \cos [(a) - \alpha] - \tan (d)) \alpha' + \sin [(a) - \alpha] (\tan \delta)' \}.$$

$$E = \frac{(\kappa)}{\cos \delta}, \quad F = \frac{\gamma \kappa}{\cos \delta}, \quad G = -F \left[\frac{1}{R^2} + \frac{m [\kappa]}{\kappa (r)^2} \right].$$

III.

SOLUTION OF THE DISTANCES OF THE DISTURBED BODY FROM THE EARTH AND ATTRACTING MASSES.

$$H = \frac{F}{[r]^2} + G, \tag{a}$$

where as a first approximation $[r] = (r)$.

$$\text{Then,} \quad r^2 = \frac{E \pm r^2 \sqrt{r^2 - (\rho)^2 \sin^2 \psi}}{(\rho) \cos \psi - H}, \tag{b}$$

where in the first approximation $\pm r^2 \sqrt{} = 0$. The equation (b) is then solved

¹Cf. footnote 2, page 482.

successively by introducing the preceding approximation of r in $\pm r^3 v'$, the two resulting solutions being kept separate. After the equation (b) has been satisfied, compute for both solutions

$$\rho = \frac{E}{r^2} + \frac{F}{[r]^3} + G \quad (c)$$

$$r^2 = (\rho)^2 + \rho^2 - 2(\rho)\rho \cos \psi, \quad [r]^2 = R^2 + \rho^2 - 2R\rho \cos [\psi]. \quad (d)$$

Then repeat the whole process with more accurate values of H from formulæ (a) until H and r remain constant.

Or, for any approximation of H , correct the first approximation of r successively by

$$\Delta r = 3r_i^2 \left[1 \mp \frac{4r_i^2 - 3(\rho)^2 \sin^2 \psi}{3K \sqrt{r_i^2 - (\rho)^2 \sin^2 \psi}} \right].$$

IV.

Compute $\sigma = \rho \cos \delta$ and σ' and σ'' from

$$\begin{aligned} 2\sigma'\sigma'' &= \frac{(\sigma)}{r^3} \sin[(\alpha) - \alpha] + \gamma S \left[\frac{1}{[r]^3} - \frac{1}{R^3} \right] \sin[A - \alpha] + \frac{(s)}{(r)^3} \sin[(a) - \alpha] - \sigma\sigma'', \\ \sigma'' &= \frac{(\sigma)}{r^3} \cos[(\alpha) - \alpha] + \gamma S \left[\frac{1}{[r]^3} - \frac{1}{R^3} \right] \cos[A - \alpha] - \frac{(s)}{(r)^3} \cos[(a) - \alpha] \\ &\quad - \sigma \left[\frac{1}{r^3} - \alpha'^2 + \frac{\gamma}{[r]^3} \right]. \end{aligned}$$

V.

Compute the residuals, the elements, and the constants to the equator as in Part 7, *Synopsis of Formulæ A* VI and VIII, using the closed expressions for f and g , if necessary as in Part 7, *Synopsis of Formulæ B*.

In case of doubt as to the nature of the disturbed body, four¹ orbits may require consideration. The orbits with reference to the Sun as primary, rest on $x = \xi - X$, $x' = \xi' - X'$, etc. The orbits with reference to the second attracting mass (planet) as primary rest on $x = \xi - (\xi)$; $x' = \xi' - (\xi)'$, etc.²

If an assumption regarding the nature of the disturbed body has been possible, then the orbits with reference to only one of the attracting masses (either Sun or planet) require consideration.

The selection of the physical solution in any case is made on the basis of the representation of the positions upon which the solutions were based.

VI.

To improve the physical solution, the special perturbations of the disturbing body are best computed by ENCKE's method of rectangular coördinates and applied to the positions computed from the osculating elements for the purpose of deriving the residuals to be removed.

¹Cf. footnote 1, page 501, where three solutions for the distances, corresponding to six mathematical orbits were found to exist.

²Provided it is intended to neglect the difference of the aberration times for the planet and satellite. For greater rigidity cf. page 478, third paragraph, which applies to the velocities as well as to the coördinates.

In the case of a disturbed satellite, the Sun being the disturbing body, ENCKE'S formulæ as given in VON OPPOLZER'S *Lehrbuch zur Bahnbestimmung*, Vol. II, pages 72 to 81, take the following form, starting with equation (3), page 73:

$$\frac{d^2 x}{(k)^2 dt^2} = -\frac{x}{r^3} - \gamma \frac{(x) + x}{[r]^3} + \gamma \frac{(x)}{(r)^3}, \quad \gamma = \frac{1}{(m)} = \frac{k^2}{(k)^2}$$

and since

$$x = x_0 + \bar{\xi}$$

and

$$\frac{d^2 x_0}{(k)^2 dt^2} = -\frac{x_0}{r_0^3},$$

$$\frac{d^2 \bar{\xi}}{(k)^2 dt^2} = -\gamma \left\{ \frac{(x) + x}{[r]^3} - \frac{(x)}{(r)^3} \right\} + \left\{ \frac{x_0}{r_0^3} - \frac{x}{r^3} \right\},$$

or

$$\frac{d^2 \bar{\xi}}{dt^2} = -k^2 \left\{ \frac{(x) + x}{[r]^3} - \frac{(x)}{(r)^3} \right\} + (k)^2 \left\{ \frac{x_0}{r_0^3} - \frac{x}{r^3} \right\}.$$

Let

$$q = (x_0 + \frac{1}{2} \bar{\xi}) \bar{\xi} + (y_0 + \frac{1}{2} \bar{\eta}) \bar{\eta} + (z_0 + \frac{1}{2} \bar{z}) \bar{z}.$$

With q as argument take f from v. OPPOLZER'S table or compute f directly. Then

$$\frac{d^2 \bar{\xi}}{dt^2} = -k^2 \left\{ \frac{(x) + x}{[r]^3} - \frac{(x)}{(r)^3} \right\} + \frac{(k)^2}{r_0^3} \left\{ f q x - \bar{\xi} \right\} - w k^2 \left\{ \frac{(x) + x}{[r]^3} - \frac{(x)}{(r)^3} \right\} = \Sigma(X);$$

or, if

$$h = \frac{w(k)^2}{r_0^3}; \quad a = \frac{x_0 + \frac{1}{2} \bar{\xi}}{r_0^2(1 + \frac{1}{2} h)}; \quad b = \frac{y_0 + \frac{1}{2} \bar{\eta}}{r_0^2(1 + \frac{1}{2} h)}; \quad c = \frac{z_0 + \frac{1}{2} \bar{z}}{r_0^2(1 + \frac{1}{2} h)},$$

$$S_{(x)} = f_{(x)}(a + iw) + \frac{1}{2} \Sigma(X) - \frac{1}{2} f^n(a + iw) \dots, \quad S_{(y)}, \quad S_{(z)}, \text{ etc.}$$

Then

$$q = \frac{a S_{(x)} + b S_{(y)} + c S_{(z)}}{1 - \frac{1}{2} f(ax + by + cz)}.$$

If further

$$h' = \frac{h}{1 + \frac{1}{2} h},$$

then

$$\frac{d^2 \bar{\xi}}{dt^2} = \Sigma(X) + h' \left\{ f q x - S_{(x)} \right\}, \quad \frac{d^2 \bar{\eta}}{dt^2}, \quad \frac{d^2 \bar{z}}{dt^2}.$$

$$R^2 = r_0^2(1 + \frac{1}{2} h); \quad S(X) = [wk]^2 \left\{ \frac{(x)}{(r)^3} - \frac{[x]}{[r]^3} \right\}.$$

The residuals are removed by differential correction by applying the proper formulæ of Part 7, *Synopsis of Formulæ*, A VII or B.

the information science community. The authors of the paper have been successful in their attempt to make a case for the importance of the information science community in the development of the information science discipline.

1. The information science community is a community of scholars who are interested in the study of information science.

2.

3.

4.

5.

6.

7.

8.

9.

10.

11.

12.

13.

14.

15.

16.

17.

18.

19.

20.

21.

22.

23.

24.

25.

26.

27.

28.

29.

30.

31.

32.

33.

34.

35.

36.

37.

38.

39.

40.

41.

42.

43.

44.

45.

46.

47.

48.

49.

50.

51.

52.

53.

54.

55.

56.

57.

58.

59.

60.

61.

62.

63.

64.

65.

66.

67.

68.

69.

70.

71.

72.

73.

74.

75.

76.

77.

78.

79.

80.

81.

82.

83.

84.

85.

86.

87.

88.

89.

90.

91.

92.

93.

94.

95.

96.

97.

98.

99.

100.

101.

102.

103.

104.

105.

106.

107.

108.

109.

110.

111.

112.

113.

114.

115.

116.

117.

118.

119.

120.

121.

122.

123.

124.

125.

126.

127.

128.

129.

130.

131.

132.

133.

134.

135.

136.

137.

138.

139.

140.

141.

142.

143.

144.

145.

146.

147.

148.

149.

150.

151.

152.

153.

154.

155.

156.

157.

158.

159.

160.

161.

162.

163.

164.

165.

166.

167.

168.

169.

170.

171.

172.

173.

174.

175.

176.

177.

178.

179.

180.

181.

182.

183.

184.

185.

186.

187.

188.

189.

190.

191.

192.

193.

194.

195.

196.

197.

198.

199.

200.

201.

202.

203.

204.

205.

206.

207.

208.

209.

210.

211.

212.

213.

214.

215.

216.

217.

218.

219.

220.

221.

222.

223.

224.

225.

226.

227.

228.

229.

230.

231.

232.

233.

234.

235.

236.

237.

238.

239.

240.

241.

242.

243.

244.

245.

246.

247.

248.

249.

250.

251.

252.

253.

254.

255.

256.

257.

258.

259.

260.

261.

262.

263.

264.

265.

266.

267.

268.

269.

270.

271.

272.

273.

274.

275.

276.

277.

278.

279.

280.

281.

282.

283.

284.

285.

286.

287.

288.

289.

290.

291.

292.

293.

294.

295.

296.

297.

298.

299.

300.

301.

302.

303.

304.

305.

306.

307.

308.

309.

310.

311.

312.

313.

314.

315.

316.

317.

318.

319.

320.

321.

322.

323.

324.

325.

326.

APPLICATION OF LEUSCHNER'S METHOD OF DIRECT
SOLUTION OF ORBITS OF DISTURBED BODIES.

Doc. 10000

COMPILED BY

RUSSELL TRACY CRAWFORD,

ASSOCIATE PROFESSOR OF PRACTICAL ASTRONOMY,

BERKELEY ASTRONOMICAL DEPARTMENT.

APPLICATIONS OF LEUSCHNER'S METHOD OF DIRECT SOLUTION OF ORBITS OF DISTURBED BODIES.

By RUSSELL TRACY CRAWFORD.

CONTENTS.

	PAGE.
INTRODUCTION - - - - -	489
THE ORBIT OF THE SEVENTH SATELLITE OF JUPITER. Computation by R. T. CRAWFORD and A. J. CHAMPREUX - - - - -	490
THE ORBIT OF THE EIGHTH SATELLITE OF JUPITER. Computation by R. T. CRAWFORD and W. F. MEYER - - - - -	499

INTRODUCTION.

Two examples are given to show the computations necessary for the determination of an orbit according to LEUSCHNER'S method of *Direct Solution of Orbits of Disturbed Bodies*, as set forth in Part 9 of this volume. The numbers of formulæ given are those of the *Synopsis of Formulæ* in Part 9.

The computation for the Seventh Satellite of Jupiter is given in full. The only difference in the direct solutions between the orbits of the Seventh and the Eighth Satellites is in the manner of solving the equation for r , so that this will be given in detail for the case of the Eighth Satellite.

The computations for the Seventh Satellite were made by R. T. CRAWFORD and A. J. CHAMPREUX; those for the Eighth Satellite were made by R. T. CRAWFORD and W. F. MEYER.

THE ORBIT OF THE SEVENTH SATELLITE OF JUPITER.

I

The observations upon which this orbit is based were made by PERRINE at the Lick Observatory. They are published in *Lick Observatory Bulletin* No. 78. They are:

	1905 Gr. M. T.	α (1905.0)	δ (1905.0)
I	Jan. 3.633333	19° 5' 55".5	+ 7° 13' 58".1
II	Jan. 28.623611	21 56 5.4	+ 8 18 11.6
III	Feb. 8.631944	23 40 9.8	+ 8 55 18.8
IV	Feb. 21.642361	26 0 45.8	+ 9 43 52.4
V	Mar. 6.652778	28 36 3.4	+ 10 35 50.7

The coördinates α and δ given here are the observed coördinates corrected for parallax corresponding to Jupiter's distance.

Use for the mass of Jupiter NEWCOMB's value, viz., $\frac{1}{m} = 1047.355$. Then

$\log \gamma = \log \frac{1}{m}$	3.0200939	$\log \frac{\sin 1''}{(k)}$	7.9600404
$\log \sqrt{\frac{1}{m}}$	1.5100470	$\log \frac{2}{(k)}$	3.5754956
$\log \sqrt{m}$	8.4899530	$\log m_{\oplus}$	4.4889
$\log k$	8.2355814	$\log (1 + m_{\oplus})$	0.000001
$\log (k)$	6.7255344	$\log \frac{k}{(k)} \sqrt{1 + m_{\oplus}}$	1.510048
$\log (k)^2$	3.4510689		

Using the first, third, and fifth observations and formulæ A II of Part 7, first approximations to the velocities and accelerations for the middle date are found to be

$\log \alpha'_0$	0.729270	$\log \delta'_0$	0.272687
$\log \alpha''_0$	2.095593	$\log \delta''_0$	1.541625

The more accurate determination of these quantities depending upon the five observations, according to the formulæ A II, footnote 2, of Part 7, is made exactly as shown in detail in example No. 2 of Part 8. The resulting values are:

$\log \alpha'_0$	0.743547	$\log \delta'_0$	0.289298	$\log (\tan \delta)_0'$	0.299872
$\log \alpha''_0$	2.069686	$\log \delta''_0$	1.496778	$\log (\tan \delta)_0''$	1.523510

From the *American Ephemeris and Nautical Almanac*, we take, for the middle date, approximate $\log (\rho) = 0.72426$. Using this to correct the middle date for aberration, we have for the corrected date, Feb. 8.601357. With this date then, we take out

$$^1 \log (\rho) = 0.724258 \quad (\alpha) = 1^h 34^m 34^s.14 \quad (\delta) = + 8^\circ 41' 1''.8$$

Then

$$\begin{aligned} (\alpha) - \alpha_0 &= - 6^s.51 \\ \tan \psi &= 7.621292 & \cos \psi &= 9.999996 & \sin \psi &= 7.621288 \end{aligned}$$

¹It would have been more accurate to use (ρ) , (α) , (δ) for the uncorrected date, and to have reduced them to the beginning of the year, including the aberration terms. Cf. also, last paragraph, page 477, and the discussion on page 478.

Again, from the *American Ephemeris and Nautical Almanac*, we take for the corrected middle date the heliocentric coördinates of Jupiter reduced to 1905.0. Whence

$$(l) = 35^{\circ} 28' 34''.6, \quad (b) = -1^{\circ} 10' 36''.0 \quad \log(r) = 0.696347$$

Transforming these to the equator in the usual way, gives

$$\begin{array}{lll} (a) \quad 33^{\circ} 34' 39''.6 & \log \sin(d) & 9.326503 \\ \log \tan(d) & 9.336497 & \log \cos(d) \quad 9.990006 \end{array}$$

Since the observations were reduced to the beginning of the year by including the aberration terms, the Sun's coördinates should be taken out for the middle date uncorrected for aberration. They are

X_0	+0.7529310	Y_0	-0.5851685	Z_0	-0.2538530
$\log X_0$	9.8767552	$\log Y_0$	9.7672810n	$\log Z_0$	9.4045823n
$R_0 \cos D \cos A$	9.876755	$\log S_0 = R_0 \cos D$	9.979360	$\sin \delta_0$	9.190578
$R_0 \cos D \sin A$	9.767281n	$R_0 \sin D$	9.404582n	$\sin D$	9.410354n
$\tan A$	9.890526n	$\tan D$	9.425222n	$\log I$	8.600932n
A	322° 8' 45''.9	$\sin D$	9.410354n	$\cos \delta_0$	9.994713
$A - \alpha_0$	298 28 36.1	$\cos D$	9.985132	$\cos D$	9.985132
$\sin A$	9.787921n	$\log R_0$	9.994228	$\cos(A - \alpha_0)$	9.678338
$\cos A$	9.897395			$\log II$	9.658183
				$\cos[\psi]$	9.618345

II

		$p; q; s$	$A; D; S$	$(\alpha); (\delta); (\sigma)$	$(a); (d); (s)$
$\log(\alpha_0)'$	2.230641	$\tan \delta_0$	9.195864	9.195864	9.195864
$\tan \delta_0$	9.195864	$\cos(p - \alpha_0)$	9.678338	0.000000	9.993473
$\log I$	1.426505	$\tan \delta_0 \cos(p - \alpha_0)$	8.874202	9.195864	9.189337
$\log \alpha_0''$	2.069686	$\tan q$	9.425222n	9.183932	9.336497
$\log(\tan \delta_0)'$	0.299872	sub	0.107610	8.444926	9.605662
$\log II$	2.369558	$\log()$	9.532832	7.628858	8.794999n
$\log \alpha_0'$	0.743547	$\log \alpha_0'$	0.743547		0.743547
$\log(\tan \delta_0)''$	1.523510	$\log I$	0.276379	8.372405	9.538546n
$\log III$	2.267057	$\sin(p - \alpha_0)$	9.943994n	6.675247n	9.235709
I	+ 26.6996	$\log(\tan \delta_0)'$	0.299872	0.299872	0.299872
- II	- 234.1844	$\log II$	0.243866n	6.975119n	9.535581
III	+ 184.9512	add	8.890629	9.982244	7.835714
Δ	- 22.5336	$\log \{ \}$	9.134495	8.354649	7.371295n
$\log \Delta$	1.352830n	$\log(- s)$	9.979360n	0.719250n	0.686353n
		$\log - s \{ \}$	9.113855n	9.073899n	8.057648
		$\log \kappa; \log(\kappa); \log[\kappa]$	7.761025	7.721069	6.704818n
$\log R_0^3$	9.982684			$\log \gamma \sec \delta_0$	3.025381
$\log \left(1 = \frac{\kappa}{R_0^3} \right)$	7.778341			$\log F$	0.786406
$\log m$	6.979906			$\log E$	7.726356
$\log[\kappa]$	6.704818n				
$\text{colog}(r)^3$	7.910959				
$\log II$	1.595683n ⁻¹⁰				
add	0				
$\log[]$	7.778341				
$\log \gamma$	3.020094				
$\sec \delta_0$	0.005287				
$\log G$	0.803722n				

III

$$\text{Let } H = \frac{F}{[r]^2} + G.$$

For first approximation to H let $(r) = [r]$.

Solution for r .

Trial I			
log F	0.786406	$(\rho) \cos (\psi)$	0.724254
colog $(r)^2$	7.910959	sub	0.264665
log I	8.697365	log denom.	1.064974
log G	0.803722n	log E	7.726356
add	9.996587	log r^3	6.661382
log H	0.800309n	log r	8.887127

Substituting this approximate value of r in the complete right member of the equation (b) for r^3 , we will hereafter get two solutions for r because of the double sign before the radical. The solution coming from the positive sign for the radical will be designated r_1 and called "first solution"; that from the negative sign r_2 and called "second solution." The trials then proceed:

Trial II		III for r_1	III for r_2
log r^2	7.774254	7.776090	7.772408
log $(\rho)^2 \sin^2 (\psi)$	6.691092	6.691092	6.691092
sub	9.962572	9.962737	9.962405
diff.	7.736826	7.738827	7.734813
log $\sqrt{\text{diff.}}$	8.868413	8.869414	8.867406
log r^3	6.661382	6.664135	6.658612
log $r^3 \sqrt{\text{diff.}}$	5.529795	5.533549	5.526018
log E	7.726356	7.726356	7.726356
add	0.002753	0.002777	
sum	7.729109	7.729133	
log r^3	6.664135	6.664159	
log r_1	8.888045	8.888053	
sub	9.997230		9.997253
diff.	7.723586		7.723609
log r_2^3	6.658612		6.658635
log r_2	8.886204		8.886212

Second Approximation for H by (c, d, a) .

	First Solution.	Second Solution.		First Solution.	Second Solution.
log r^3	6.664159	6.658635	I	+ 0.97377	+ 0.97377
log E	7.726356	7.726356	II	+ 27.30800	+ 28.87353
log $\frac{E}{r^3}$	1.062197	1.067721	- III	- 4.28300	- 4.40406
log H	0.800309n	0.800309n	$[r]^2$	+ 23.99877	+ 25.44324
add	9.917836	9.929941	log $[r]^2$	1.380189	1.405572
log ρ	0.718145	0.730250	log $[r]$	0.690094	0.702786
log ρ^2	1.436290	1.460500	log $[r]^3$	2.070283	2.108358
log 2	0.301030		log F	0.786406	0.786406
log R	9.994228		log $\frac{F}{[r]^3}$	8.716123	8.678048
cos $[\psi]$	9.618345		log G	0.803722n	0.803722n
log III	0.631748	0.643853	add	9.996436	9.996736
log R^2	9.988456		log H	0.800158n	0.800458n
			$(\rho) \cos (\psi)$	0.724254	0.724254
			sub	0.264734	0.264597
			log $[(\rho) \cos (\psi) - H]$	1.064892	1.065055

Using Second Approximation for H by (δ).

Trial	IV for r_1	IV for r_2
$\log r^2$	7.776106	7.772424
$(\rho)^2 \sin^2 (\psi)$	6.691092	6.691092
sub	9.962738	9.962407
diff.	7.738844	7.734831
$\log 1/\text{diff.}$	8.869422	8.867415
$\log r^3$	6.664159	6.658635
$\log r^3 1/\text{diff.}$	5.533581	5.526050
$\log E$	7.726356	7.726356
add	0.002777	
sum	7.729133	
$\log \text{denom.}$	1.064892	
$\log r_1^3$	6.664241	
$\log r_1$	8.888080	

Trial.	IV for r_2
sub	9.997253
diff.	7.723609
$\log \text{denom.}$	1.065055
$\log r_2^3$	6.658554
$\log r_2$	8.886185

Substituting these values of r in the equation for $[\rho]$ we find (details omitted)

$\log \rho$	0.718147	0.730251
$\log [r]$	0.690096	0.702787

which agree so closely with those found in the second approximation for H that another approximation is not necessary and the solution for r is ended.

IV

Determination of σ , σ' , and σ'' .

	First Solution.	Second Solution.
$\log \rho_0$	0.718147	0.730251
$\cos \delta_0$	9.994713	9.994713
$\log \sigma_0$	0.712860	0.724964
$\log \alpha_0''$	2.069686	2.069686
$\log I$	2.782546	2.794650
$\text{colog } r_0^3$	3.335759	3.341446
$\log (\sigma)$	0.719250	0.719250
$\sin [(\alpha) - \alpha_0]$	6.675247n	6.675247n
$\log II$	0.730256n	0.735943n
$\text{colog } [r]^3$	7.929712	7.891639
$\text{colog } R_0^3$	0.017316	0.017316
sub	9.996436	9.996736
$\log []$	0.013752n	0.014052n
$\sin (A - \alpha_0)$	9.943994n	9.943994n
$\log S$	9.979360	9.979360
$\log \gamma$	3.020094	3.020094
$\log III$	2.957200	2.957500
$\log (s)$	0.686353	
$\text{colog } r^3$	7.910959	
$\sin [(a) - \alpha_0]$	9.235709	
$\log IV$	7.833021	
- I	- 606.1029	- 623.2329
+ II	- 5.3735	- 5.4443
+ III	+ 906.1500	+ 906.7760
- IV	- 0.0068	- 0.0068
sum	+ 294.6668	+ 278.0920
$\log \text{sum}$	2.469331	2.444188
$\log 2 \alpha_0'$	1.044577	1.044577
$\log \sigma_0'$	1.424754	1.399611

	First Solution.	Second Solution.
$\text{colog } r_0^3$	3.335759	3.341446
$\log (\alpha_0')^2$	1.487094	1.487094
sub	9.993802	9.993884
diff.	3.329561	3.335330
$\log \frac{\gamma}{[r]^3}$	0.949806	0.911733
add	0.001808	0.001635
$\log []$	3.331369	3.336965
$\log \sigma_0$	0.712860	0.724964
$\log I$	4.044229	4.061929
$\log \frac{(\sigma_0)}{r_0^3}$	4.055009	4.060696
$\cos [(\alpha) - \alpha_0]$	0.000000	
$\log II$	4.055009	4.060696
$\log \gamma S \left[\frac{1}{[r]^3} - \frac{1}{R_0^3} \right]$	3.013206n	3.013506n
$\cos (A - \alpha_0)$	9.678338	9.678338
$\log III$	2.691544n	2.691844n
$\log \frac{(s)}{[r]^3}$	8.597312	
$\cos [(a) - \alpha_0]$	0.000000	
$\log IV$	8.597312	
- I	- 11072.08	- 11532.65
+ II	+ 11350.34	+ 11499.95
+ III	- 491.52	- 491.86
- IV	- 0.04	- 0.04
σ_0''	- 213.30	- 524.60
$\log \sigma_0''$	2.328991n	2.719828n

Determination of the Velocities of Jupiter's Geocentric Rectangular Coördinates at the Middle Date.

1905 Gr. M. T.	Feb. 1.0	5.0	9.0	13.0	Feb. 17.0
log (ρ)	0.7147895	0.7198293	0.7246981	0.7293847	0.7338806
(α) 1905.0	22° 33' 23".4	23° 6' 36".0	23° 41' 51".8	24° 19' 3".3	24° 58' 3".8
(δ) 1905.0	+ 8 13 42.1	+ 8 27 42.8	+ 8 42 24.5	+ 8 57 43.6	+ 9 13 36.5
cos (δ)	9.9955060	9.9952463	9.9949661	9.9946653	9.9943442
cos (α)	9.9654377	9.9636712	9.9617430	9.9596504	9.9573897
log (ξ)	0.6757332	0.6787468	0.6814072	0.6837004	0.6856145
sin (α)	9.5838721	9.5938371	9.6041303	9.6146801	9.6254232
log (η)	0.2941676	0.3089127	0.3237945	0.3387301	0.3536480
sin (δ)	9.1556969	9.1677647	9.1800634	9.1925153	9.2050500
log (ζ)	9.8704864	9.8875940	9.9047615	9.9219000	9.9389306

	(ξ)	f^I	f^{II}	f^{III}	f^{IV}
Feb. 1.0	+4.739508				
5.0	4.772510	+0.033002	-0.003678		
9.0	4.801834	29324	3901	-0.000223	
13.0	4.827257	25423	-0.004101	-0.000200	
Feb. 17.0	+4.848579	+0.021322			+0.000023

	(η)				
Feb. 1.0	+1.968646				
5.0	2.036633	+0.067987	+0.003011		
9.0	2.107631	70998	2745	-0.000266	
13.0	2.181374	73743	+0.002488	-0.000257	
Feb. 17.0	+2.257605	+0.076231			+0.000009

	(ζ)				
Feb. 1.0	+0.7421410				
5.0	0.7719586	+0.0298176	+0.0013088		
9.0	0.8030850	311264	11992	-0.0001096	
13.0	0.8354106	323256	+0.0010854	-0.0001138	
Feb. 17.0	+0.8688216	+0.0334110			-0.0000042

Date: = Feb. 8.631944

 $n = -0.092014$ $N_1^3(n) = -0.16243$ $N_1^4(n) = -0.0819$

(φ)	(ξ)	(η)	(ζ)
$f^I (a + iw)$	+0.0273735	+0.0723705	+0.0317260
$n f^{II} (a + iw)$	+ 3589	- 2526	- 1103
$N_1^3(n) f^{III} (a + iw)$	+ 342	+ 424	+ 181
$N_1^4(n) f^{IV} (a + iw)$	- 19	- 07	+ 3
sum	+0.0277647	+0.0721594	+0.0316341
log sum	8.443493	8.858293	8.500155
log (φ) ₀	1.115899	1.530699	1.172561

V

	First Solution.	Second Solution.
$\log \sigma_0$	0.712860	0.724964
$\cos \alpha_0$	9.961837	9.961837
$\sin \alpha_0$	9.603641	9.603641
$\tan \delta_0$	9.195864	9.195864
$\log \sigma_0 \cos \alpha_0$	0.674697	0.686801
$\log (\xi)_0$	0.681177	0.681177
sub	8.177067	8.115077
$\log x_0$	8.851764n	8.796254
$\log \sigma_0 \sin \alpha_0$	0.316501	0.328605
$\log (\eta)_0$	0.322421	0.322421
sub	8.137500	8.156571
$\log y_0$	8.454001n	8.478992
$\log \sigma_0 \tan \delta_0$	9.908724	9.920828
$\log (\zeta)_0$	9.903183	9.903183
sub	8.108583	8.617692
$\log z_0$	8.011766	8.520875
$\log x_0^2$	7.703528	7.592508
$\log y_0^2$	6.908002	6.957984
add	0.064507	0.090608
sum	7.768035	7.683116
$\log z_0^2$	6.023532	7.041750
add	0.007752	0.089328
$\log r_0^2$	7.775787	7.772444
$\log r_0^*$	8.887893	8.886222

* These values would have checked more closely with the final values resulting from the trials, if the inaccuracy indicated in the foot note 2, page 482, had not been committed.

	First Solution.	Second Solution.
$\log \sigma'_0$	1.424754	1.399611
$\log I$	1.386591	1.361448
$\log \alpha'_0$	0.743547	0.743547
$\log II$	1.060048	1.072152
I	+ 24.35517	+ 22.98517
- II	- 11.48281	- 11.80733
- III = - $(\xi)'_0$	- 13.05867	- 13.05867
x'_0	- 0.18631	- 1.88083
$\log x'_0$	9.270236n	0.274350n
$\log I$	1.028395	1.003252
$\log II$	1.418244	1.430348
I	+ 10.67566	+ 10.07516
II	+ 26.19653	+ 26.93694
- III = - $(\eta)'_0$	- 33.93900	- 33.93900
y'_0	+ 2.93319	+ 3.07310
$\log y'_0$	0.467340	0.487577
$\log I$	0.620618	0.595475
$\log (\tan \delta)'_0$	0.299872	0.299872
$\log II$	1.012732	1.024836
I	+ 4.17463	+ 3.93981
II	+ 10.29750	+ 10.58854
- III = - $(\zeta)'_0$	- 14.87855	- 14.87855
z'_0	- 0.40642	- 0.35020
$\log z'_0$	9.608975n	9.544316n
$\log x_0 x'_0$	8.122000	9.070604n
$\log y_0 y'_0$	8.921341n	8.966569
add	9.924936	9.432450
sum	8.846277n	8.399019n
$\log z_0 z'_0$	7.620741n	8.065191n
add	0.025098	0.165431
$\log r_0 r'_0$	8.871375n	8.564450n
$\log r'_0$	9.983295n	9.678265n

Preliminary Osculating Elements (Equatorial).

	First Solution.	Second Solution.		First Solution.	Second Solution.
$\log y_0 z_0'$	8.062976	8.023308n	$\log x_0 \cos \Omega$	8.784405n	8.326032
$\log z_0 y_0'$	8.479106	9.008452	$\log y_0 \sin \Omega$	8.167013	8.452538n
sub	9.789868	0.042765	add	9.880054	9.529115
$1/p \sin i \sin \Omega$	8.268974n	9.051217n	$\log r_0 \cos u_0$	8.664459n	7.855147n
$\log x_0 z_0'$	8.460739	8.340570n	$\log r_0 \sin u_0$	8.791870	8.884332
$\log z_0 x_0'$	7.282002n	8.795225n	$\tan u_0$	0.127411n	1.029185n
sub	0.027864	9.812224	u_0	126° 42' 48".4	95° 20' 30".0
$1/p \sin i \cos \Omega$	8.488603	8.607449	v_0	208 2 4 .4	266 5 11 .3
$\tan \Omega$	9.780371n	0.443768n	ω	278 40 44 .0	189 15 18 .7
Ω	328° 54' 24".8	289° 47' 45".3	$\sin u_0$	9.903977	9.998110
$\sin \Omega$	9.713012n	9.973546n	$\cos u_0$	9.776566n	8.968925n
$\cos \Omega$	9.932641	9.529778	$\log r_0$	8.887893	8.886222
$1/p \sin i$	8.555962	9.077671	$\log e^2$	9.294520	8.240814
$\log x_0 y_0'$	9.319104n	9.283831	$\log (1 - e^2)$	9.904702	9.992372
$\log y_0 x_0'$	7.724237	8.753342n	$\log a$	8.767430	8.889884
sub	0.010900	0.112199	$\log a^3$	6.302290	6.669652
$1/p \cos i$	9.330004n	9.396030	$\log a^{3/2}$	8.151145	8.334826
$\tan i$	9.225958n	9.681641	$\log \frac{k}{(k)} \frac{1}{1 + m_0}$	1.510048	1.510048
i	170° 26' 57".8	25° 39' 41".7	$\log 1 \text{ year (days)}$	2.562581	2.562581
$\sin i$	9.219896	9.636543	$\log P \text{ (days)}$	2.223774	2.407455
$\cos i$	9.993938n	9.954902	P	167 ^d .4073	255 ^d .5376
$\log 1/p$	9.336066	9.441128	$\log 360^\circ$	2.556302	2.556302
$\log p$	8.672132	8.882256	$\log \mu^\circ$	0.332528	0.148847
$\log \frac{p}{r_0}$	9.784052	9.996071	μ°	2°.150445	1°.408793
sub	9.809003	7.958444			
$e \cos v_0$	9.593055n	7.954515n	$\log \frac{1-e}{1+e}$	9.792822	9.942359
$e \sin v_0$	9.319361n	9.119393n	$\tan \frac{1}{2} v_0$	0.602671n	0.029687n
$\tan v_0$	9.726306	1.164878	$\tan \frac{1}{2} E_0$	0.395493n	9.972046n
v_0	208° 2' 4".4	266° 5' 11".3	$\frac{1}{2} E_0$	111° 54' 46".7	136° 50' 33".7
$\sin v_0$	9.672101n	9.998986n	E_0	223 49 33 .4	273 41 7 .4
$\cos v_0$	9.945795n	8.834108n	$\sin E_0$	9.840400n	9.999101n
$\log e$	9.647260	9.120407	$\log e'' \sin E_0$	4.802085n	4.433933n
$\log e''$	4.961685	4.434832	$e'' \sin E_0$	- 63399".4	- 27160".2
φ	26° 21' 9".5	7° 34' 56".1	M_0	- 17° 36' 39".4 241 26 12 .8	- 7° 32' 40".2 281 13 47 .6
$\frac{1}{2} v_0$	104° 1' 2".2	133° 2' 35".6	$\log ab. T$	8.47943n	8.49153n
$\log (1 - e)$	9.745173	9.938545	$ab. T$	- 0.030160	- 0.031012
$\log (1 + e)$	0.159529	0.053827	$t_0 = \text{Feb.}$	8.631944	8.631944
$\log \frac{1-e}{1+e}$	9.585644	9.884718	Epoch = Feb.	8.601784	8.600932

Constants for the Equator 1905.0.

(Preliminary Osculating Elements).

	First Solution.	Second Solution.		First Solution.	Second Solution.
$\sin a \sin A_a$	9.932641	9.529778	$\sin b \sin B_a$	9.713012n	9.973546n
$\sin a \cos A_a$	9.706950n	9.928448	$\sin b \cos B_a$	9.926579n	9.484680
$\tan A_a$	0.225691n	9.601330	$\tan B_a$	9.786433	0.488866n
A_a	120° 44' 26".0	21° 46' 5".6	B_a	211° 26' 52".5	287° 55' 30".4
ω	278 40 44.0	189 15 18.7	B'	130 7 36.5	117 13 49.1
A'	39 25 10.0	211 1 24.3			
$\sin A_a$	9.934241	9.569202	$\sin B_a$	9.717441n	9.978268n
$\cos A_a$	9.708550n	9.967872	$\cos B_a$	9.931008n	9.489402
$\sin a$	9.998400	9.960576	$\sin b$	9.995571	9.995278

First Solution.	Second Solution.
$x = r [9.998400] \sin (39^\circ 25' 10".0 + v)$	$x = r [9.960576] \sin (211^\circ 1' 24".3 + v)$
$y = r [9.995571] \sin (130 7 36.5 + v)$	$y = r [9.995278] \sin (117 13 49.1 + v)$
$z = r [9.219896] \sin (278 40 44.0 + v)$	$z = r [9.636543] \sin (189 15 18.7 + v)$

In order to determine which of the two solutions is the physical solution, a place other than those upon which the orbits are based was computed from each orbit. An observation of August 9 was available at the time this work was done. The position was determined without taking into account the perturbations due to the action of the Sun.

The representation of the August 9 observation in position angle and distance (with respect to Jupiter) is

	First Solution.	Second Solution.
$(O - C) \left\{ \begin{array}{l} \Delta p \\ \Delta s \end{array} \right.$	$\begin{array}{l} + 216".9 \\ + 15'.6 \end{array}$	$\begin{array}{l} + 2".7 \\ + 5'.2 \end{array}$

These residuals lead us to reject the first solution (which is retrograde) and accept the second solution (which is direct).

VI

Using the second solution, the special perturbations due to the action of the Sun were then computed by ENCKE's method for the period January 4 to March 9 (using eight day intervals).

Then with the osculating elements and these perturbations, the positions for the dates of the fundamental places, January 3, January 28, February 21 and March 6 were computed. The representation of these places is:

	Jan. 3	Jan. 28	Feb. 21	March 6
$(O - C) \left\{ \begin{array}{l} \Delta \alpha \\ \Delta \delta \end{array} \right.$	$\begin{array}{l} - 9".6 \\ + 23.9 \end{array}$	$\begin{array}{l} - 2".3 \\ - 0.4 \end{array}$	$\begin{array}{l} + 2".6 \\ - 0.6 \end{array}$	$\begin{array}{l} + 1".0 \\ - 11.3 \end{array}$

Using the residuals of January 3 and March 6, with the use of the series for $\Delta f, \Delta g$, *Synopsis of Formulae*, [VII], Part 7, a differential correction was computed, giving an orbit which represented these places by

	Jan. 3	March 6
$(O - C) \left\{ \begin{array}{l} \Delta \alpha \\ \Delta \delta \end{array} \right.$	$\begin{array}{l} - 3".5 \\ + 3.8 \end{array}$	$\begin{array}{l} - 1".4 \\ + 1.2 \end{array}$

This orbit is given by the following:

Elements.

Epoch 1905 Feb. 8.6009 Gr. M. T.

M_0	283° 4' 4"	} Mean Equinox and Equator 1905.0.
ω	187 29 41	
Ω	288 19 59	
i	25 39 24	
e	0.121519	
$\log a$	8.893716	
μ^*	1°.39027	
P	258.9424 days.	

Constants for the Equator 1905.0.

$$\begin{aligned} x &= r [9.959820] \sin (207^\circ 40' 43'' + v) \\ y &= r [9.995934] \sin (114 \quad 7 \quad 31 + v) \\ z &= r [9.636463] \sin (187 \quad 29 \quad 41 + v) \end{aligned}$$

Messrs. CRAWFORD and CHAMPREUX have also computed an orbit *irrespective of the perturbations* for this satellite. For the sake of comparison all of their results are tabulated below, together with the preliminary orbit derived by PERRINE¹ and the final orbit derived by ROSS.²

The three orbits by CRAWFORD and CHAMPREUX are designated below respectively by (Cr. & Ch.)₁, (Cr. & Ch.)₂, and (Cr. & Ch.)₃. The orbit (Cr. & Ch.)₁ represents the *solution irrespective of the perturbations*; the orbit (Cr. & Ch.)₂ represents the *direct solution of the disturbed orbit*; the orbit (Cr. & Ch.)₃ represents the same after the removal of the residuals of the first and last places by *differential correction*. All of these orbits are freely referred to by Professor LEUSCHNER in the *Introduction* to Part 9.

³ELEMENTS OF THE SEVENTH SATELLITE OF JUPITER (DIRECT MOTION) REFERRED TO THE EARTH'S EQUATOR.

Computer	Epoch 1905	M			Ω		
(Cr. & Ch.) ₁	February 8.6009	83°	17'	57"	279°	45'	8"
(Cr. & Ch.) ₂	February 8.6009	281	13	48	289	47	45
(Cr. & Ch.) ₃	February 8.6009	283	19	59	288	19	59
PERRINE (preliminary).					275	47	
ROSS (final)					281	7.8	

i			ω	e	Period	a	
26°	27'	14"	6° 35' 42"	0.12576	251.1415	49'	48"
25	39	42	189 15 19	0.13195	255.5376	50	20
25	39	23	187 29 41	0.12152	258.9424	50	47
26	15		182 6	0.24	200	43	48
26	12		331 16.8	0.0246	265.0	52	54

a (Cr. & Ch.) for $\log (\mu) = 0.72124$; a (Ross) for $\log (\mu) = 0.71624$.

¹L. O. Bulletin No. 78.

²L. O. Bulletin No. 82.

³Cf. Addendum, page 503.

THE ORBIT OF THE EIGHTH SATELLITE OF JUPITER.

I

The observations upon which this orbit is based are the following:

	1908 Gr. M. T.	α (1908.0)	δ (1908.0)	Reference
I	Jan. 27.5288	131° 27' 50".2	+ 18° 5' 6".3	<i>M. N.</i> Vol. LXVIII, No. 8.
II	Feb. 22.4560	128 23 42 .1	+ 19 5 54 .3	<i>M. N.</i> Vol. LXVIII, No. 8.
III	March 8.8486	127 9 49 .1	+ 19 31 36 .7	<i>L. O. Bulletin</i> No. 156.
IV	Apr. 1.7021	126 41 24 .4	+ 19 49 27 .6	<i>L. O. Bulletin</i> No. 156.
V	Apr. 29.7023	128 26 3 .9	+ 19 35 49 .6	<i>L. O. Bulletin</i> No. 156.

The coördinates α and δ are the observed coördinates corrected for parallax corresponding to Jupiter's distance.

The mass factors that are used are the same as those used for the Seventh Satellite.

The first approximation for the velocities and accelerations, using the first, third, and fifth observations, gives:

$$\begin{array}{ll} \log \alpha_0' & 2.228880n \\ \log \alpha_0'' & 0.997216 \end{array} \quad \begin{array}{ll} \log \delta_0' & 1.857764 \\ \log \delta_0'' & 0.413583n \end{array}$$

The more accurate values of these quantities depending upon the five observations are:

$$\begin{array}{lll} \log \alpha_0' & 2.322919n & \log \delta_0' & 1.902567 & \log (\tan \delta)_0' & 9.914059 \\ \log \alpha_0'' & 1.039044 & \log \delta_0'' & 0.447788n & \log (\tan \delta)_0'' & 1.730332n \end{array}$$

From the *American Ephemeris and Nautical Almanac* we take, for the middle date, approximate $\log (\rho) = 0.658699$. With this (ρ) , the corrected middle date becomes March 8.82230. For this date, we have the following:

$$\begin{array}{lll} {}^1(\alpha) & 126^\circ 48' 37''.6 & (\alpha) - \alpha_n & -21' 11''.5 & \cos \psi & 9.999978 \\ (\delta) & +19 59 30 .5 & \tan \psi & 7.998946n & \sin \psi & 7.998924n \end{array}$$

The heliocentric coördinates of Jupiter reduced to 1908.0 for the corrected middle date are

$$(l) = 131^\circ 44' 47''.0 \quad (b) = +0^\circ 41' 55''.2 \quad \log (r) = 0.725533$$

Hence

$$\begin{array}{lll} (a) & 134^\circ 24' 47''.1 & \sin (d) & 9.488674 \\ \tan (d) & 9.510330 & \cos (d) & 9.978344 \end{array}$$

The Sun's coördinates for the uncorrected middle date are

$$X_0 = +0.9728883 \quad Y_0 = -0.1833903 \quad Z_0 = -0.0795554$$

Then

$$\log R_0 = 9.997043 \quad \cos [\psi] = 9.859229n$$

Jupiter's geocentric rectangular velocities are given by

$$\log (\dot{x})' = 0.569466n \quad \log (\dot{y})' = 1.322412 \quad \log (\dot{z})' = 0.973888$$

II

$$\begin{array}{lll} \log \kappa & 7.570066n & \log (\kappa) & 7.001973 & \log [\kappa] & 7.67395 \\ \log G & 0.624754 & \log F & 0.615886n & \log E & 7.027699 \end{array}$$

¹ See footnote, page 490.

III

With a first approximation to II , letting $(r) = [r]$, we get for the first trial $\log r = 9.153204$.

Using this r in the complete equation for r^3 ,

$$r^3 = E \pm \frac{r^3 \sqrt{r^2 - (\rho)^2 \sin^2(\psi)}}{(\rho) \cos(\psi) - H},$$

we get for a second trial¹ for each solution

$$\log r_1 = 9.198200$$

$$\log r_2 = 9.087567$$

and from a third trial

$$\log r_1 = 9.217353$$

$$\log r_2 = 9.121781$$

The solutions for r from this point on differ from the method used in the computation for the Seventh Satellite, so the computation will be given in detail.

Solutions.	Trial IV		Trial V	
	1	2	1	2
$\log r^2$	8.43171	8.24356	8.48248	8.22483
$\log 4 r^2$	9.03677	8.84562	9.08454	8.82689
$\log 3 (\rho)^2 \sin^2(\psi)$	7.79237	7.79237	7.79237	7.79237
sub	9.97453	9.95978	9.97725	9.95792
$\log [\text{numer.}]$	9.01130	8.80540	9.06179	8.78481
$\log 3 K \sqrt{r^2 - (\rho)^2 \sin^2(\psi)}$	9.24541	9.13974	9.27115	9.12908
$\log II$	9.76589	9.66566	9.79064	9.65573
sub ; add	9.85394		9.79197	
$\log []$	9.61983	0.16527	9.58261	0.16215
$\log 3 r^2$	8.91183	8.72068	8.95960	8.70195
$\log \text{denom.}$	8.53166	8.88595	8.54221	8.86410
$\log (-M)$	6.50140	6.33689n	4.6272n	4.7227n
$\log \partial r$	7.96974	7.45094n	6.0850n	5.8586n
old $\log r_1$; $\log r_2$	9.217353	9.121781	9.241241	9.112413
add	0.023888	9.990632	9.999696	9.999758
$\log r_1$; $\log r_2$	9.241241	9.112413	9.240937	9.112171
$\log r^2$	8.482482	8.224826	8.481874	8.224342
$(\rho)^2 \sin^2(\psi)$	7.315246	7.315246	7.315246	7.315246
sub	9.969398	9.942927 +	9.969353 +	9.942859 +
diff.	8.451880	8.167753 +	8.451227 +	8.167201 +
$\log 1$ diff.	9.225940	9.083877	9.225614	9.083601
$\log r^3$	7.723723	7.337239	7.722811	7.336513
$\log r^3 1$ diff.	6.949663	6.421116	6.948425	6.420114
$\log E$	7.027699	7.027699	7.027699	7.027699
add ; sub	0.263762	9.876570	0.263799	9.876887
sum ; diff.	7.291461	6.904269	7.290898	6.904586
$\log K$	9.568086	9.568086	9.568086	9.568086
$\log r_1^3$	7.723375	7.336183	7.722812	7.336500
$\log r_1$; $\log r_2$	9.241125	9.112061	9.24037	9.112167
sub	6.9038	7.3865		5.47
$\log M$	4.6172	4.7227		2.81
			$\log \text{denom.}$	8.8641
			$\log (-M)$	2.81n
			$\log \partial r$	3.95n
			old $\log r_1$	9.112171
			add	9.999997
			$\log r_2$	9.112168

¹ After the first trial the differential formulae, the application of which follows, might have been applied at once.

With these values of r_1 and r_2 , we get for a second approximation for H , as shown in detail for the Seventh Satellite, the following:

	First Solution.	Second Solution.
$\log [r]$	0.712058	0.735589
$\log H$	0.621633	0.622103
$\log K = \{(\rho) \cos (\psi) - H\}$	9.571221	9.565903

With this second approximation for H , we get new values for r_1 and r_2 by the differential method:

$$\log r_1 = 9.238241 \quad \log r_2 = 9.112669$$

A third approximation for H gave the following final values¹:

$$\log r_1 = 9.238259 \quad \log r_2 = 9.112669$$

IV

	First Solution.	Second Solution.
$\log \sigma$	0.616736	0.644368
$\log \sigma'$	1.300139	1.542730
$\log \sigma''$	2.897837	2.850649

V

	First Solution.	Second Solution.		First Solution.	Second Solution.
$\log x_0$	8.823340	8.989527n	$\log x'_0$	0.305495n	1.026380n
$\log y_0$	9.119019n	8.929802	$\log y'_0$	9.489438n	1.075799
$\log z_0$	8.957631n	7.755080	$\log z'_0$	0.023899	0.817936
$\log r_0$	9.238270	9.112645	$\log r'_0$	0.039975n	1.207024

Preliminary Osculating Elements (Equatorial).

	First Solution.	Second Solution.		First Solution.	Second Solution.
ω	61° 40' 10".7	289° 46' 19".6	$\log a$	9.203422	7.548979n
Ω	235 55 28.0	139 46 52.9	P	2.06624 years	
i	144 51 14.9	108 44 14.9	μ_0	0°.477023	
e	0.48212	13.55153	M_0	287° 13' 34".4	
			Epoch =	March 8.82326	T = Feb. 24.5842

Constants for the Equator (1908.0).

	First Solution.	Second Solution.		First Solution.	Second Solution.
A'	281° 16' 7".4	214° 58' 9".6	$\sin a$	9.943989	9.898318
B'	0 37 1.1	358 58 17.4	$\sin b$	9.976146	9.839306
C'	61 40 10.7	289 46 19.6	$\sin c$	9.760166	9.976351

The observations were represented by means of the Elements and Constants for the Equator for each solution, with the following results:

Representation from the First Solution.

	Jan. 27	Feb. 22	April 1	April 29
$(O - C) \left\{ \begin{array}{l} \Delta \alpha \\ \Delta \delta \end{array} \right.$	-46".2 +10.7	-10".9 +4.7	+34".9 -13.3	+136".0 -46.5

¹ By the graphical solution, cf. footnote 1, page 290, Dr. E. S. HAYNES has found the number of solutions to be three, the third solution being close to the second.

Representation from the Second Solution.

	Jan. 27	Feb. 22	April 1	April 29
$(O - C) \left\{ \begin{array}{l} \Delta\alpha \\ \Delta\delta \end{array} \right\}$	$\begin{array}{l} - 4' \ 27''.4 \\ - 21 \ 4.4 \end{array}$	$\begin{array}{l} + 0' \ 12''.1 \\ 2 \ 3.2 \end{array}$	$\begin{array}{l} + 2' \ 24''.9 \\ - 0 \ 14.1 \end{array}$	$\begin{array}{l} + 18' \ 17''.2 \\ - 2 \ 49.2 \end{array}$

It is evident from a comparison of the residuals by the two solutions that the first solution is the physical one.

VI

Using the first solution the special perturbations due to the action of the Sun were computed by the modification of ENCKE'S method over the period from January 25 to April 30.

The osculating Elements with the perturbations gave the following representation:

	Jan. 27	Feb. 22	April 1	April 29
$(O - C) \left\{ \begin{array}{l} \Delta\alpha \\ \Delta\delta \end{array} \right\}$	$\begin{array}{l} - 42''.7 \\ + 6.7 \end{array}$	$\begin{array}{l} - 9''.9 \\ + 4.2 \end{array}$	$\begin{array}{l} + 37''.8 \\ - 14.4 \end{array}$	$\begin{array}{l} + 146''.7 \\ - 51.2 \end{array}$

Using the residuals of January 27 and April 29, a differential correction by the use of the closed expressions as shown in Part 8, Example No. 10, gave the following results:

	Jan. 27	April 29
$(O - C) \left\{ \begin{array}{l} \Delta\alpha \\ \Delta\delta \end{array} \right\}$	$\begin{array}{l} + 5''.5 \\ - 0.3 \end{array}$	$\begin{array}{l} + 11''.2 \\ - 3.0 \end{array}$

These residuals were substituted for the constant terms in the equations arising from the first differential correction. This procedure gave the final results as follows:

Representation of the Observations.

	Jan. 27	Feb. 22	April 1	April 29
$(O - C) \left\{ \begin{array}{l} \Delta\alpha \cos \delta + 4''.3 \\ \Delta\delta \end{array} \right\}$	$\begin{array}{l} + 0.8 \\ + 0.8 \end{array}$	$\begin{array}{l} + 4''.5 \\ + 0.8 \end{array}$	$\begin{array}{l} + 0''.8 \\ - 0.4 \end{array}$	$\begin{array}{l} + 2''.8 \\ + 0.3 \end{array}$

Elements.

Epoch 1908 March 8.82326 Gr. M. T.

M_0	266° 3' 51"	} Mean equinox and equator 1908.0.
ω	67 45 54	
Ω	240 1 9	
i	144 51 21	
e	0.351958	
$\log a$	9.215643	
μ°	0°.457308	
P	2.15532 years	

Constants for the Equator 1908.0.

$$\begin{aligned} x &= r [9.937932] \sin (282^\circ \ 58' \ 7'' + \tau) \\ y &= r [9.981246] \sin (\ 3 \ 1 \ 12 + \tau) \\ z &= r [9.760148] \sin (\ 67 \ 45 \ 54 + \tau) \end{aligned}$$

Below is given a tabulation of the original elements derived by COWELL and CROMMELIN and of those derived above, together with the corresponding residuals, from which it would appear that the results obtained by LEUSCHNER'S method are all that could possibly be desired.

Elements.

COWELL and CROMMELIN		CRAWFORD and MEYER
i	148°.86	144°.86
Ω	277°.46	240°.02
e	0.333 (about)	0.352
a	0.1702 (about)	0.1643
P	2.167 years.	2.155

Residuals (O - C).

Date 1908	C. & C.	$\Delta\alpha$	C. & M.	C. & C.	$\Delta\delta$	C. & M.
Jan. 27	+0".6		+4".3	+0".6		+0".8
Feb. 22	+1.0		+4.5	+0.3		+0.8
March 8	-2.4		0.0	+0.4		0.0
April 1	+2.0		+0.8	-1.6		-0.4
April 29	+8.2		+2.8	-2.6		+0.3

ADDENDUM.

Before this volume was ready to go to press the following set of elements for the Seventh Satellite by ROSS appeared in *A. N.* 4175:

$$\left. \begin{array}{l} \Omega = 291^\circ.5 \\ i = 25^\circ.3 \\ \omega = 182^\circ.8 \end{array} \right\} \text{Earth's Equator 1905.0} \quad \begin{array}{l} e = 0.208 \\ P = 260.6 \text{ days.} \end{array}$$

These elements depend "upon twelve observations distributed uniformly over the observed arc from January 3, 1905, to September 25, 1906. The principal perturbations have been included." This set of elements, based upon two oppositions, is to be regarded as the best set of elements, at this time, for the orbit of the seventh satellite. Comparison shows that, with the exception of the eccentricity, the elements derived above by the direct solution agree better with these last elements by ROSS than any of the others.

AUG 11 1953

